

12, March 2013, NR/HEP<sup>2</sup> @ Instituto Superior Técnico

---

# Initial Conditions for Numerical Relativity

~ Introduction to numerical methods  
for solving elliptic PDEs ~

Part I

Hirotsada Okawa  
CENTRA/Instituto Superior Técnico



## Goal of lectures

---

# Let's solve the elliptic PDEs!

● **Why?**  $\longrightarrow$  Part I

● **How?**  $\longrightarrow$  Part II

A black spiral binding is visible on the left side of the page, consisting of a series of black rings connected by a grey wire.

# Numerical Relativity

# ADM formalism

- **General Relativity** :  $4_{(N)}$ -dimensional spacetimes

- **ADM formalism** : Arnowitt-Deser-Misner(1962),

re-printed in arxiv:gr-qc/0405109

@ Cf. Hamiltonian formalism  $\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q^i}, \quad \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p^i}$

$\mathcal{H}$  : *Hamiltonian*,  $q^i$  : *coordinates*,  $p^i$  : *momenta*

@ Spacetimes = **Evolution** of **3D spaces**(**3+1** decomposition)

# ADM formalism

- **General Relativity** :  $4_{(N)}$ -dimensional spacetimes

- **ADM formalism** : Arnowitt-Deser-Misner(1962),

re-printed in arxiv:gr-qc/0405109

@ Cf. Hamiltonian formalism  $\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q^i}$ ,  $\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p^i}$

$\mathcal{H}$  : *Hamiltonian*,  $q^i$  : *coordinates*,  $p^i$  : *momenta*

@ Spacetimes = **Evolution** of **3D spaces**(**3+1** decomposition)



# ADM formalism

- **General Relativity** :  $4_{(N)}$ -dimensional spacetimes

- **ADM formalism** : Arnowitt-Deser-Misner(1962),

re-printed in arxiv:gr-qc/0405109

@ Cf. Hamiltonian formalism  $\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q^i}$ ,  $\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p^i}$

$\mathcal{H}$  : *Hamiltonian*,  $q^i$  : *coordinates*,  $p^i$  : *momenta*

@ Spacetimes = **Evolution** of **3D spaces**(**3+1** decomposition)



a 3-dimensional Cabbage

# ADM formalism

- **General Relativity** :  $4_{(N)}$ -dimensional spacetimes
- **ADM formalism** : Arnowitt-Deser-Misner(1962),  
re-printed in arxiv:gr-qc/0405109

@ Cf. Hamiltonian formalism  $\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q^i}$ ,  $\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p^i}$   
 $\mathcal{H}$  : *Hamiltonian*,  $q^i$  : *coordinates*,  $p^i$  : *momenta*

@ Spacetimes = **Evolution** of **3D spaces**(**3+1** decomposition)

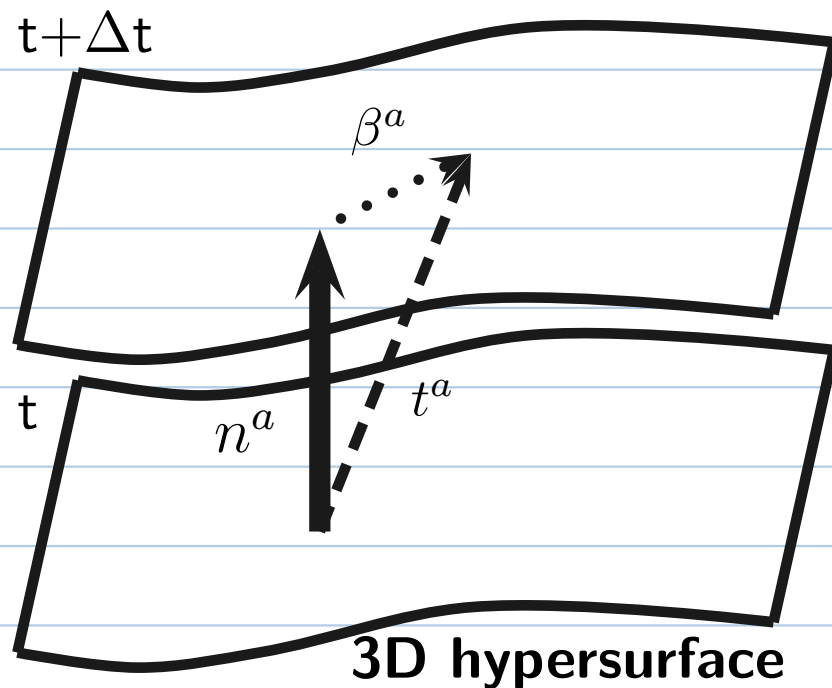


a 3-dimensional Cabbage



2-dimensional leaves

# ADM decomposition



$n^a$  is a timelike normal vector  
defined as  $n^a = (1/\alpha, \beta^i/\alpha)$ ,  $n^a n_a = -1$ .

## 3D spatial metric

(regarded as coordinate)

$$\gamma_{ab} \equiv g_{ab} + n_a n_b$$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij} (\beta^i dt + dx^i) (\beta^j dt + dx^j)$$

## Projection tensor

$$\gamma_b^a = \delta_b^a + n^a n_b$$

## 3D Derivatives

$$D_i \psi = \gamma_i^a \nabla_a \psi,$$

$$D_i W^j = \gamma_i^a \gamma_b^j \nabla_a W^b.$$

## Extrinsic curvature

(as momentum)

$$K_{ab} \equiv -\frac{1}{2\alpha} (\partial_t \gamma_{ab} - D_b \beta_a - D_a \beta_b)$$

## Einstein equations

$$G_{ab} \equiv \mathcal{R}_{ab} - \frac{1}{2} g_{ab} \mathcal{R} = \kappa T_{ab},$$

$$\kappa = 8\pi G/c^4.$$

## Projection of Einstein equations

$$\mathcal{C} \equiv (G_{ab} - \kappa T_{ab}) n^a n^b = 0,$$

$$\mathcal{D}_i \equiv \gamma_i^c (G_{cb} - \kappa T_{cb}) n^b = 0,$$

$$\mathcal{E}_{ij} \equiv \gamma_i^c \gamma_j^d (G_{cd} - \kappa T_{cd}) = 0.$$



# Einstein equations

$\rho$ :energy density,  $j_i$ :current density

$$G_{ab}n^an^b = \kappa T_{ab}n^an^b \rightarrow {}^{(3)}R + K^2 - K_{ij}K^{ij} = 2\kappa\rho,$$

$$\gamma_i^c G_{cb}n^b = \gamma_i^c \kappa T_{cb}n^b \rightarrow D_j K_i^j - D_i K = \kappa j_i.$$

# Einstein equations

## Hamiltonian Constraint

$\rho$ :energy density,  $j_i$ :current density

$$G_{ab}n^an^b = \kappa T_{ab}n^an^b \rightarrow {}^{(3)}R + K^2 - K_{ij}K^{ij} = 2\kappa\rho,$$

$$\gamma_i^c G_{cb}n^b = \gamma_i^c \kappa T_{cb}n^b \rightarrow D_j K_i^j - D_i K = \kappa j_i.$$

## Momentum Constraints

All quantities are spatial.  
(They must be satisfied on each 3D hypersurface.)

# Einstein equations

## Hamiltonian Constraint

$\rho$ :energy density,  $j_i$ :current density

$$G_{ab}n^an^b = \kappa T_{ab}n^an^b \rightarrow {}^{(3)}R + K^2 - K_{ij}K^{ij} = 2\kappa\rho,$$

$$\gamma_i^c G_{cb}n^b = \gamma_i^c \kappa T_{cb}n^b \rightarrow D_j K_i^j - D_i K = \kappa j_i.$$

## Momentum Constraints

All quantities are spatial.  
(They must be satisfied on each 3D hypersurface.)

## Evolution equation

$$\gamma_i^c \gamma_j^d G_{cd} = \gamma_i^c \gamma_j^d \kappa T_{cd} \longrightarrow$$

$$\begin{aligned} \partial_t K_{ij} = & \beta^k D_k K_{ij} + K_{kj} D_i \beta^k + K_{ki} D_j \beta^k - D_j D_i \alpha \\ & + \alpha \left( R_{ij} - 2K_{ik} K_j^k + K_{ij} K - \kappa \left[ S_{ij} + \frac{\rho - S}{3} \gamma_{ij} \right] \right), \\ & S_{ij} \equiv \gamma_i^c \gamma_j^d T_{cd}, S \equiv \gamma^{ij} S_{ij}. \end{aligned}$$

# Einstein equations

## Hamiltonian Constraint

$\rho$ :energy density,  $j_i$ :current density

$$G_{ab}n^a n^b = \kappa T_{ab}n^a n^b \rightarrow {}^{(3)}R + K^2 - K_{ij}K^{ij} = 2\kappa\rho,$$

$$\gamma_i^c G_{cb}n^b = \gamma_i^c \kappa T_{cb}n^b \rightarrow D_j K_i^j - D_i K = \kappa j_i.$$

## Momentum Constraints

All quantities are spatial.  
(They must be satisfied on each 3D hypersurface.)

## Evolution equation

$$\gamma_i^c \gamma_j^d G_{cd} = \gamma_i^c \gamma_j^d \kappa T_{cd} \longrightarrow$$

$$\begin{aligned} \partial_t K_{ij} = & \beta^k D_k K_{ij} + K_{kj} D_i \beta^k + K_{ki} D_j \beta^k - D_j D_i \alpha \\ & + \alpha \left( R_{ij} - 2K_{ik} K_j^k + K_{ij} K - \kappa \left[ S_{ij} + \frac{\rho - S}{3} \gamma_{ij} \right] \right), \end{aligned}$$

$\gamma_{ij}$  and  $K_{ij}$  are variables to evolve in ADM system.  $S_{ij} \equiv \gamma_i^c \gamma_j^d T_{cd}$ ,  $S \equiv \gamma^{ij} S_{ij}$ .

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_j \beta_i + D_i \beta_j$$

(Definition of  $K_{ij}$ )

**Cf. Hamiltonian formalism**

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q^i}, \quad \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p^i}$$

# Propagation of Constraints

In principle, we can solve only the **evolution equation** in ADM system for an appropriate initial condition, so that we can save numerical costs.

$$\nabla^b G_{ab} = \kappa \nabla^b T_{ab}$$

# Propagation of Constraints

In principle, we can solve only the **evolution equation** in ADM system for an appropriate initial condition, so that we can save numerical costs.

$$0 = \nabla^b G_{ab} = \kappa \nabla^b T_{ab} = 0$$

**Bianchi Identity** 

Nature of Riemann tensor

 **Energy Conservation**

# Propagation of Constraints

In principle, we can solve only the **evolution equation** in ADM system for an appropriate initial condition, so that we can save numerical costs.

$$0 = \nabla^b G_{ab} = \kappa \nabla^b T_{ab} = 0$$

**Bianchi Identity** 

Nature of Riemann tensor

 **Energy Conservation**

$$G_{ab} - \kappa T_{ab} = \mathcal{E}_{ab} + n_a \mathcal{D}_b + n_b \mathcal{D}_a + n_a n_b \mathcal{C}$$

$$\mathcal{C} \equiv (G_{ab} - \kappa T_{ab}) n^a n^b,$$

$$\mathcal{D}_a \equiv \gamma_a^c (G_{cb} - \kappa T_{cb}) n^b,$$

$$\mathcal{E}_{ab} \equiv \gamma_a^c \gamma_b^d (G_{cd} - \kappa T_{cd}).$$

# Propagation of Constraints

In principle, we can solve only the **evolution equation** in ADM system for an appropriate initial condition, so that we can save numerical costs.

$$0 = \nabla^b G_{ab} = \kappa \nabla^b T_{ab} = 0$$

**Bianchi Identity** 

Nature of Riemann tensor

 **Energy Conservation**

$$G_{ab} - \kappa T_{ab} = \mathcal{E}_{ab} + n_a \mathcal{D}_b + n_b \mathcal{D}_a + n_a n_b \mathcal{C}$$

$$\begin{aligned} \mathcal{C} &\equiv (G_{ab} - \kappa T_{ab}) n^a n^b, \\ \mathcal{D}_a &\equiv \gamma_a^c (G_{cb} - \kappa T_{cb}) n^b, \\ \mathcal{E}_{ab} &\equiv \gamma_a^c \gamma_b^d (G_{cd} - \kappa T_{cd}). \end{aligned}$$

$$n^a \nabla^b (G_{ab} - \kappa T_{ab}) = 0$$

Hamiltonian Constraint along  $n^a$

$$\dashrightarrow n^a \nabla_a \mathcal{C} = -\mathcal{E}_{ab} D^b n^a - D^a \mathcal{D}_a - \mathcal{D}_a n_b \nabla^b n^a - \mathcal{C} D^a n_a$$

$$\gamma_a^c \nabla^b (G_{cb} - \kappa T_{cb}) = 0$$

Momentum Constraints along  $n^a$

$$\dashrightarrow n^b \nabla_b \mathcal{D}_a = -D^b \mathcal{E}_{ba} - \mathcal{E}_{ab} n^c \nabla_c n^b - \mathcal{D}_a \nabla^b n_b + \mathcal{D}_b n_a n^c \nabla_c n^b - \mathcal{D}_b \nabla^b n_a$$



A vertical spiral binding on the left side of the page, consisting of a black wire with silver-colored loops.

**How to cook Black Holes in NR**

# Schwarzschild Black Hole

$$ds^2 = - \left(1 - \frac{2M}{R}\right) dt^2 + \frac{1}{1 - \frac{2M}{R}} dR^2 + R^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

@ 3D metric

$$\gamma_{RR} = \frac{1}{1 - \frac{2M}{R}}, \quad \gamma_{\theta\theta} = R^2, \quad \gamma_{\phi\phi} = R^2 \sin^2 \theta$$

@ Extrinsic curvature

$$K_{ij} = \dot{\gamma}_{ij} = 0$$

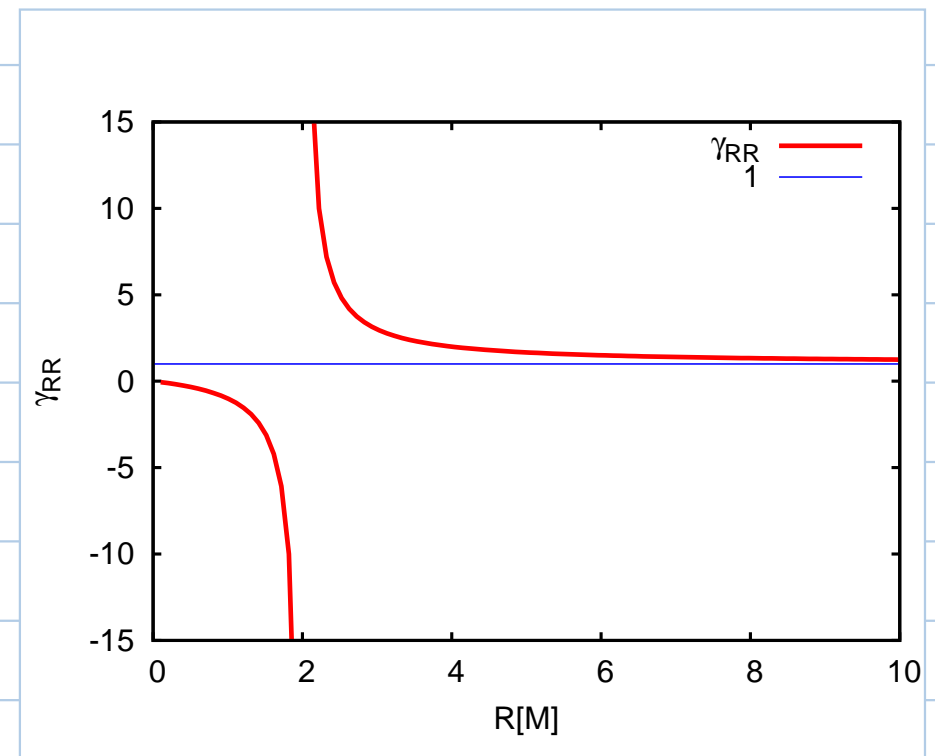
Hamiltonian Constraint

$${}^{(3)}R + K^2 - K_{ij}K^{ij} = 0,$$

$$D_j K_i^j - D_i K = 0.$$

Momentum Constraints

But how do we treat the coordinate singularity?



# Schwarzschild Black Hole

$$ds^2 = - \left(1 - \frac{2M}{R}\right) dt^2 + \frac{1}{1 - \frac{2M}{R}} dR^2 + R^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

@ 3D metric

$$\gamma_{RR} = \frac{1}{1 - \frac{2M}{R}}, \quad \gamma_{\theta\theta} = R^2, \quad \gamma_{\phi\phi} = R^2 \sin^2 \theta$$

@ Extrinsic curvature

$$K_{ij} = \dot{\gamma}_{ij} = 0$$

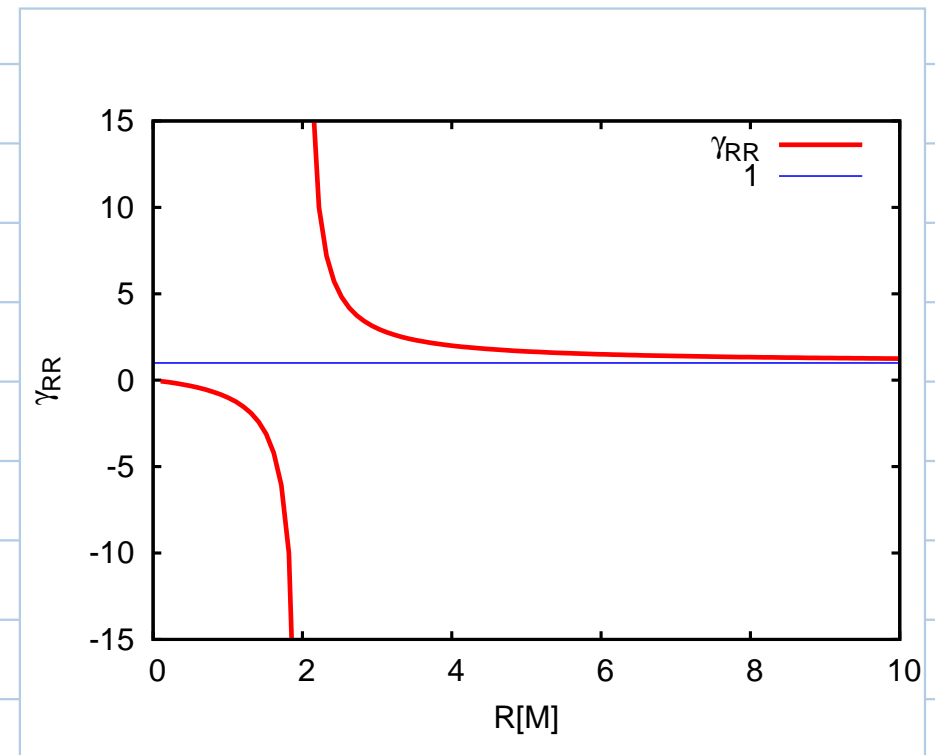
Hamiltonian Constraint

$${}^{(3)}R + K^2 - K_{ij}K^{ij} = 0,$$

$$D_j K_i^j - D_i K = 0.$$

Momentum Constraints

But how do we treat the coordinate singularity?



**We should change the coordinates!**

# Isotropic Coordinates

- isotropic radial coordinate  $r$  defined as  $R = r \left(1 + \frac{M}{2r}\right)^2$ .

- $\frac{dR}{dr} = \left(1 + \frac{M}{2r}\right) \left(1 - \frac{M}{2r}\right), \quad R = [2M, \infty] \rightarrow r = [\frac{M}{2}, \infty]$

$$ds^2 = - \left(1 - \frac{2M}{R}\right) dt^2 + \frac{1}{1 - \frac{2M}{R}} dR^2 + R^2 d\Omega^2,$$

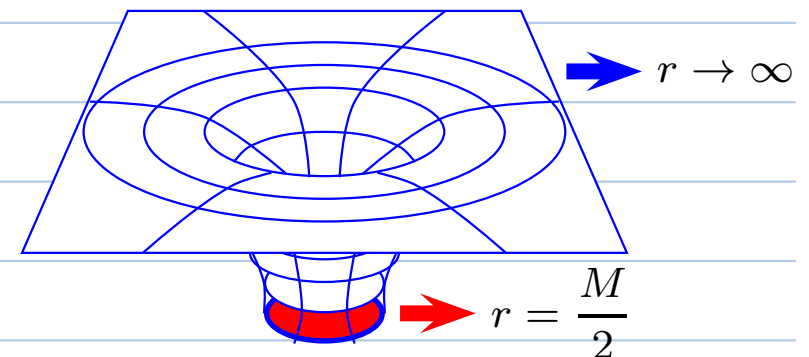
$$= - \left(\frac{1 - M/2r}{1 + M/2r}\right)^2 dt^2 + \left(1 + \frac{M}{2r}\right)^4 [dr^2 + r^2 d\Omega^2] \equiv -\alpha_0^2 dt^2 + \psi_0^4 \eta_{ij} dx^i dx^j$$

$$r \rightarrow \left(\frac{M}{2}\right)^2 \frac{1}{\bar{r}}, \quad dr \rightarrow - \left(\frac{M}{2}\right)^2 \frac{1}{\bar{r}^2} d\bar{r}$$

$$ds^2 = -\alpha_0^2 dt^2 + \left(1 + \frac{M}{2\bar{r}}\right)^4 [d\bar{r}^2 + \bar{r}^2 d\Omega^2]$$

**Same form as coordinate  $r$ !**

$$\bar{r} = [\frac{M}{2}, \infty] \rightarrow r = [0, \frac{M}{2}]$$



# Isotropic Coordinates

- isotropic radial coordinate  $r$  defined as  $R = r \left(1 + \frac{M}{2r}\right)^2$ .

- $\frac{dR}{dr} = \left(1 + \frac{M}{2r}\right) \left(1 - \frac{M}{2r}\right), \quad R = [2M, \infty] \rightarrow r = [\frac{M}{2}, \infty]$

$$ds^2 = - \left(1 - \frac{2M}{R}\right) dt^2 + \frac{1}{1 - \frac{2M}{R}} dR^2 + R^2 d\Omega^2,$$

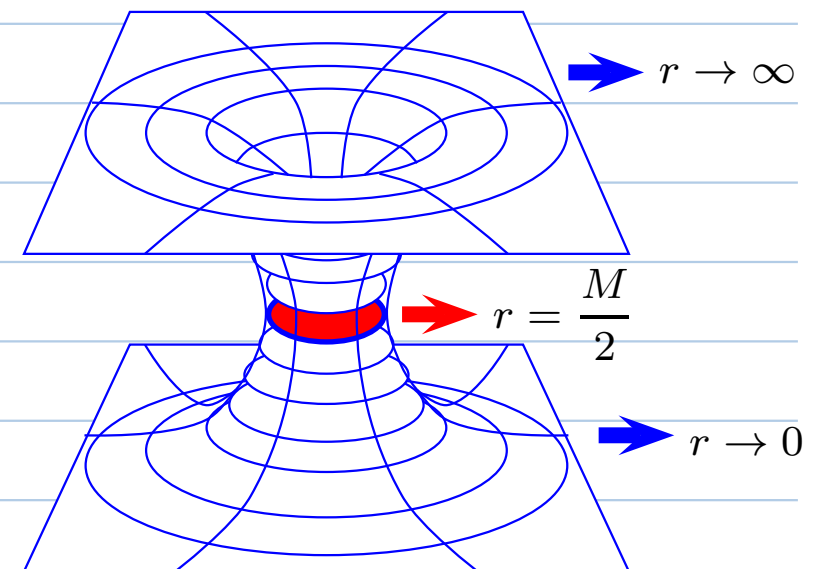
$$= - \left(\frac{1 - M/2r}{1 + M/2r}\right)^2 dt^2 + \left(1 + \frac{M}{2r}\right)^4 [dr^2 + r^2 d\Omega^2] \equiv -\alpha_0^2 dt^2 + \psi_0^4 \eta_{ij} dx^i dx^j$$

$$r \rightarrow \left(\frac{M}{2}\right)^2 \frac{1}{\bar{r}}, \quad dr \rightarrow - \left(\frac{M}{2}\right)^2 \frac{1}{\bar{r}^2} d\bar{r}$$

$$ds^2 = -\alpha_0^2 dt^2 + \left(1 + \frac{M}{2\bar{r}}\right)^4 [d\bar{r}^2 + \bar{r}^2 d\Omega^2]$$

**Same form as coordinate  $r$ !**

$$\bar{r} = [\frac{M}{2}, \infty] \rightarrow r = [0, \frac{M}{2}]$$



**Einstein-Rosen Bridge**

# Initial Value Problem

- Initial data must satisfy the constraints.  
(Hamiltonian constraint and Momentum constraints)
- We have 4 constraints and 12 variables.
- We can set the situation we want to solve and must solve the constraint equations.

## York-Lichnerowicz conformal decomposition

$$\gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}.$$

### Relation of conformal variables

$$\begin{aligned}\Gamma_{jk}^i &= \tilde{\Gamma}_{jk}^i + \frac{2}{\psi} \left[ \delta_j^i \tilde{D}_k \psi + \delta_k^i \tilde{D}_j \psi - \tilde{\gamma}^{il} \tilde{\gamma}_{jk} \tilde{D}_l \psi \right], \\ R &= \tilde{R} \psi^{-4} - 8 \psi^{-5} \tilde{\Delta} \psi.\end{aligned}$$

## Transverse-Traceless decomposition

$$K_{ij} = A_{ij} + \frac{1}{3} \gamma_{ij} K,$$

$$A_{ij} = \psi^{-2} \tilde{A}_{ij}.$$

### Relation of conformal variables

$$\begin{aligned}D_j A^{ij} &= \partial_j A^{ij} + \Gamma_{kj}^i A^{kj} + \Gamma_{jk}^j A^{ik} \\ &= \psi^{-10} \tilde{D}_j \tilde{A}^{ij},\end{aligned}$$

$$\tilde{A}_{ij} \equiv \tilde{D}_i W_j + \tilde{D}_j W_i - \frac{2}{3} \tilde{\gamma}_{ij} \tilde{D}_k W^k.$$

# Constraints

- Constraint equations in ADM formalism

$${}^{(3)}R + K^2 - K_{ij}K^{ij} = 2\kappa\rho,$$

$$D_j K_i^j - D_i K = \kappa j_i.$$

- Conformal decomposition of Constraints

$$\tilde{\Delta}\psi - \frac{1}{8}\psi\tilde{R} - \frac{1}{12}\psi^5 K^2 + \frac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} = -\frac{\kappa}{8}\psi^5\rho,$$

$$\tilde{\Delta}W_i + \frac{1}{3}\tilde{\nabla}_i(\tilde{\nabla}_j W^j) + \tilde{R}_{ij}W^j = \frac{2}{3}\psi^6\tilde{\nabla}_i K + \kappa\psi^{10}j_i$$

- We must solve four elliptic differential equations for initial data.

# Multi Black Holes

## @ Single Black Hole (Schwarzschild BH)

- Momentum constraints are trivially satisfied.
- Hamiltonian constraint is also satisfied.

$$\gamma_{ij} = \psi^4 \eta_{ij}^{(flat)},$$

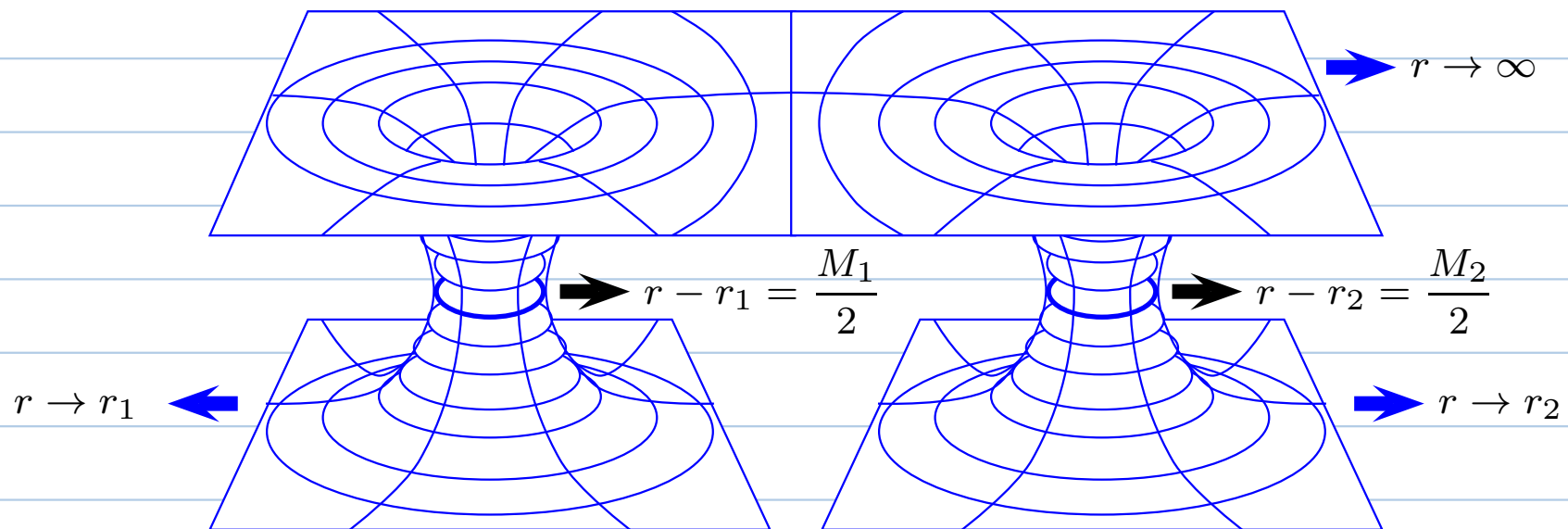
$$K_{ij} = 0,$$

$$\psi = 1 + \frac{M}{2r},$$

$$\Delta_{flat} \psi = 0.$$

## @ Multi-Black Hole (time symmetric)

$$\psi = 1 + \sum_{a=1}^N \frac{M_a}{2|r - r_a|},$$





# Puncture Black Hole

## @ Bowen-York initial data

- Assumptions

$$K = 0, \quad \text{Maximal condition}$$

$$\tilde{\gamma}_{ij} = \eta_{ij}, \quad \text{Conformally flatness}$$

$$\psi \rightarrow 1, \quad \text{Asymptotically flatness}$$

- Constraints

$$\Delta W_i + \frac{1}{3} \nabla_i (\nabla_j W^j) = 0,$$

$$\Delta \psi + \frac{1}{8} \psi^{-7} \tilde{A}_{ij} \tilde{A}^{ij} = 0,$$

- A solution

$$W^i = -\frac{1}{4r} [7P^i + \bar{n}^i \bar{n}_j P^j] + \frac{1}{r^2} \epsilon^{ijk} \bar{n}_j S_k,$$

$$\tilde{A}_{ij}^{BY} = \frac{3}{2r^2} [P_i \bar{n}_j + P_j \bar{n}_i - (\eta_{ij} - \bar{n}_i \bar{n}_j) P^k \bar{n}_k] + \frac{3}{r^3} [\epsilon_{kil} S^l \bar{n}^k \bar{n}_j + \epsilon_{kjl} S^l \bar{n}^k \bar{n}_i].$$

## @ Puncture initial data

$$\psi = 1 + \sum_{a=1}^N \frac{M_a}{2|r - r_a|} + u,$$

$$\Delta u = -\frac{1}{8} \psi^{-7} \sum_{a=1}^N \tilde{A}_{ij}^{BY,(a)} \tilde{A}^{ij}_{BY,(a)},$$

We can make the data contained many BHs.

This is powerful method for constructing initial data for NR.

Parameter  $P^i$  and  $S^k$  correspond to the momentum and spin of BH.  $\bar{n}^i = \frac{x^i}{r}$ ,



**How to find Black Holes in NR**

# Apparent Horizon(AH)

- AH is a surface where outgoing light rays cannot expand.
- We can locally check if we have AH in our hypersurface.
- Event Horizon exists outside Apparent Horizon. (Hawking and Ellis, 1973)

2D surface defined

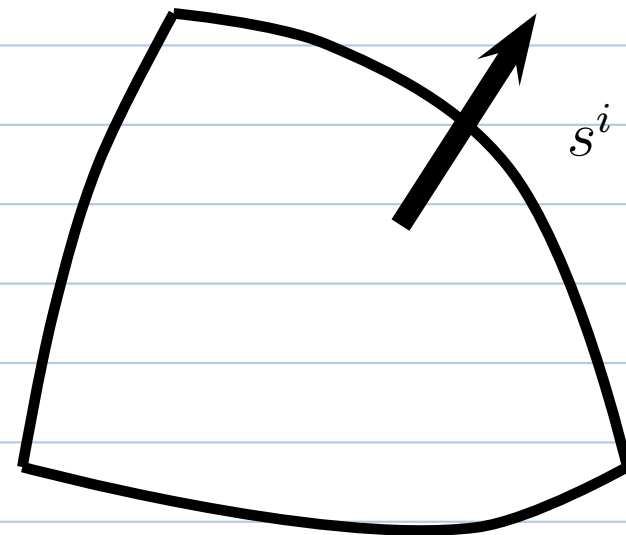
with normal vector  $s^i$ ,

$$m_{\mu\nu} = \gamma_{\mu\nu} - s_\mu s_\nu,$$

$$\ell^\mu = \frac{1}{\sqrt{2}} [s^\mu + n^\mu],$$

$$\begin{aligned}\Theta &= \nabla_\mu \ell^\mu \\ &= D_i s^i - K + K_{ij} s^i s^j \\ &= 0\end{aligned}$$

$\Theta$  : expansion



2D surface in 3D space

# Apparent Horizon(AH)

- AH is a surface where outgoing light rays cannot expand.
- We can locally check if we have AH in our hypersurface.
- Event Horizon exists outside Apparent Horizon. (Hawking and Ellis, 1973)

2D surface defined  
with normal vector  $s^i$ ,

$$m_{\mu\nu} = \gamma_{\mu\nu} - s_\mu s_\nu,$$

$$\ell^\mu = \frac{1}{\sqrt{2}} [s^\mu + n^\mu],$$

$$\begin{aligned}\Theta &= \nabla_\mu \ell^\mu \\ &= D_i s^i - K + K_{ij} s^i s^j \\ &= 0\end{aligned}$$

$\Theta$  : expansion



$$\nabla_i v^i > 0$$

# Apparent Horizon(AH)

- AH is a surface where outgoing light rays cannot expand.
- We can locally check if we have AH in our hypersurface.
- Event Horizon exists outside Apparent Horizon. (Hawking and Ellis, 1973)

2D surface defined  
with normal vector  $s^i$ ,

$$m_{\mu\nu} = \gamma_{\mu\nu} - s_\mu s_\nu,$$

$$\ell^\mu = \frac{1}{\sqrt{2}} [s^\mu + n^\mu],$$

$$\begin{aligned}\Theta &= \nabla_\mu \ell^\mu \\ &= D_i s^i - K + K_{ij} s^i s^j \\ &= 0\end{aligned}$$

$\Theta$  : expansion



# AH finder for NR

- Nakamura, Oohara and Kojima(1987)
- Shibata(1997)
- Thornburg(2003) → AHFinderDirect(Einstein Toolkit)
- Lin and Novak(2007) → LORENE(France group)

Here, we define the location of AH as  $r = h(\theta, \phi)$ .

$$s^i = C\psi^2\tilde{s}^i, \quad \Delta_{\theta\phi}h - h = \underline{h_{,\theta\theta}} + \frac{\cos\theta}{\sin\theta}h_{,\theta} + \frac{1}{\sin^2\theta}\underline{h_{,\phi\phi}} - h = S(\theta, \phi)$$

$$\tilde{s}_i = (1, \underline{-h_{,\theta}}, \underline{-h_{,\phi}}),$$

$$C = \frac{1}{\sqrt{\tilde{\gamma}_{ij}\tilde{s}^i\tilde{s}^j}},$$

$$D_i s^i = \frac{1}{\sqrt{\gamma}}\partial_i \left[ \sqrt{\gamma}\gamma^{ij}\underline{s_j} \right],$$

# AH finder for NR

- Nakamura, Oohara and Kojima(1987)
- Shibata(1997)
- Thornburg(2003) → AHFinderDirect(Einstein Toolkit)
- Lin and Novak(2007) → LORENE(France group)

Here, we define the location of AH as  $r = h(\theta, \phi)$ .

$$\begin{aligned}
 s^i &= C\psi^2 \tilde{s}^i, & \Delta_{\theta\phi}h - h &= \underline{h_{,\theta\theta}} + \frac{\cos\theta}{\sin\theta}h_{,\theta} + \frac{1}{\sin^2\theta}\underline{h_{,\phi\phi}} - h = S(\theta, \phi) \\
 \tilde{s}_i &= (1, \underline{-h_{,\theta}}, \underline{-h_{,\phi}}), & S(\theta, \phi) &= 2h\xi^{rr} - 2h\tilde{\gamma}^{r\theta}h_{,\theta} - 2h\tilde{\gamma}^{r\phi}h_{,\phi} + h^2 \cot\theta\tilde{\gamma}^{\theta r} \\
 C &= \frac{1}{\sqrt{\tilde{\gamma}_{ij}\tilde{s}^i\tilde{s}^j}}, & & -h^2 \cot\theta \tilde{\gamma}^{\theta\phi}h_{,\phi} - h^2\xi^{\theta\theta}h_{,\theta\theta} - h^2\xi^{\phi\phi}h_{,\phi\phi} \\
 D_i s^i &= \frac{1}{\sqrt{\gamma}}\partial_i \left[ \sqrt{\gamma}\gamma^{ij}\underline{s_j} \right], & & + \frac{1-C^2}{C^2} \left[ 2h\tilde{s}^r + h^2 \cot\theta\tilde{s}^\theta - h^2\tilde{\gamma}^{\theta\theta}h_{,\theta\theta} - h^2\tilde{\gamma}^{\phi\phi}h_{,\phi\phi} \right] \\
 & & & + \frac{h^2 C_{,i}\tilde{s}^i}{C^3} + \frac{4h^2\psi_{,i}\tilde{s}^i}{C^2\psi} - \frac{h^2\tilde{\Gamma}^j\tilde{s}_j}{C^2} - \frac{2h^2\tilde{\gamma}^{\theta\phi}h_{,\theta\phi}}{C^2} \\
 & & & + \frac{h^2\psi^2}{C}\tilde{A}_{ij}\tilde{s}^i\tilde{s}^j - \frac{2h^2\psi^2}{3C^3}K. \quad \xi^{ij} \equiv \gamma^{ij} - \eta^{ij}
 \end{aligned}$$

# Kerr BH

## Boyer-Lindquist coordinates

$$ds^2 = - \left( 1 - \frac{2Mr_{BL}}{\Sigma} \right) dt^2 - \frac{4aMr_{BL} \sin^2 \theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr_{BL}^2 + \Sigma d\theta^2 + \frac{A}{\Sigma} \sin^2 \theta d\phi^2,$$
$$A = (r_{BL}^2 + a^2)^2 - \Delta a^2 \sin^2 \theta, \quad \Sigma = r_{BL}^2 + a^2 \cos^2 \theta, \quad \Delta = r_{BL}^2 - 2Mr_{BL} + a^2,$$

## Quasi isotropic coordinates

$$r_{BL} = r \left( 1 + \frac{M+a}{2r} \right) \left( 1 + \frac{M-a}{2r} \right),$$
$$ds^2 = - \frac{a^2 \sin^2 \theta - \Delta}{\Sigma} dt^2 - \frac{4aMr_{BL} \sin^2 \theta}{\Sigma} dt d\phi + \frac{\Sigma}{r^2} dr^2 + \Sigma d\theta^2 + \frac{A}{\Sigma} \sin^2 \theta d\phi^2,$$

### 3D metric

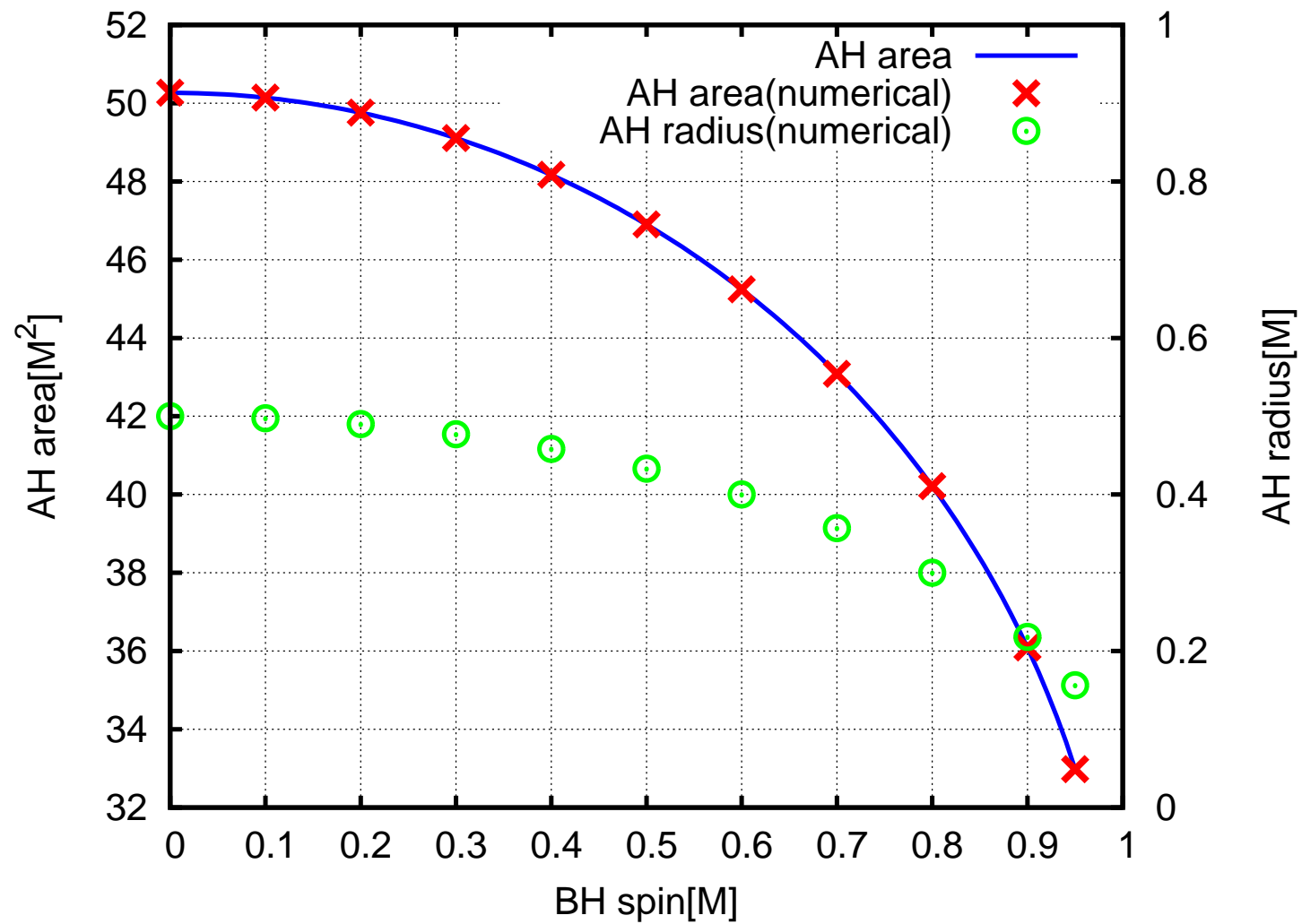
$$\gamma_{ij} = \text{diag} \left[ \frac{\Sigma}{r^2}, \Sigma, \frac{A \sin^2 \theta}{\Sigma} \right],$$

### Extrinsic curvature

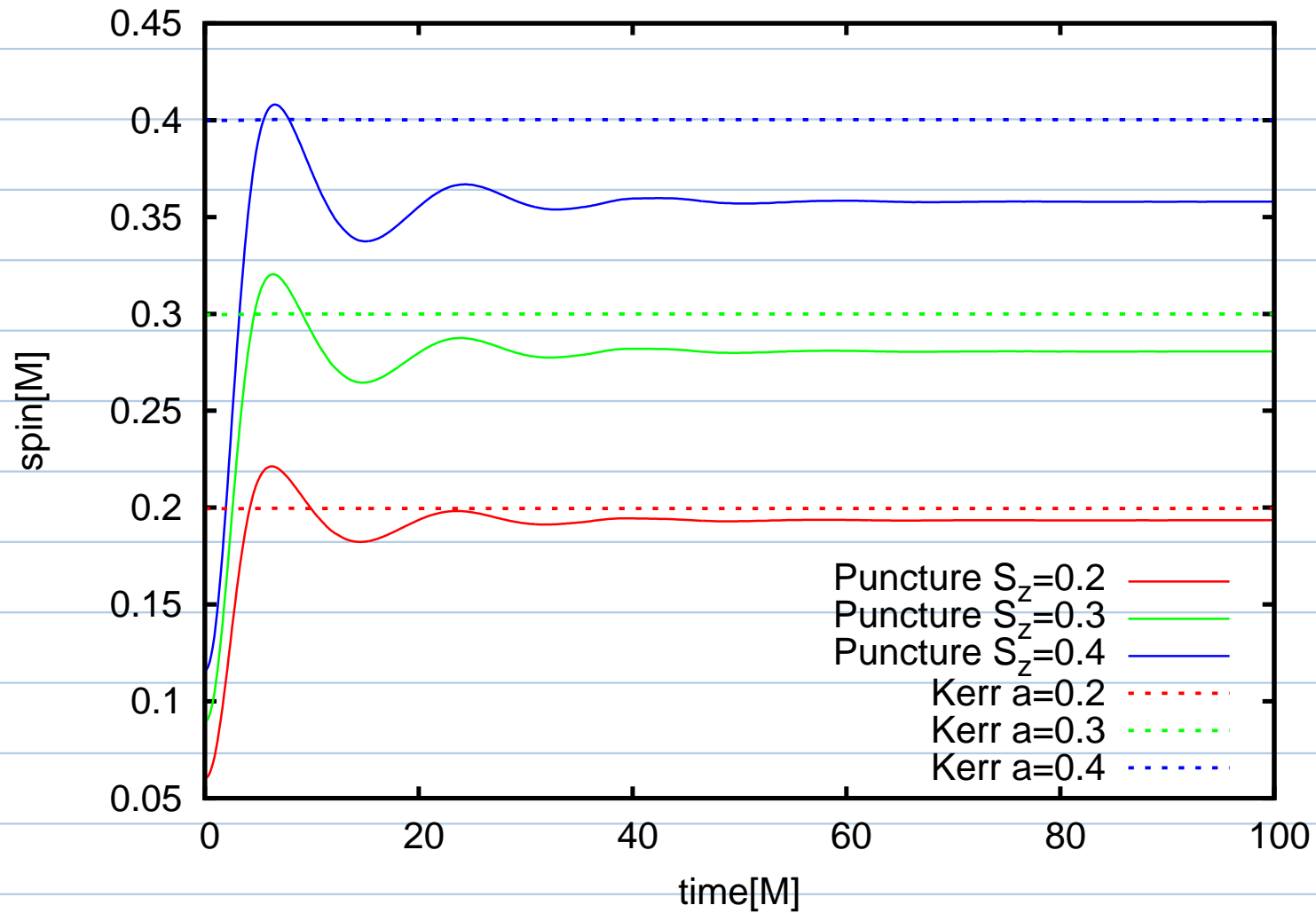
$$K_{r\phi} = \frac{aM \left[ 2r_{BL}^2 (r_{BL}^2 + a^2) + \Sigma (r_{BL}^2 - a^2) \right] \sin^2 \theta}{r \Sigma \sqrt{A \Sigma}},$$
$$K_{\theta\phi} = \frac{-2a^3 M r_{BL} \sqrt{\Delta} \cos \theta \sin^3 \theta}{\Sigma \sqrt{A \Sigma}}.$$



# AH of Kerr BH



# Puncture BH vs Kerr BH



# Summary

---

- Numerical Relativity became one of powerful tools to investigate the gravity.
- I briefly reviewed ADM formalism.
- Constraints must be satisfied at initial, but in general, they are complicated elliptic PDEs.
- Puncture method is a simple method to make initial data including BHs and it's an elliptic PDE.
- Also, I talked about Apparent Horizon, which is important to search for BH in NR simulations. This is also described as an elliptic PDE.
- Next, I'd like to talk about how to solve it.