Black Holes in Braneworlds

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Numerical Relativity and High Energy Physics (Maderia ’11)

Plan

- Review of ADD (and RSI) and RSII braneworlds
- The numerical static black hole problem
- Black holes in ADD
- An aside; AdS/CFT on a black hole metric (+ some details...)
- Black holes in RSII

- Caveat: this talk will only discuss static solutions, vacuum gravity (ie. no moduli stabilization).

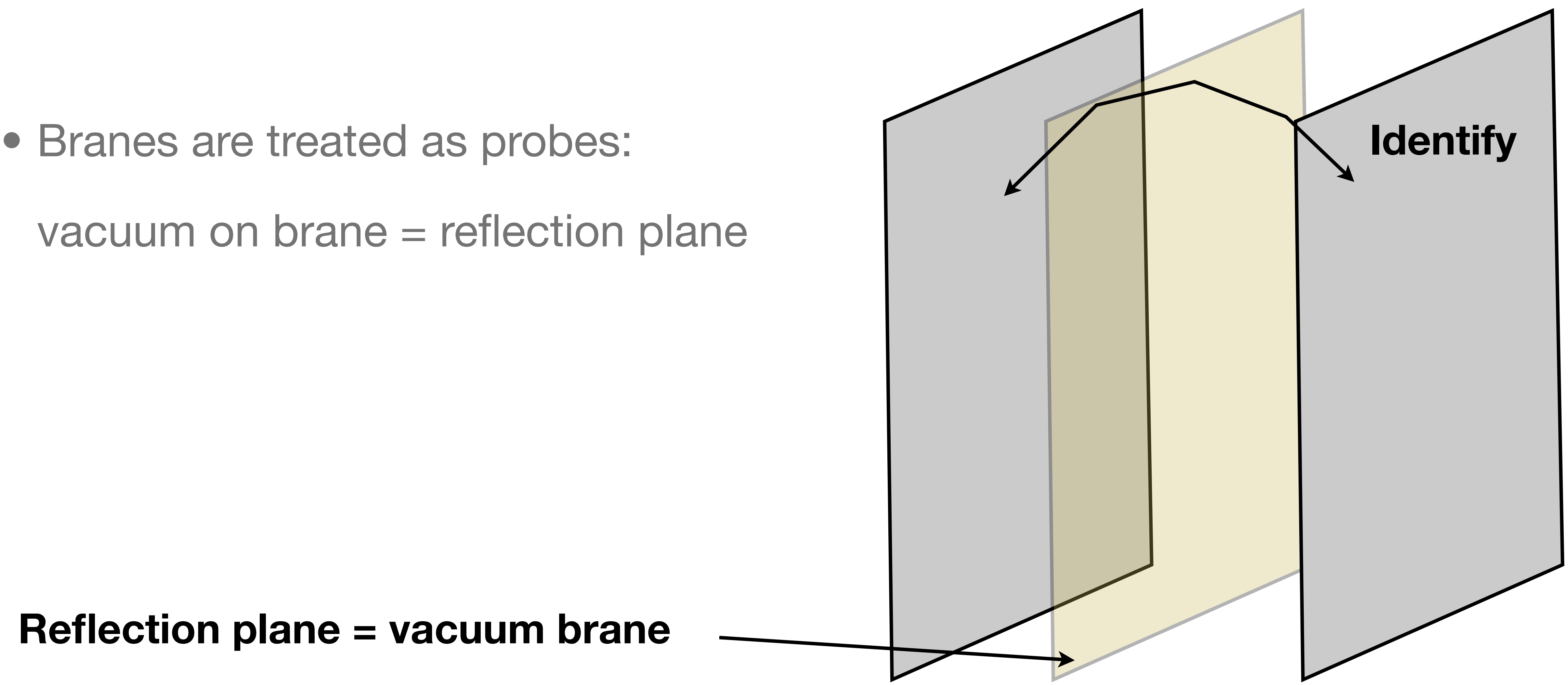
Talk based on...

- Framework for static numerical construction and KK results: with M. Headrick & S. Kitchen [0905.1822]
- For RSII results:
  - related solution: with P. Figueras & J. Lucietti [1104.4489]
  - results: with P. Figueras [1105.2558]
- Also for stationary solutions: with A. Adam & S. Kitchen [1105.6347]
- Review on KK: with G. Horowitz [1107.5563]
- Review on numerical methods: [1107.5513]

Part I: Review of ADD & RSII brane worlds

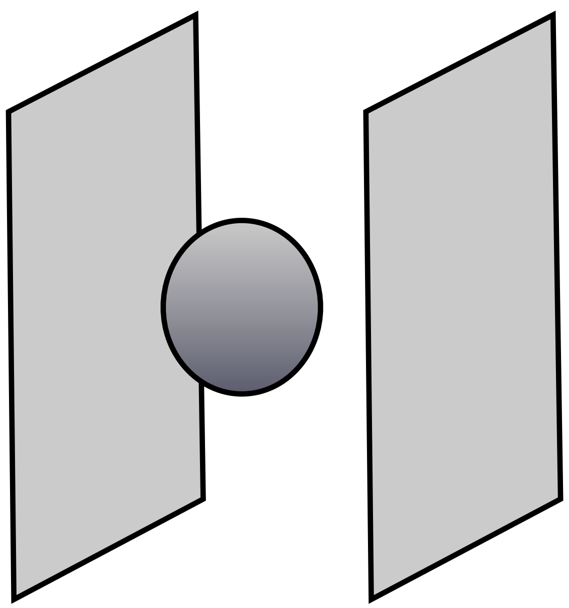
- Braneworlds essentially fall into two categories: compact and non-compact.
- The canonical compact model is `ADD'
- The simplest toy model to study is Kaluza-Klein theory ie. gravity in 5-d, where one dimension is compact.
- Branes are treated as probes:

  vacuum on brane = reflection plane



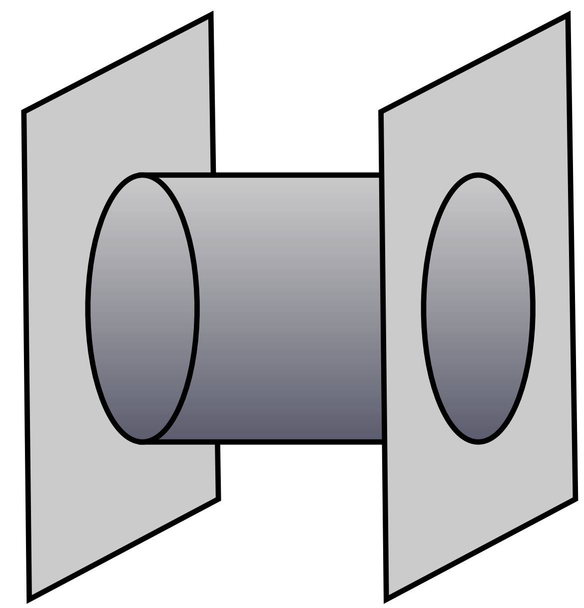
KK (or ADD)

- By KK we mean: gravity restricted to geometries asymptoting to $Mink^4 \times S^1$
- At linear level we understand well the transition from 5d to 4d behaviour as one probes scales smaller/larger than the circle size L . Consider L fixed.
- There are two obvious black hole solutions in KK; the homogeneous black string and the localized black hole.



$$ds^2 \simeq ds^2_{5dSch}$$

$$ds^2 = ds^2_{4dSch} + dz^2$$



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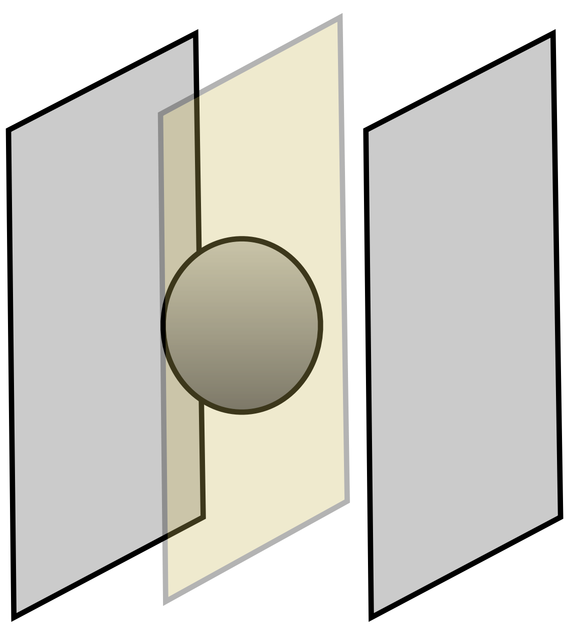

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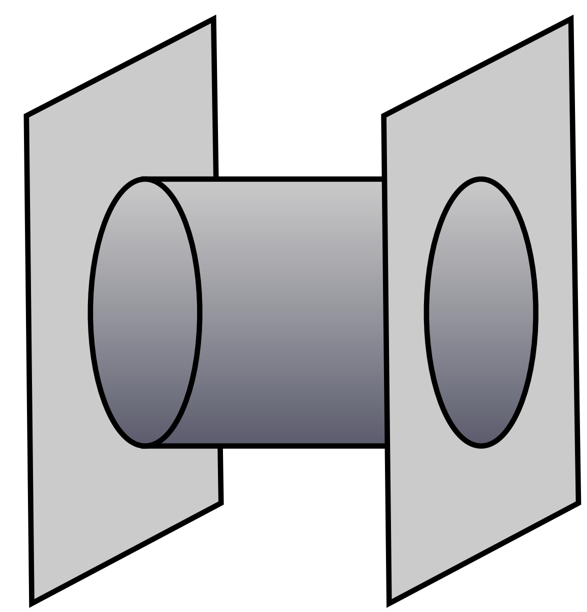

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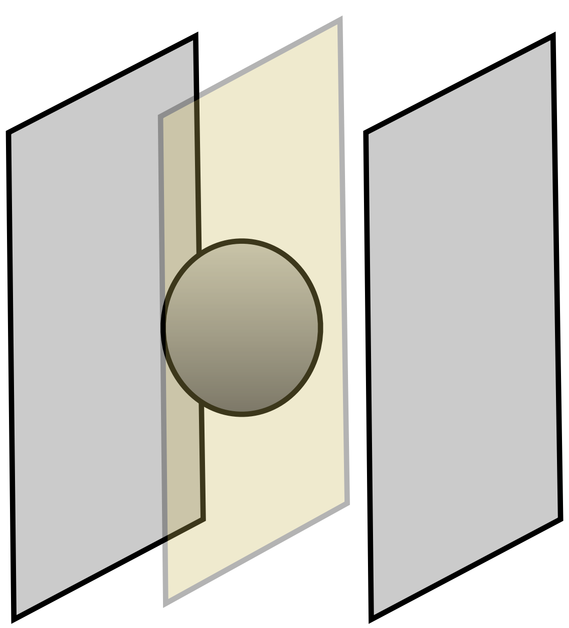

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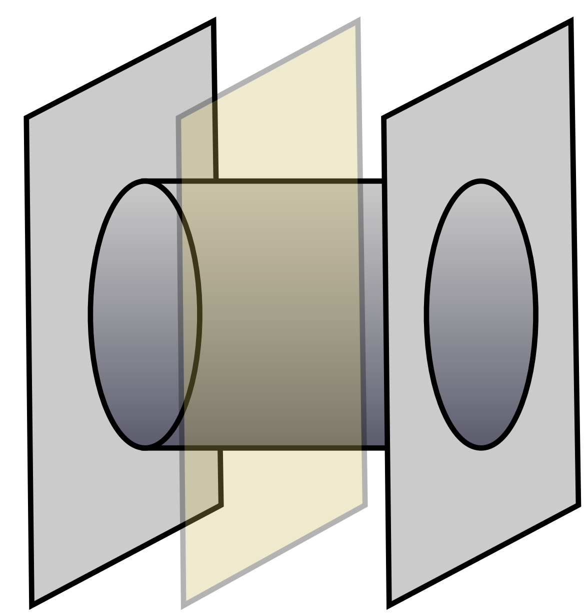
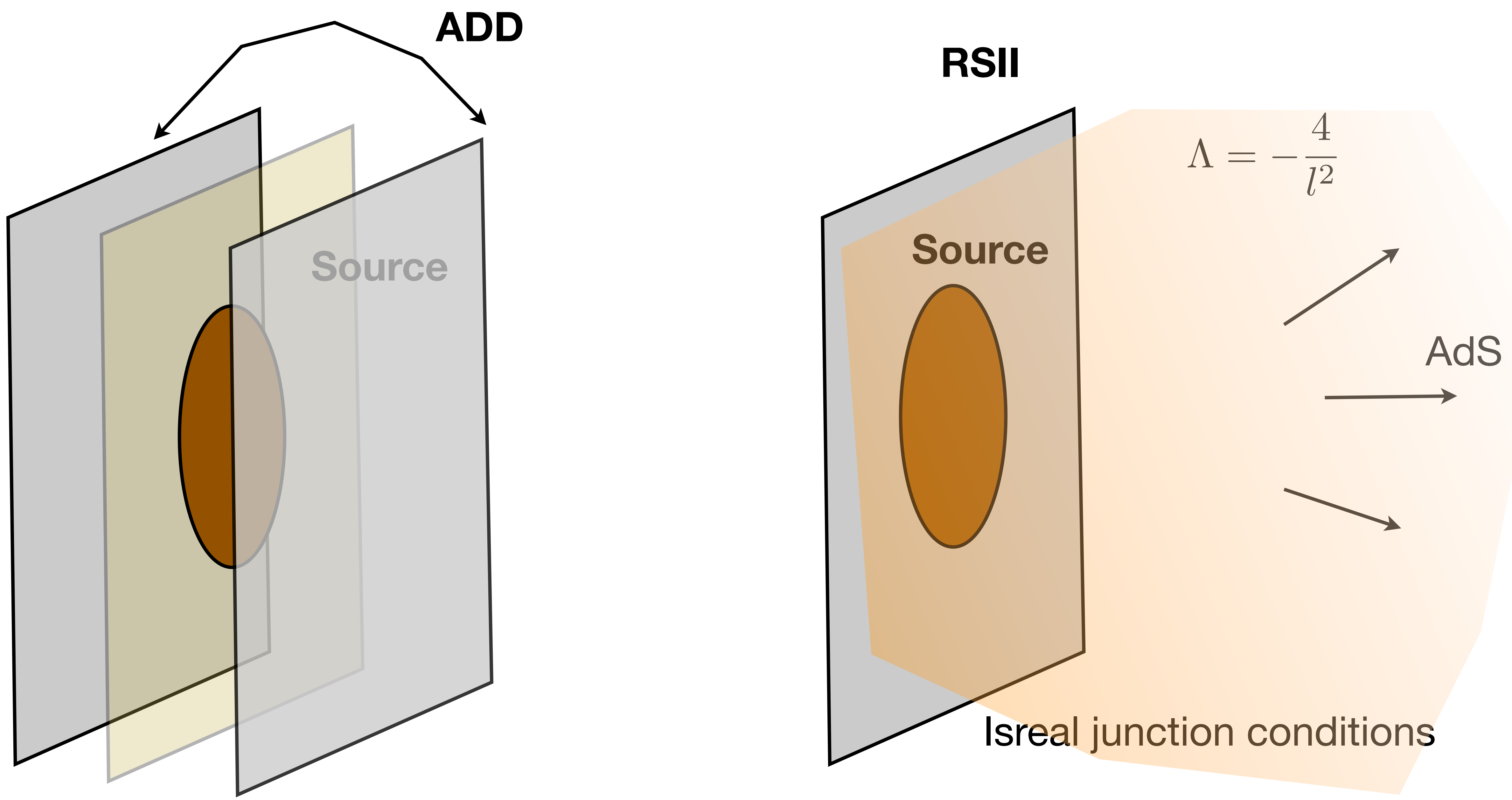


KK (or ADD)

- The important questions in this scenario are;
  - what happens as one makes the localized black hole larger?
  - what is the nature of the transition from 5d to 4d behaviour
  - are there any other solutions that might impact phenomenology?

RS II

- The 5d Randall-Sundrum II model is a remarkable ‘compactification’. 4d gravity is recovered on the brane, at scales larger than l;



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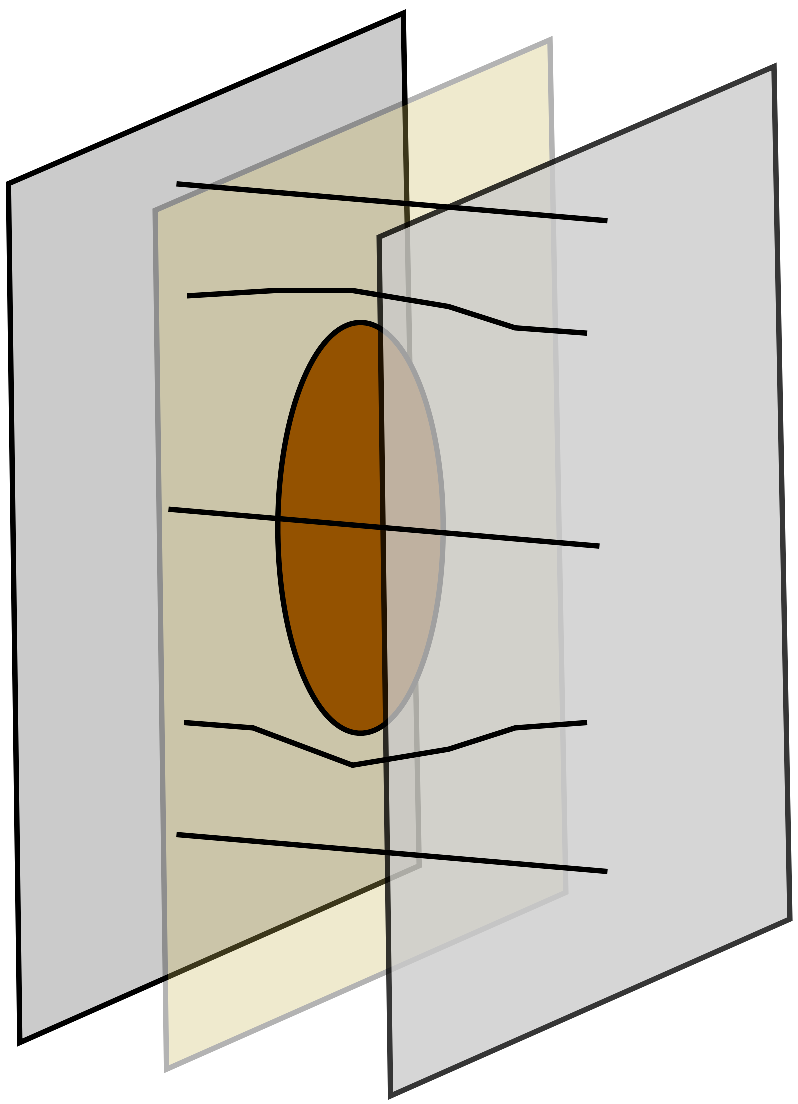
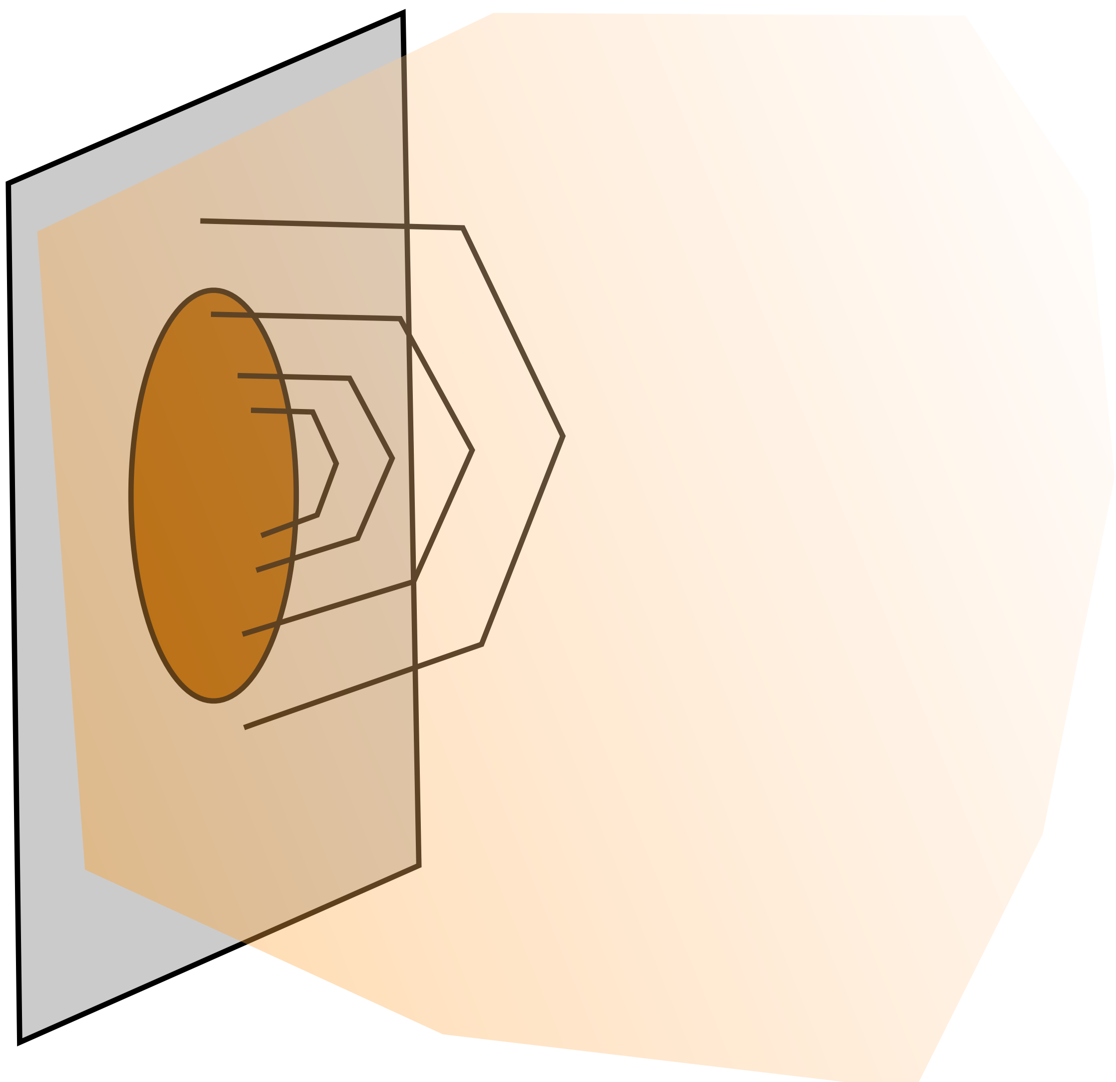




RS II

- The 5d Randall-Sundrum II model is a remarkable 'compactification'. 4d gravity is recovered on the brane, at scales larger than l.
- 4d propagator goes as; $\sim \frac{1}{r} + \frac{l^2}{r^3} + \dots$ so there is no mass gap.
- Relation to AdS/CFT: RS II ~ 4d gravity + strongly coupled CFT
- For small scales, 5d gravity is recovered. In particular a small ($R_4 \ll l$) black hole on the brane looks like 5d Schwarschild.
- Key question: is there a 4d limit for large black holes?
- Claim [Tanaka; Emparan, Kaloper, Fabbri '02]: For black holes, radius $R_4 > l$ there exist no static solutions. Counter argument [Fitzpatrick, Randall, TW '06]
- Previous numerical attempts; Kudoh $R_4 \sim 0.6\, l$, Yoshino claimed **no** solns

Part II: Static numerical problem

- static problem should be elliptic; specify asymptotics and horizon regularity

- use a characteristic version of the einstein eq - `harmonic einstein eq' - to manifest this character: $R^H_{\mu\nu} = 0$

$$R^H_{\mu\nu} \equiv R_{\mu\nu} - \nabla_{(\mu}\xi_{\nu)} \qquad\qquad \xi^\alpha \equiv g^{\mu\nu}\left(\Gamma^\alpha{}_{\mu\nu} - \bar{\Gamma}^\alpha{}_{\mu\nu}\right)$$

- reference connection - $\bar{\Gamma}^\alpha{}_{\mu\nu}$ for simplicity take ref metric $\bar{\Gamma}^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{\mu\nu}[\bar{g}_{\rho\sigma}]$

- now; $R^H_{\mu\nu} \sim -\frac{1}{2} g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu}$

- analogous to harmonic coordinates;

$$\xi^\alpha = 0 \qquad\qquad \implies \quad \nabla^2_S x^\alpha = H^\alpha \equiv -g^{\mu\nu}\bar{\Gamma}^\alpha{}_{\mu\nu}$$

Static problem

- assume one static killing horizons with the same surface gravity.
- simplest way to manifest ellipticity is to perform a euclidean contination of time, choosing period to render horizon smooth.
- ie. find smooth ricci flat metrics with U(1) isometry
- problem is now elliptic (since $R^H_{\mu\nu} \sim -\frac{1}{2} g^{\alpha\beta}\partial_\alpha\partial_\beta g_{\mu\nu}$).
- Choose the reference metric to have the U(1) isometry.
- then $R^H_{\mu\nu}$ is invariant under static U(1), so Harm. Ein. Eq. consistently truncates to static metrics.

Static problem

- elegant formulation as a boundary value problem;

- only boundary is asymptotic one where size of time circle (~temperature) is fixed.

- certainly a solution $R_{\mu\nu} = 0$ in gauge $\xi^\alpha = 0$ $\implies$ $R^H_{\mu\nu} = 0$

- may be other solutions, $R_{\mu\nu} = \nabla_{(\mu}\xi_{\nu)}$ with non-trivial ξ^α called 'ricci solitons'.

Ricci solitons

- since $R^H_{\mu\nu} = 0$ is elliptic then a solution should be locally unique.
- hence can always distinguish a soliton from a ricci flat solution.
- however, there may exist only ricci flat solutions;
- Bourguignon ('79) proves on compact manifold no solitons exist.
- For various asymptotics one can prove that for appropriate choices of reference metric that no solitons can exist [**Figueras, Lucietti, TW '11**]

Solving the elliptic system I.

- local relaxation (eg jacobi) = diffusion
- DeTurck's flow: $\frac{d}{d\lambda} g_{\mu\nu} = -2R^H_{\mu\nu} = -2R_{\mu\nu} + 2\nabla_{(\mu}\xi_{\nu)}$
- diffeomorphic to Ricci flow $\frac{d}{d\lambda} g_{\mu\nu} = -2R_{\mu\nu}$ since $\delta g_{\mu\nu} = \nabla_{(\mu}\xi_{\nu)}\delta\lambda$ is an infinitessimal diffeo.
- An important implication is that the flow is a geometric flow; the trajectory in the space of geometries is independent of the choice of $\tilde{\Gamma}$

Solving the elliptic system I.

- About a fixed pt $Ric[g_0] = 0$ we perturb; $g = g_0 + \delta g$
- In a suitable gauge, Ricci flow implies; $\delta \dot{g}_{\mu\nu} = -2 \triangle_L \delta g_{\mu\nu}$
- Thus the fixed point is stable if $\triangle_L$ is a positive operator.
- Gross, Perry and Yaffe should that for Euc. Schwarzschild $\triangle_L$ has a single negative eigenmode.
- may still use method, but if 'n' negative modes must tune 'n' parameter set of initial data.

Solving the elliptic system II.

- Solve $R^H_{\mu\nu} = 0$ using Newton method ie. $\Delta g_{\mu\nu} = -\mathcal{O}^{-1}[g]_{\mu\nu}{}^{\alpha\beta} R^H_{\alpha\beta}$ where, $\mathcal{O}$, is the linearization of $R^H_{\mu\nu}$

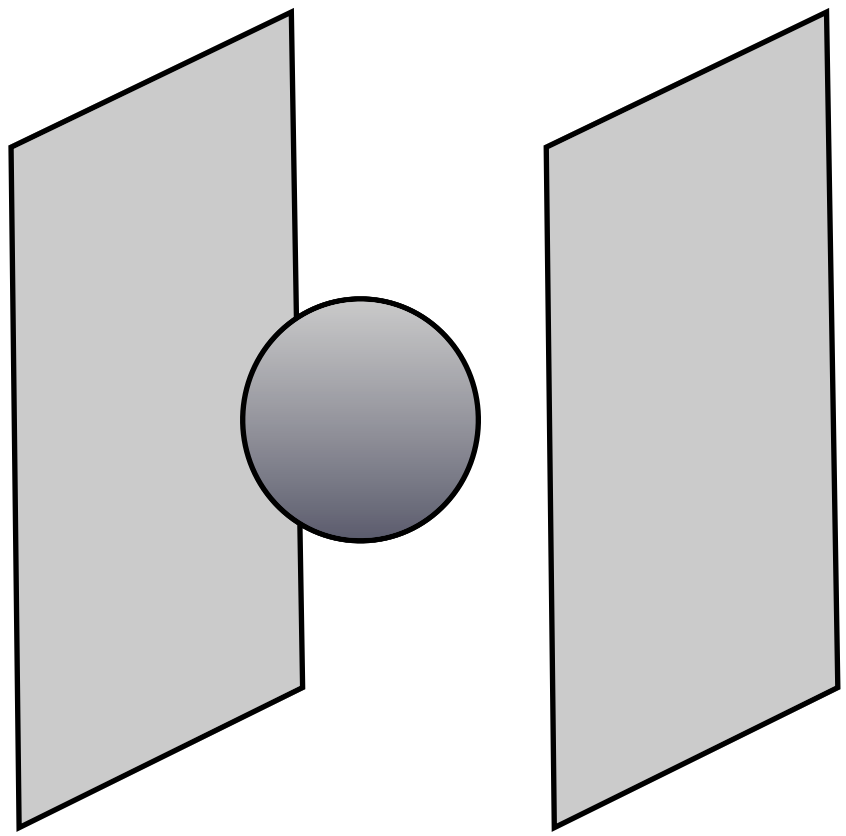
- Again we require reference connection $\bar{\Gamma}$, but now the trajectory taken in solution does depend on this choice.

- Advantages: works very well, no problem -ve modes

- Disadvantages: more complicated than Ricci flow, and it is not geometric. In particular basin of attraction depends on choice of $\bar{\Gamma}$. Also basin of attraction typically rather small in practice.

Part III: Kaluza-Klein black holes

- Can apply static numerical methods to find these
- Use Newton method
  - Ricci flow less useful due to negative modes
- No solitons exist for the system



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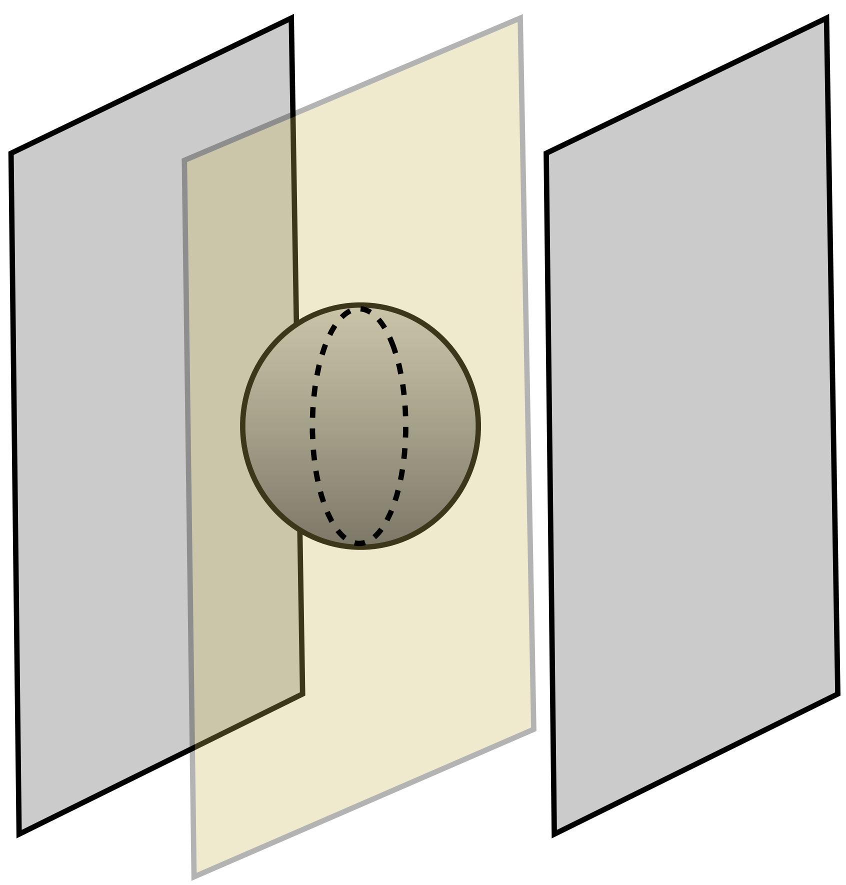
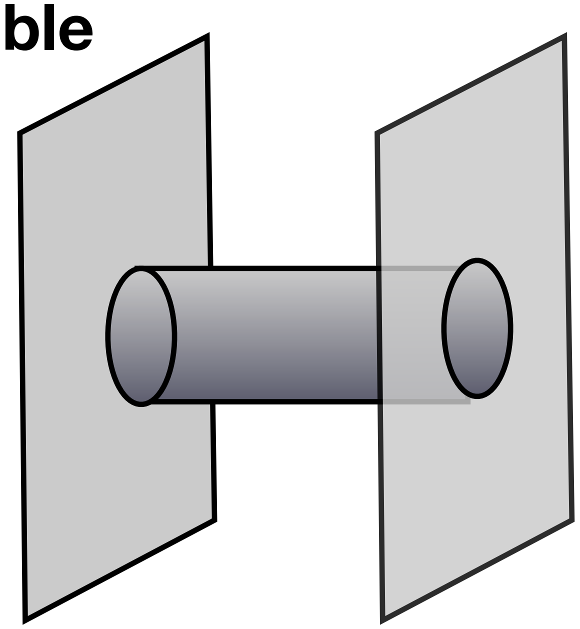


GL instability

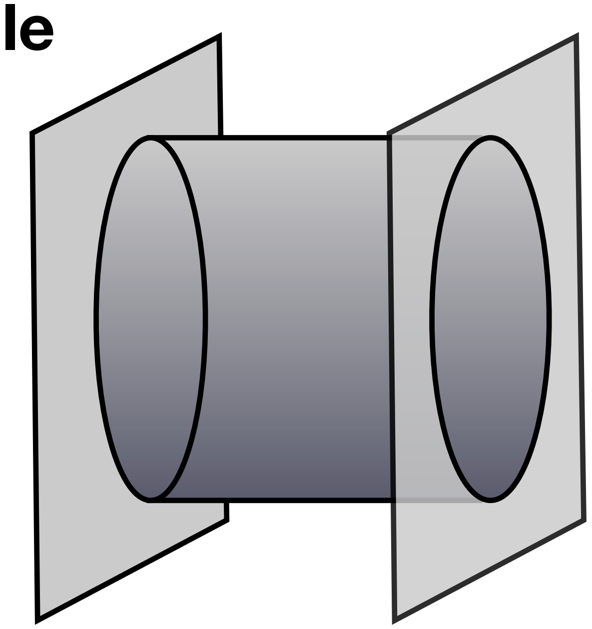
- Space of solutions is mediated by two effects;
- Thin black string is unstable to GL instability;

Unstable



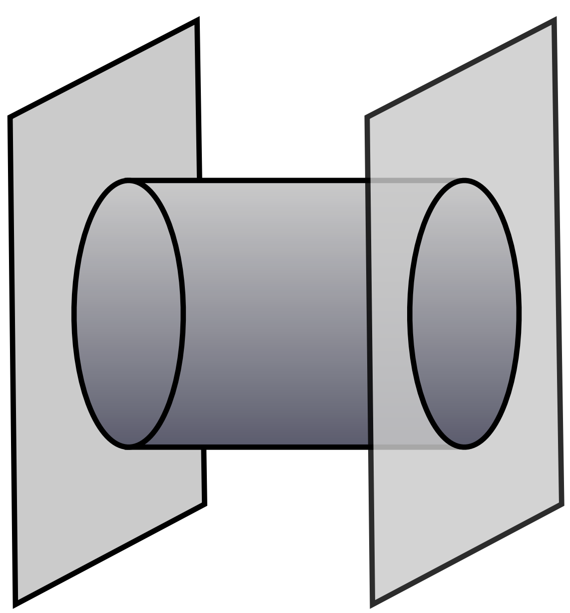
$$\delta g_{\mu\nu} \sim e^{\Omega t} e^{ikz} f_{\mu\nu}(r)$$

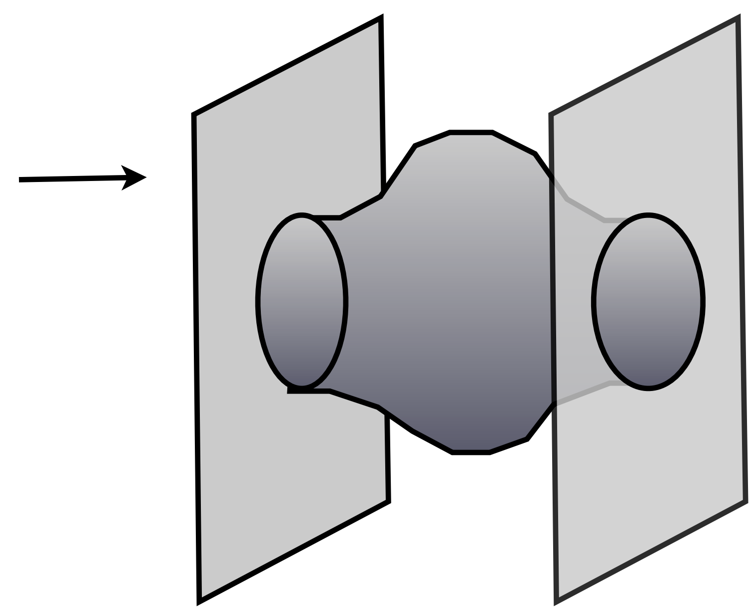
Stable



$$\delta g_{\mu\nu} \sim e^{ik_c z} f_{\mu\nu}(r)$$

Marginal

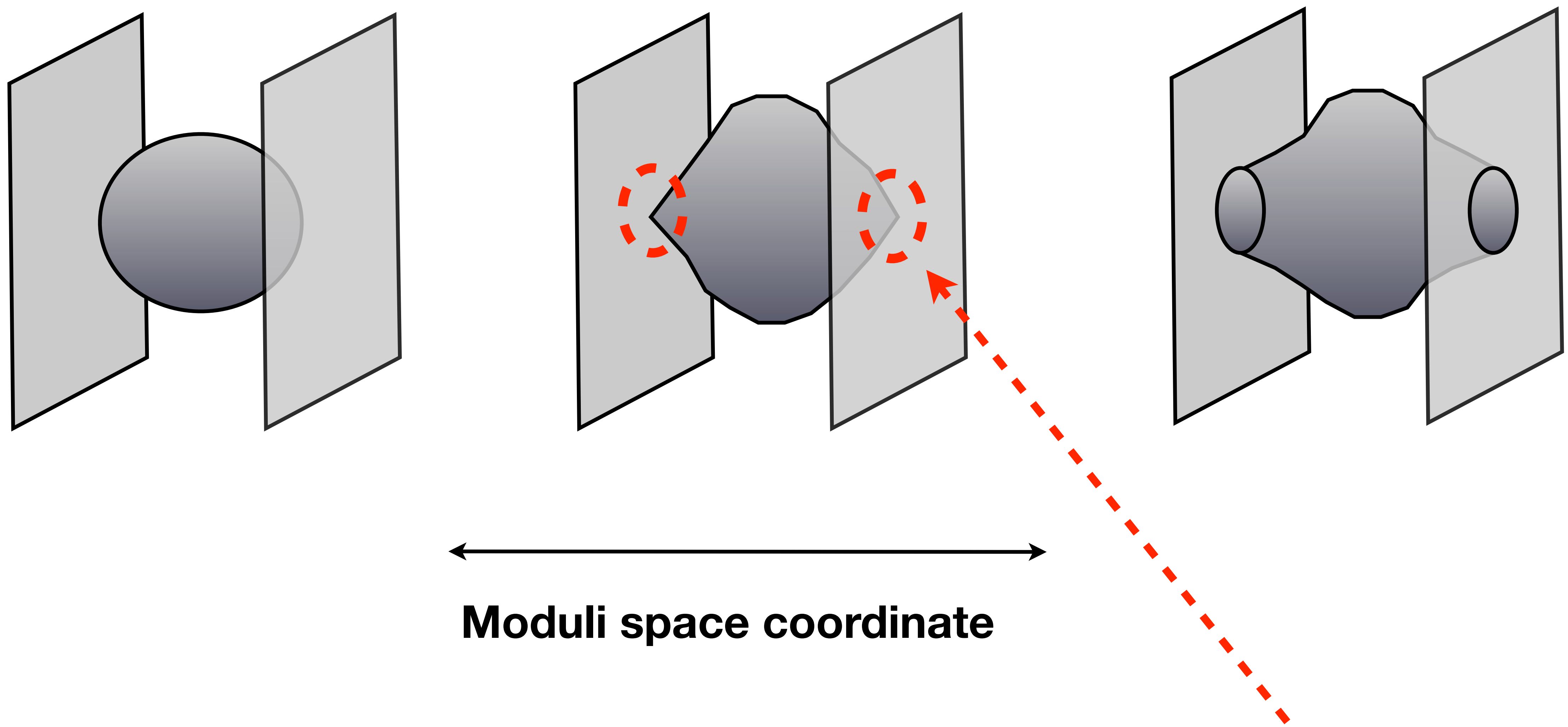




`Inhomogeneous black strings'

Topology change

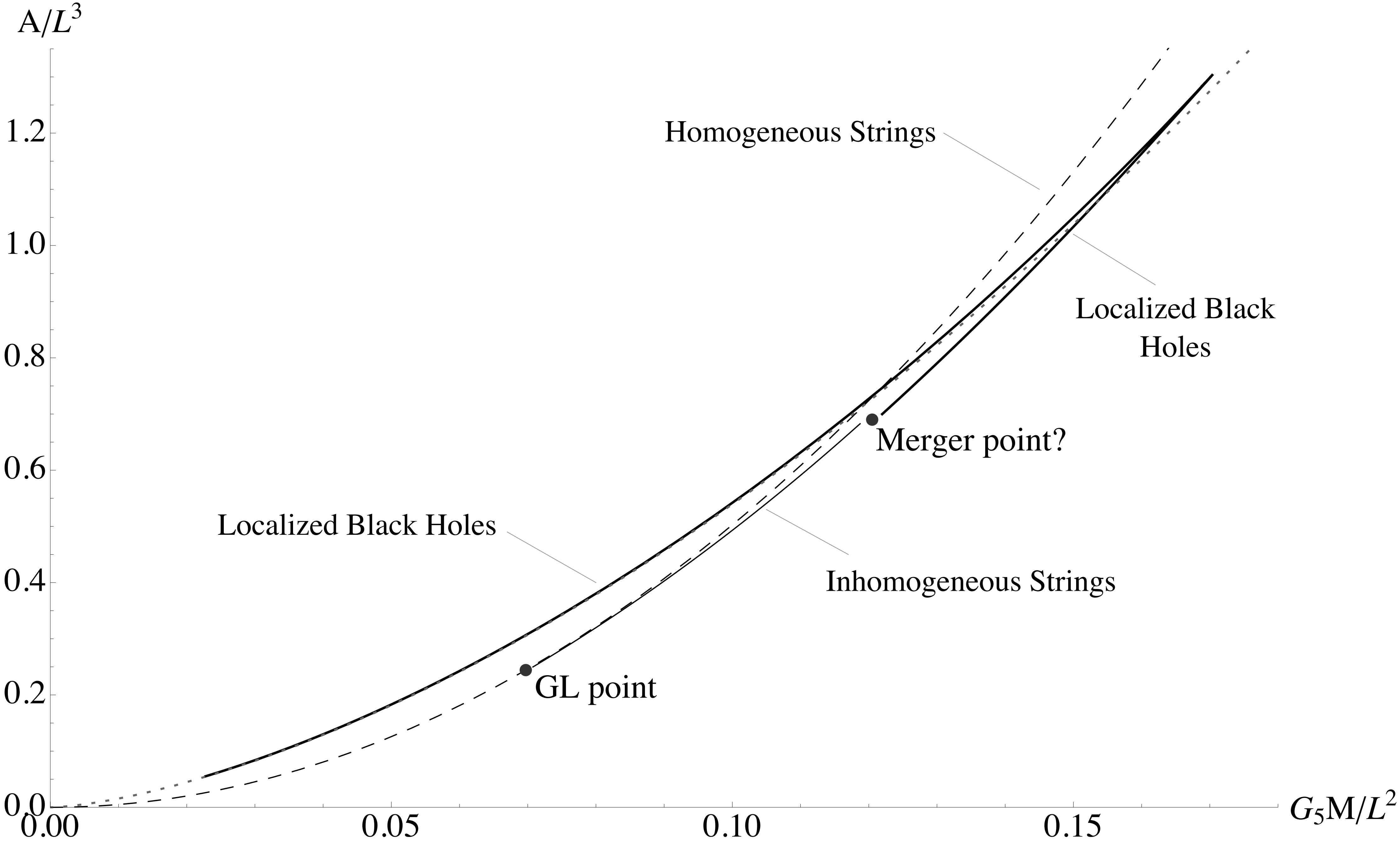
- Kol proposed a localized black hole and inhomogeneous string may meet in the space of solutions at a topology changing solution.



- Conical singular geometry: $ds^2_{cone} = d\alpha^2 + \frac{1}{3}\alpha^2 \left(d\beta^2 - \sin^2\beta dt^2\right) + \frac{1}{3}\alpha^2 d\Omega^2_2$

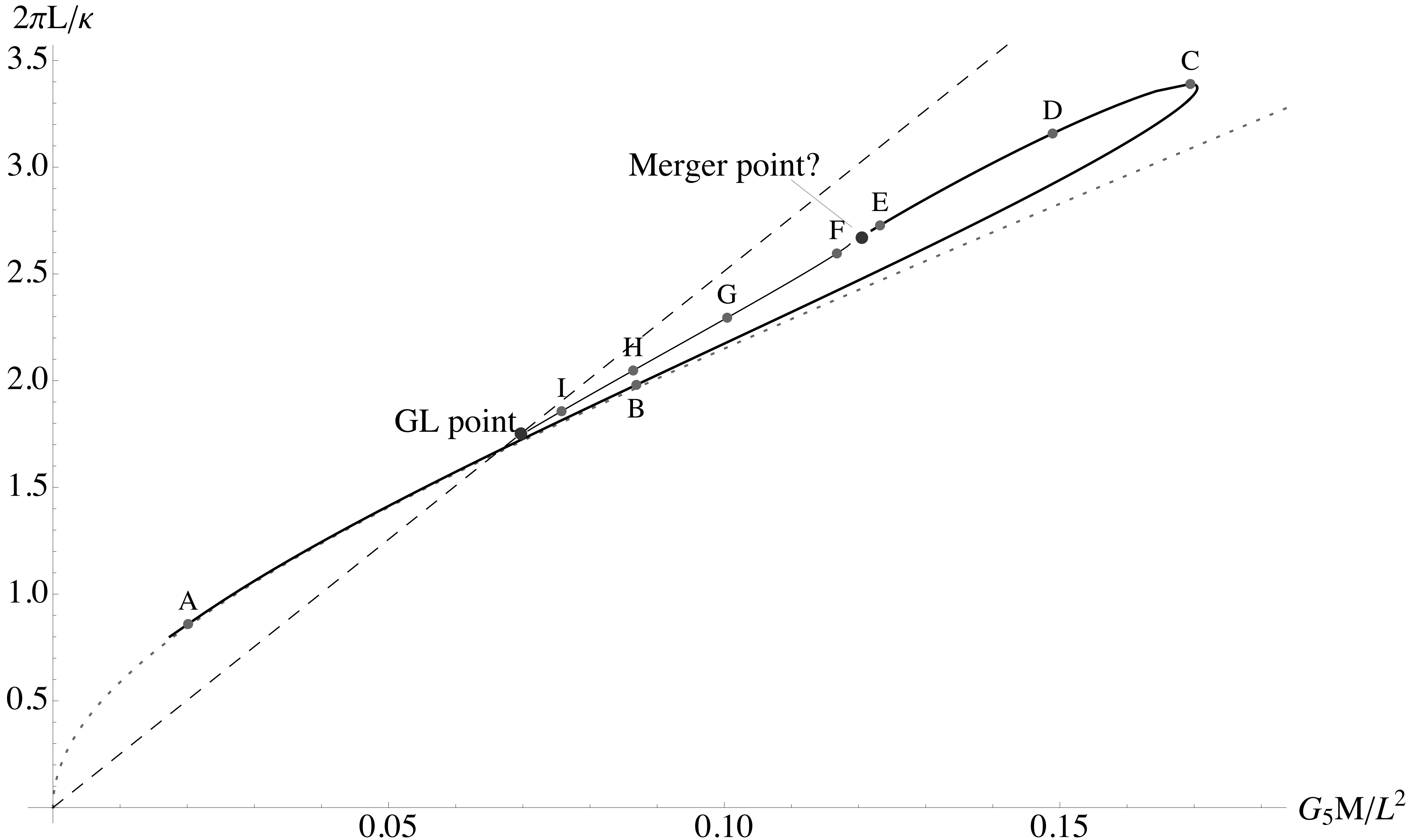
Results: Kaluza-Klein black holes

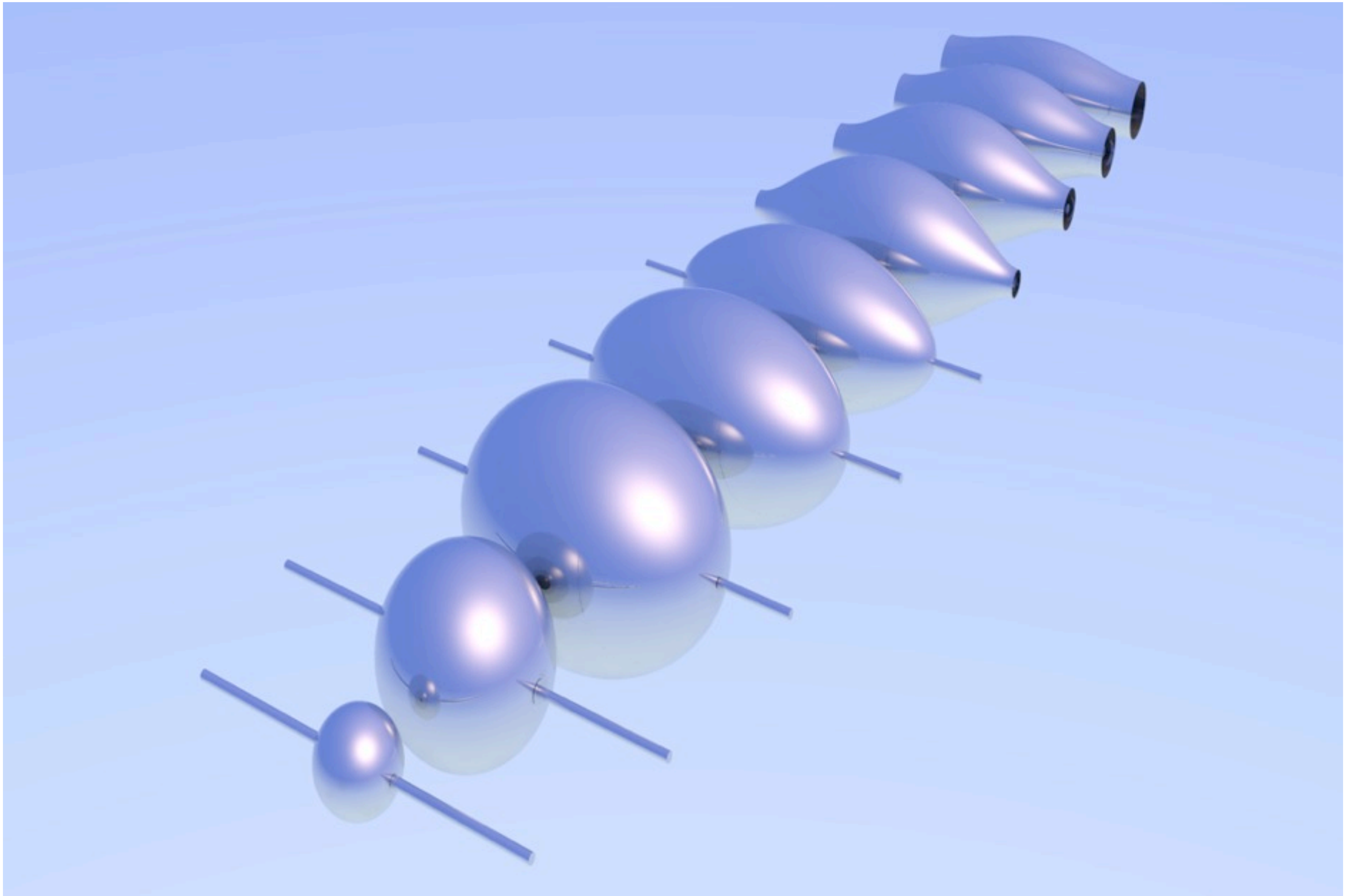
- Area against mass:



Results: Kaluza-Klein black holes

- Inv temperature against mass

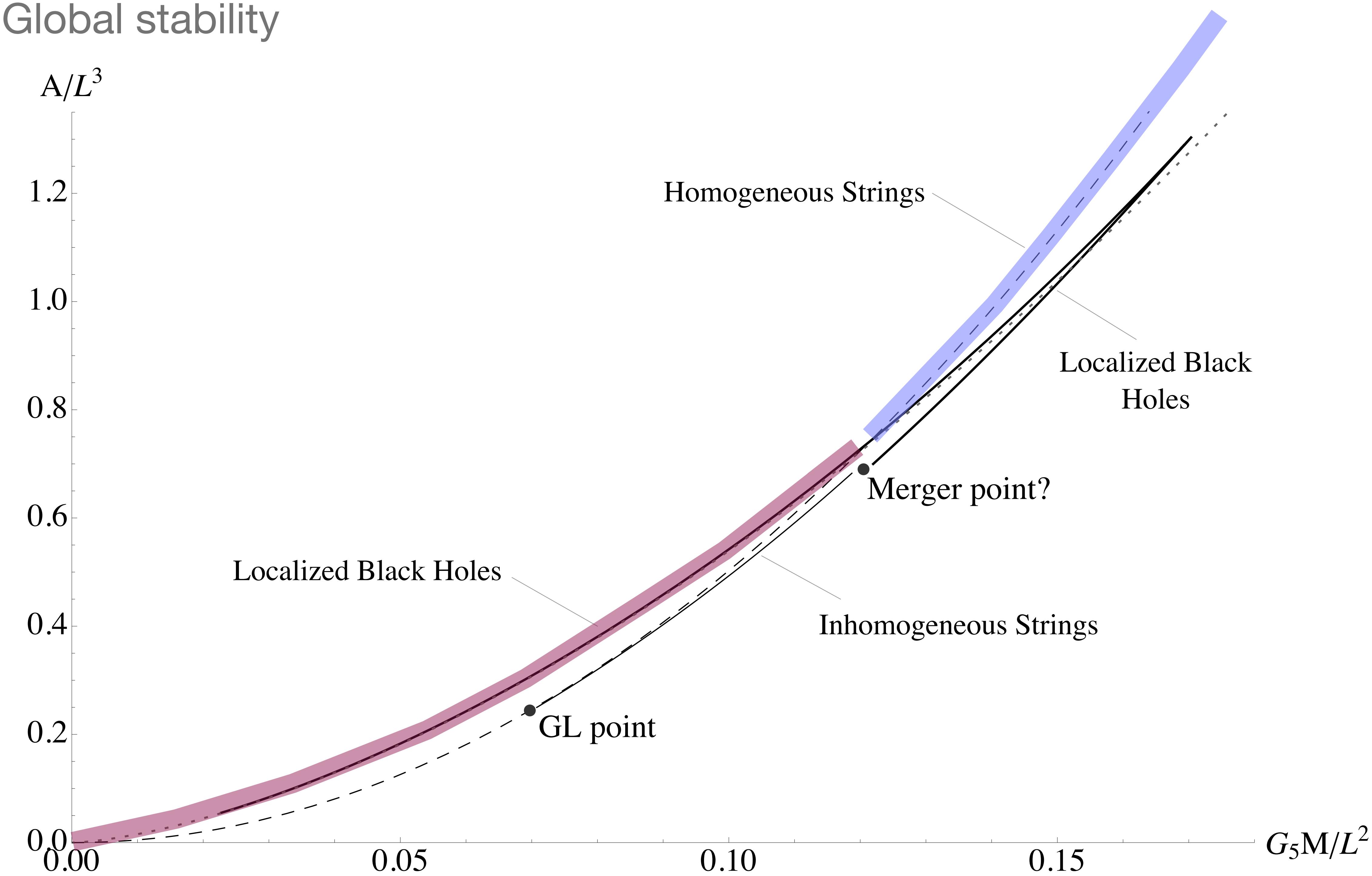




Kaluza-Klein embeddings

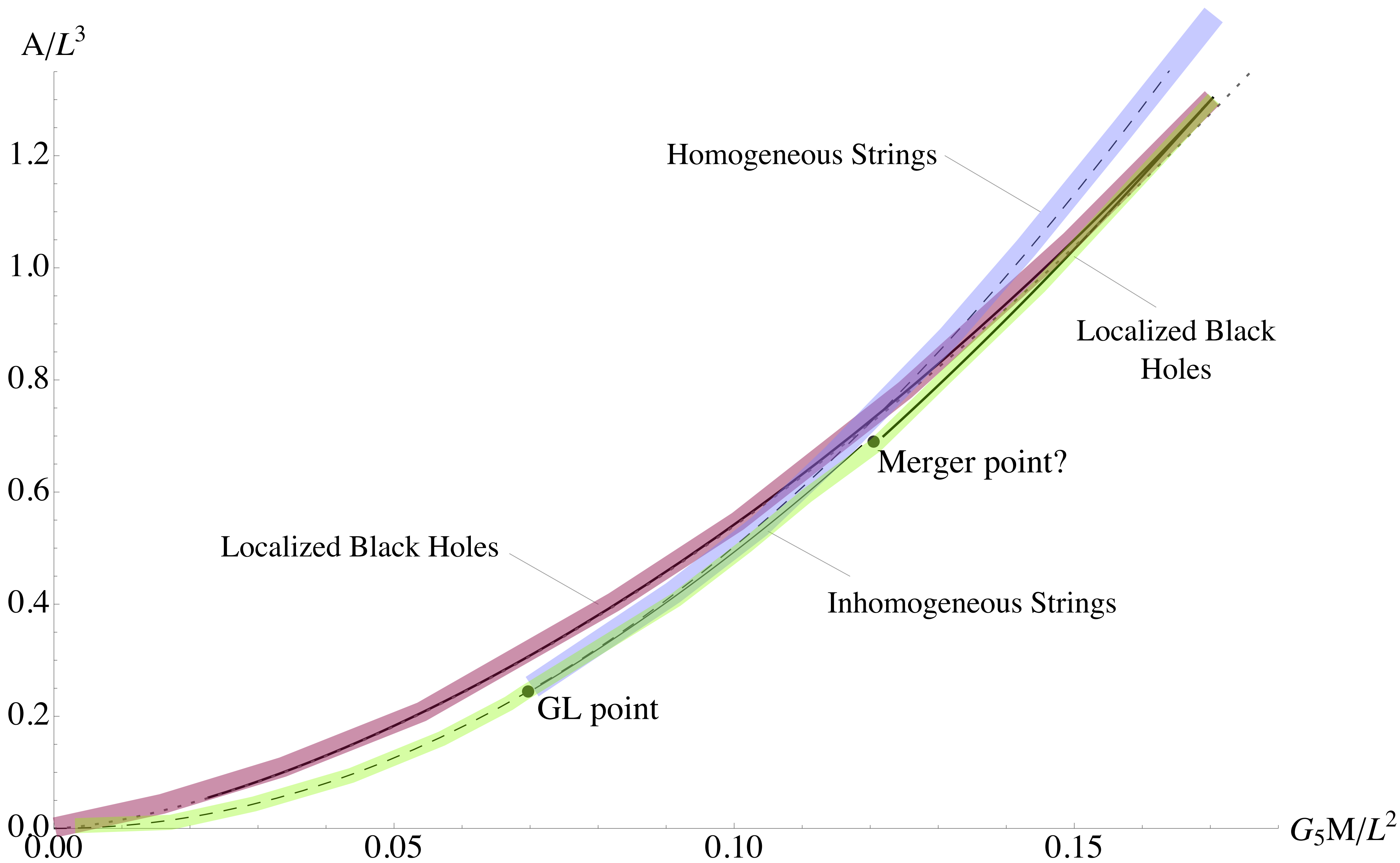
Results: Kaluza-Klein black holes

- Global stability



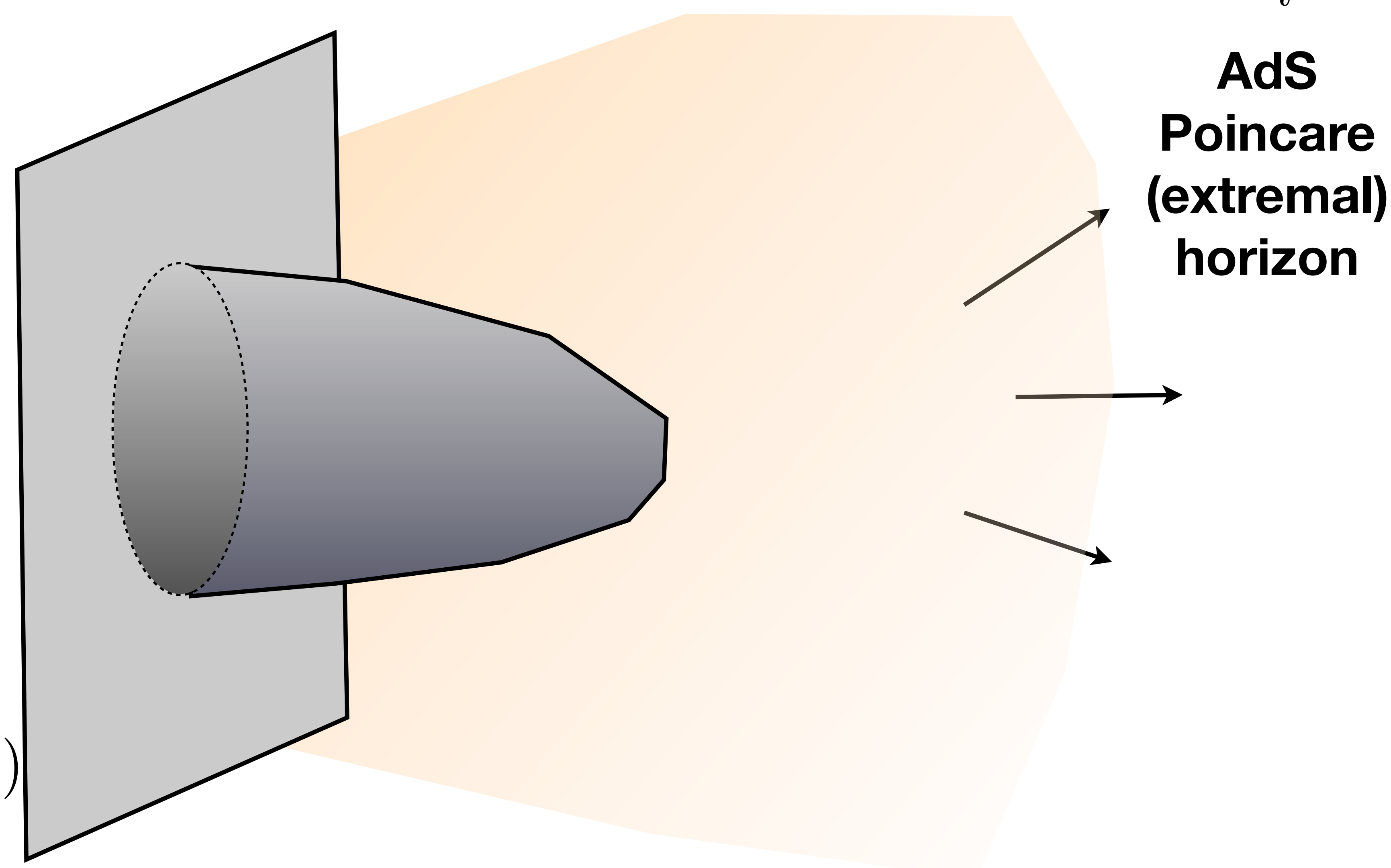
Results: Kaluza-Klein black holes

- Local stability



Part III: AdS-CFT with a black hole boundary

- Suppose we wish to understand AdS-CFT where the dual 4d CFT is placed on a Schwarzschild background. Aim to solve $R_{\mu\nu} = \Lambda g_{\mu\nu}$ such that; $\Lambda = -\frac{4}{l^2}$



$$ds^2 = \frac{l^2}{z^2}\left(dz^2 + h_{\mu\nu}(z,x)dx^\mu dx^\nu\right)$$

Fefferman-Graham expansion about $z = 0$

$$\implies \quad h_{\mu\nu}(z,x)dx^\mu dx^\nu = \left(-f dt^2 + \frac{1}{f}dr^2 + r^2 d\Omega^2\right) + z^4 t_{\mu\nu}(x)dx^\mu dx^\nu + O(z^6)$$

AdS-CFT with a black hole boundary

- We may use the isometry, $\partial/\partial\tau$, and any others (eg. axisymmetry) to simplify the problem, by reducing on these directions. At fixed points, boundary conditions are required, and are determined from smoothness of the original metric.

- We use a metric ansatz of the form;

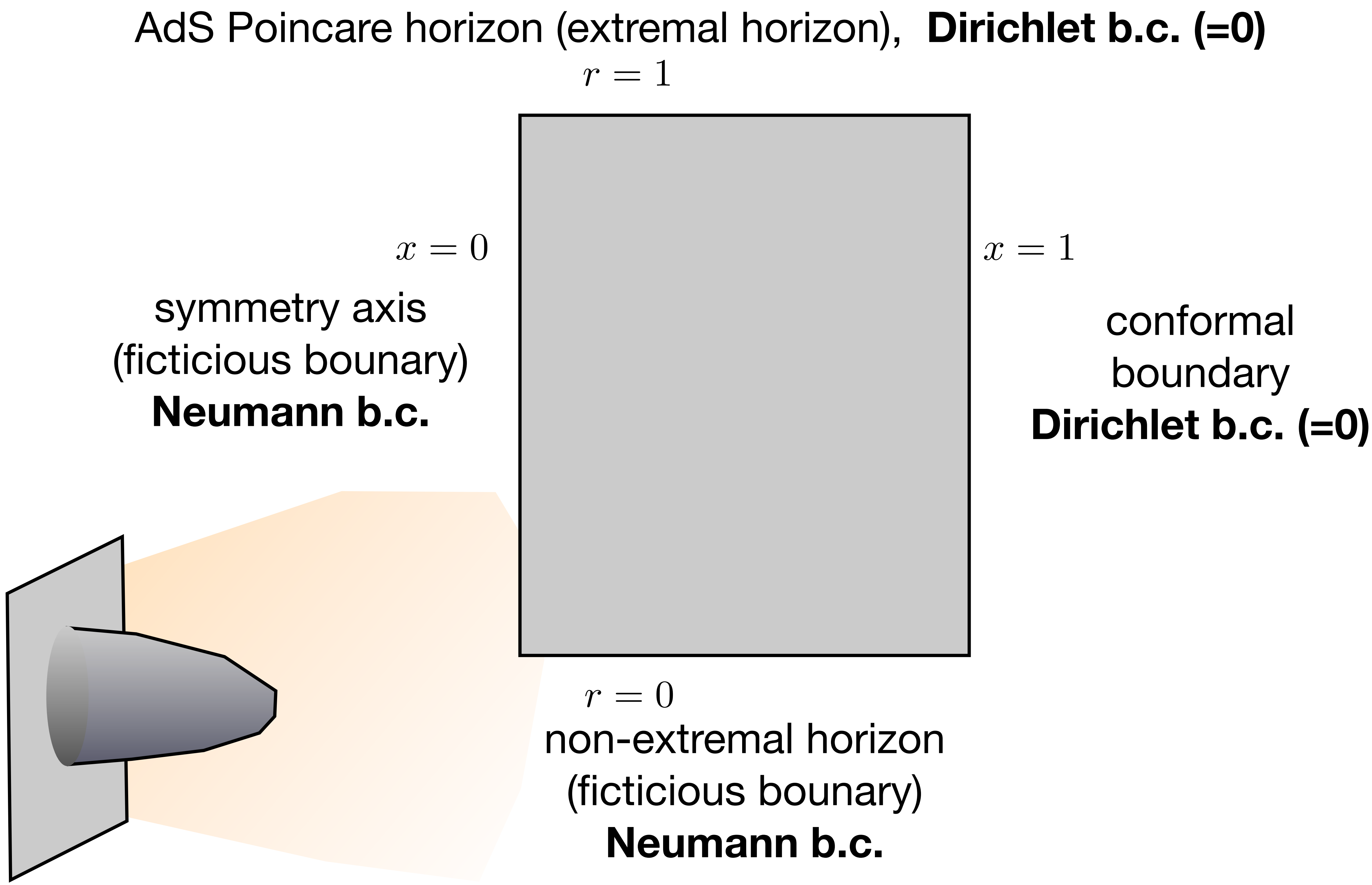
$$ds^2 = \frac{l^2}{1-x^2}\left(4r^2 f^2 e^T d\tau^2 + x^2\, g\, e^S d\Omega^2_{(2)} + \frac{4}{f^2} e^{T+r^2 f A} dr^2 + \frac{4}{g} e^{S+x^2 B} dx^2 + \frac{2rx}{f} F dr dx\right)$$

$$g = 2 - x^2\ ; \quad f = 1 - r^2$$

- Then T, A, B, F, S are functions of $r,\, x$ and their behaviour at boundaries in the $r,\, x$ domain specifies the topology of the metric.

- Initial guess = reference metric = $(T = A = B = S = F = 0)$

AdS-CFT with a black hole boundary

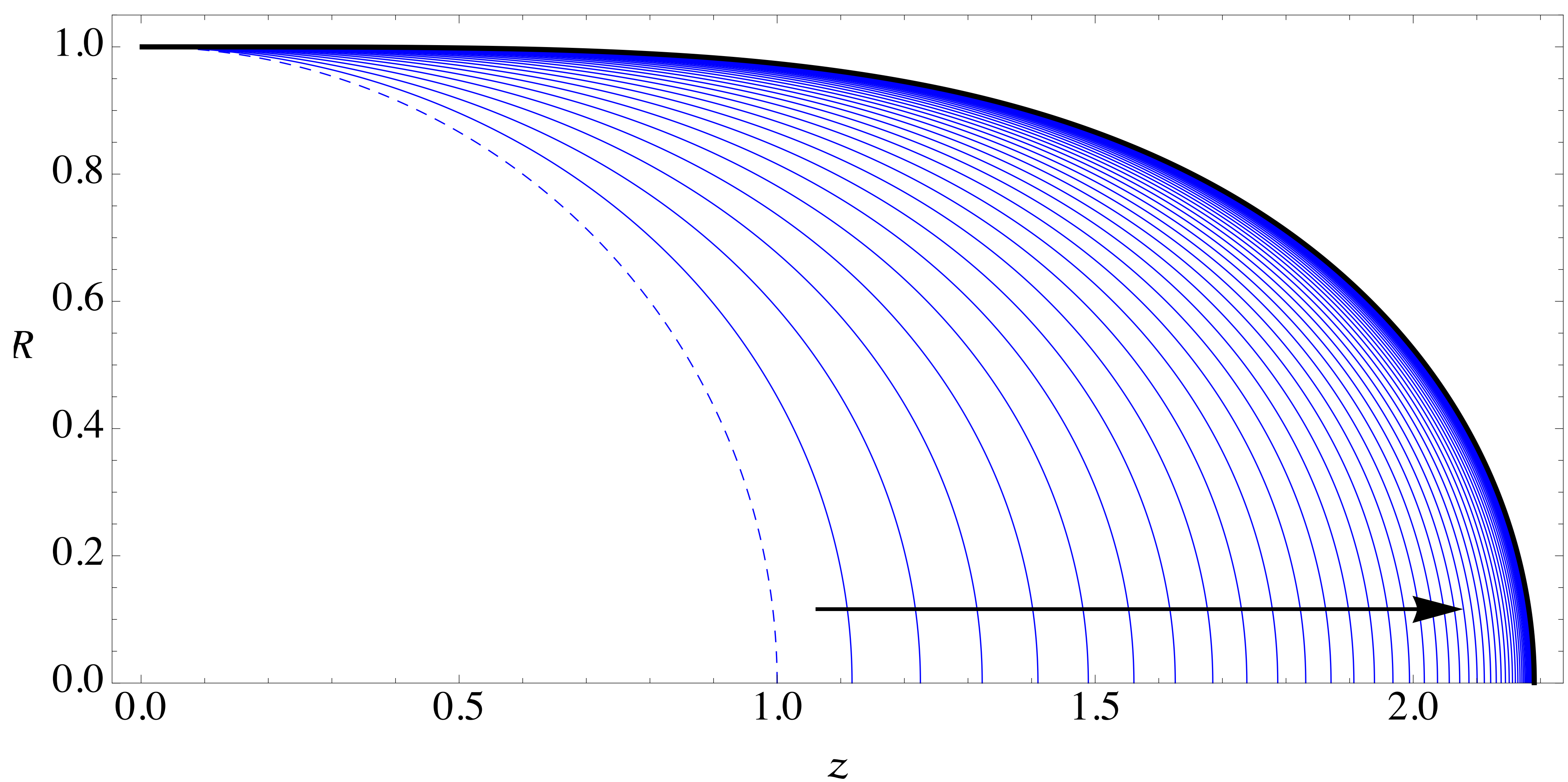


AdS-CFT with a black hole boundary

- Two very nice features of this system:
  - There exist no solitons
  - Since the black hole is not ‘dynamical’ there are no negative modes. One can simply use Ricci flow to relax to the solution.
- Thus it is very straightforward to find this solution!

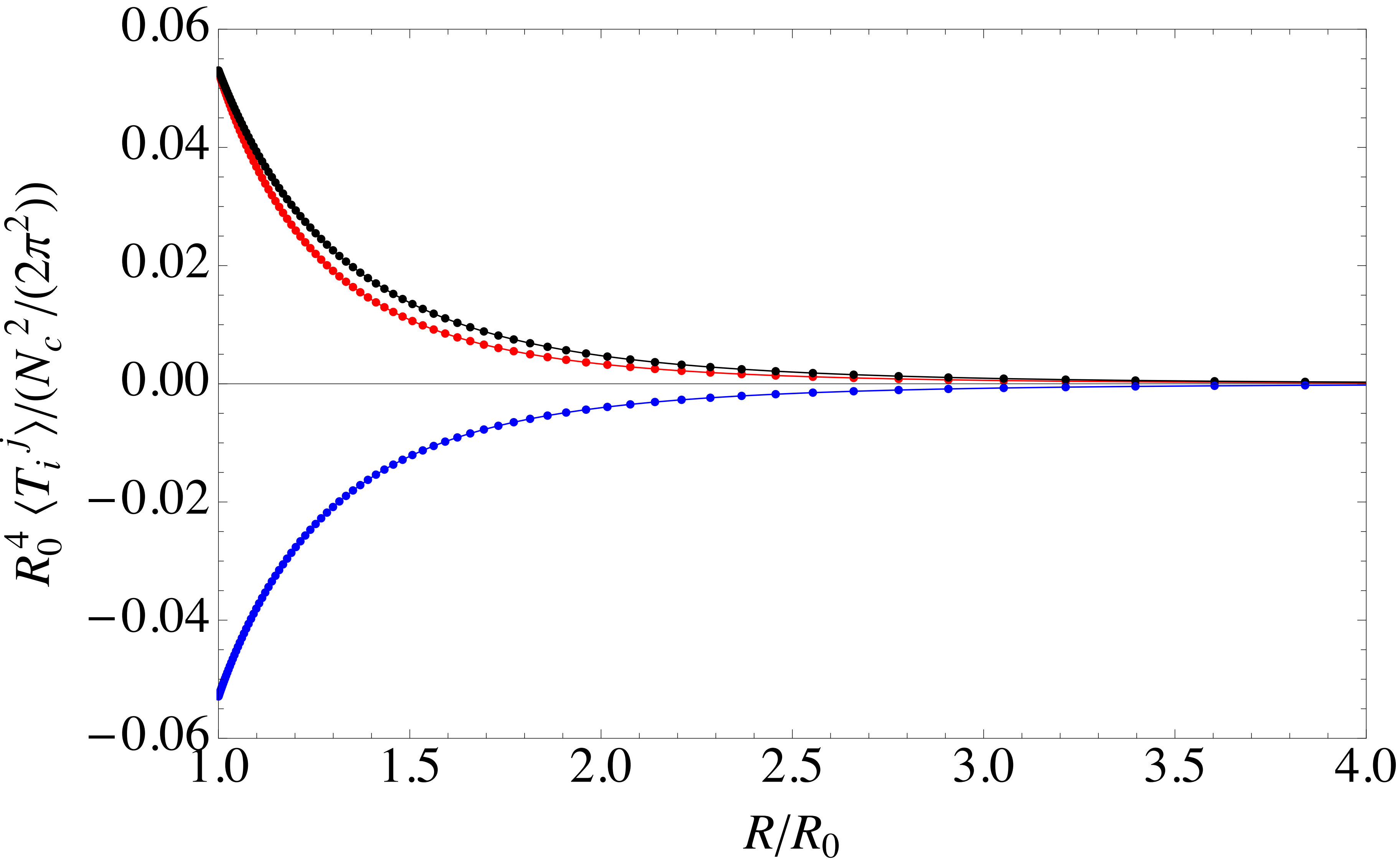
AdS-CFT with a black hole boundary

- Solve using ricci flow (or newton method)
- Embedding into hyperbolic space $ds^2 = \frac{l^2}{z^2}\left(dz^2 + dr^2 + r^2 d\Omega^2_{(2)}\right)$ as $r = R(z)$



AdS-CFT with a black hole boundary

- extract O(N^2) part of boundary stress tensor



AdS-CFT with a black hole boundary

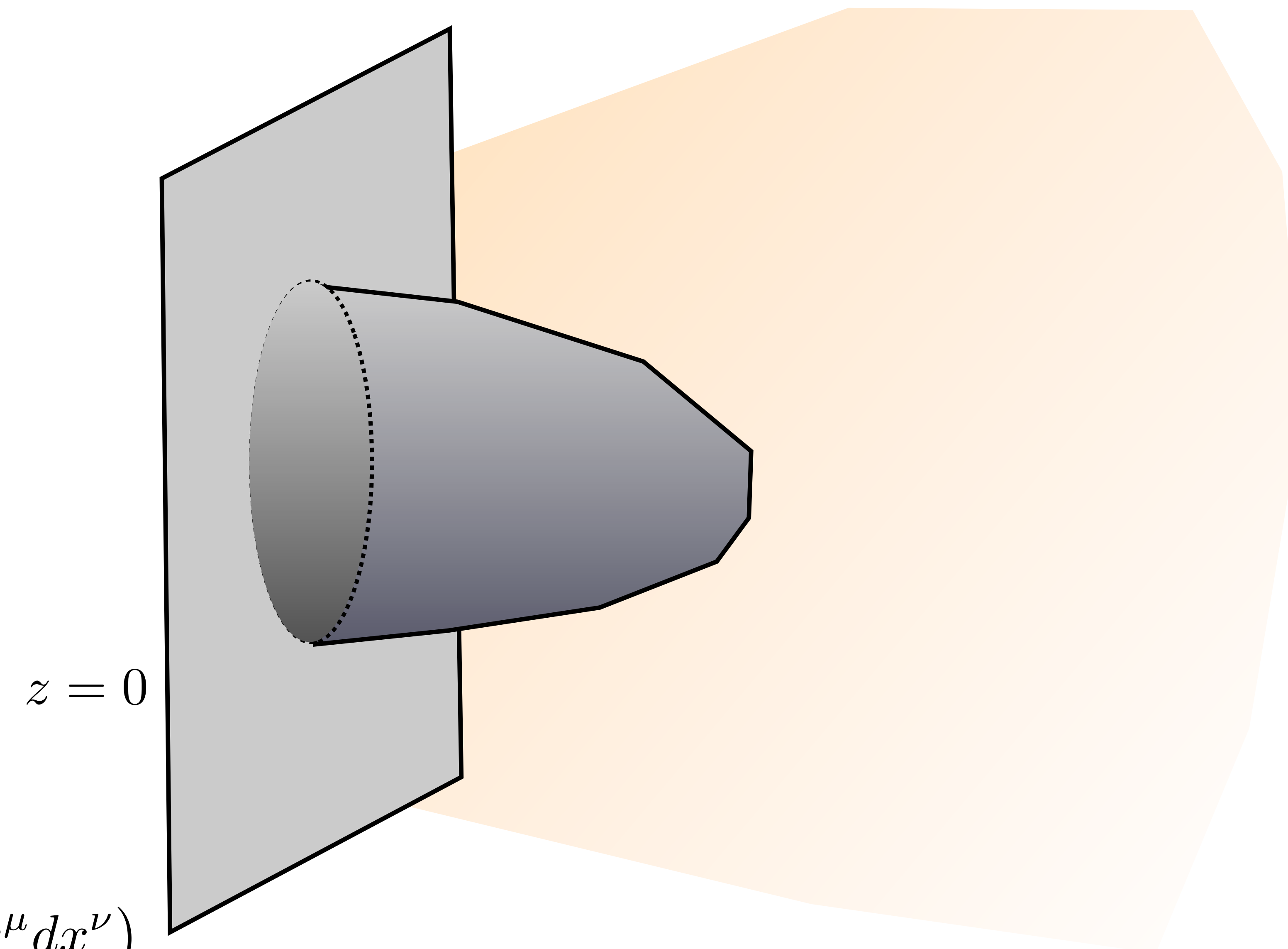
- Interpretation - many of issues similar to those of large RS black holes.
  - Found classical dual to CFT on Schwarzschild metric in standard (zero temp) vacuum   ie. not the Hartle-Hawking vacuum.
  - Believe it describes the leading order (ie. planar) behaviour of the CFT in the Unruh (and possibly Boulware) vacua.
  - Determines its O(N^2) stress tensor which is regular everywhere.
  - In order to see the usual divergences on past horizon for Unruh vacuum one should include bulk quantum/string corrections.
  - Presumably see singularity in O(1) component of stress tensor.

AdS-CFT with a black hole boundary

- Physical picture: Black hole act as both a thermal source and an energy sink
  - Black hole heats up vacuum to form plasma surrounding it
  - Radiation pressure of plasma counteracts the (attractive) strong interaction
  - At O(N^2) reaches equilibrium configuration with energy emitted from horizon equaling energy radiated back in.
  - At infinity only O(1) flux of radiation

Part IV: RSII black holes from AdS/CFT

- From a soln of AdS/CFT one can construct an approximate `large' RSII soln



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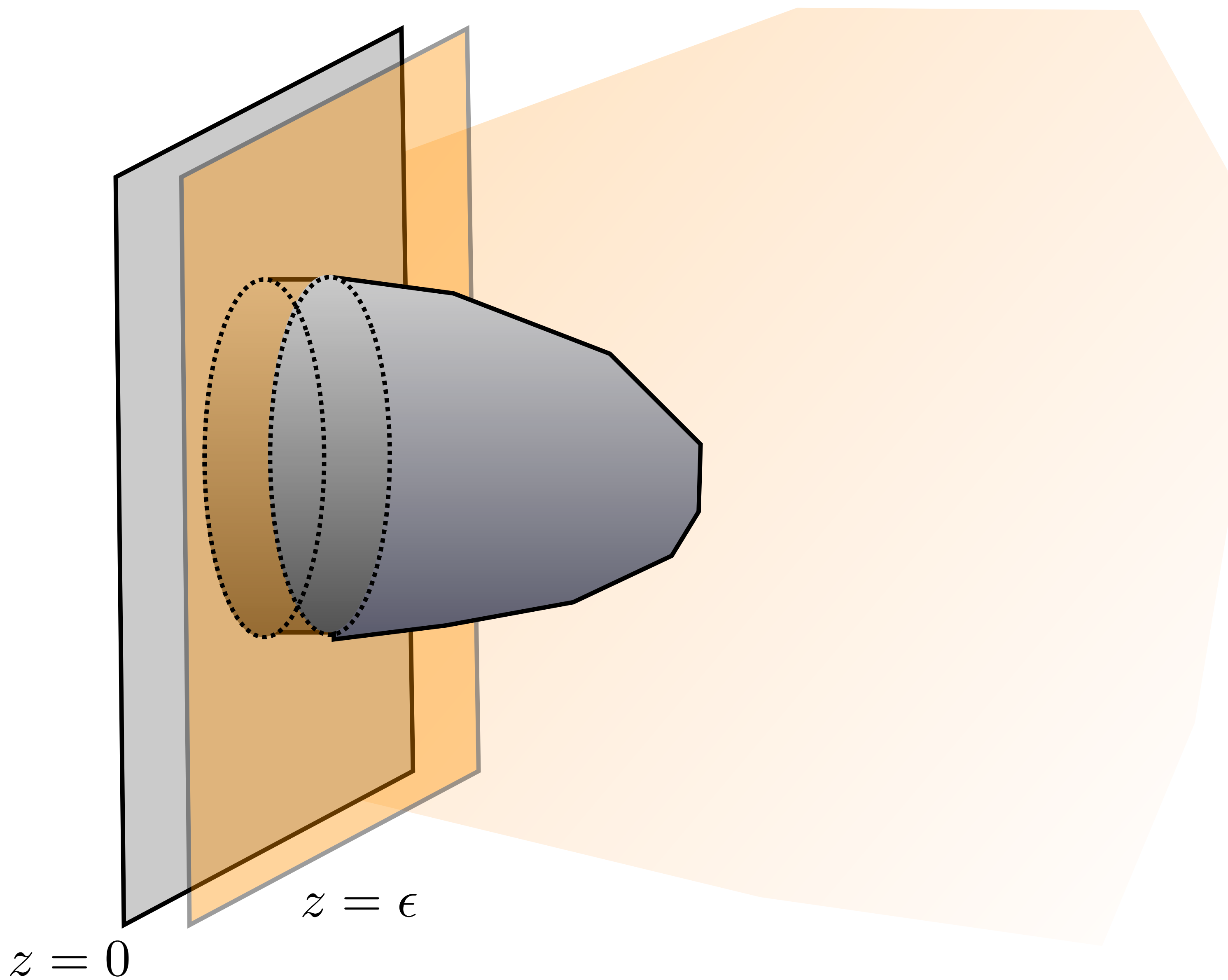

$\epsilon \ll 1$

RSII black holes from AdS/CFT

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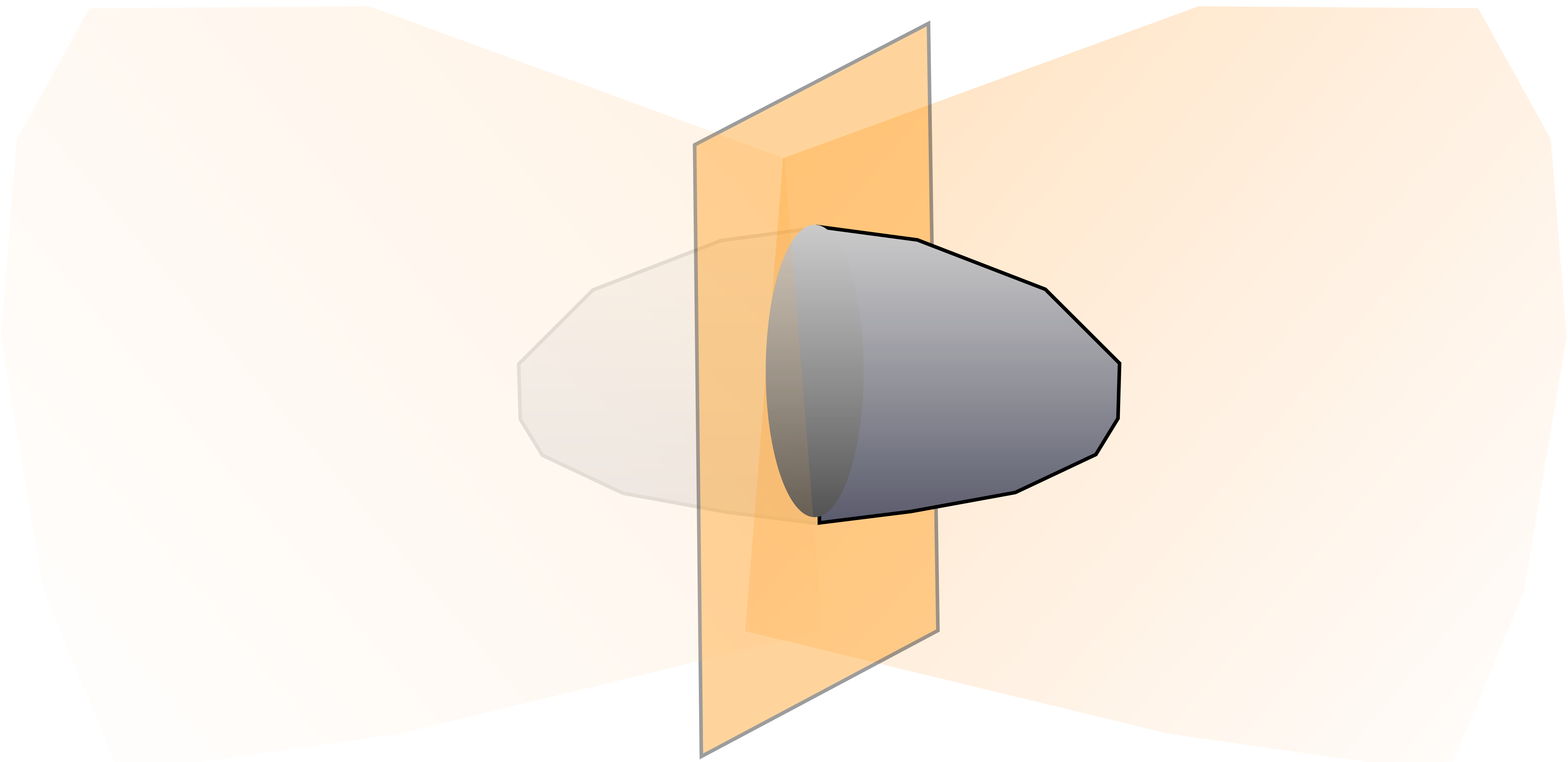
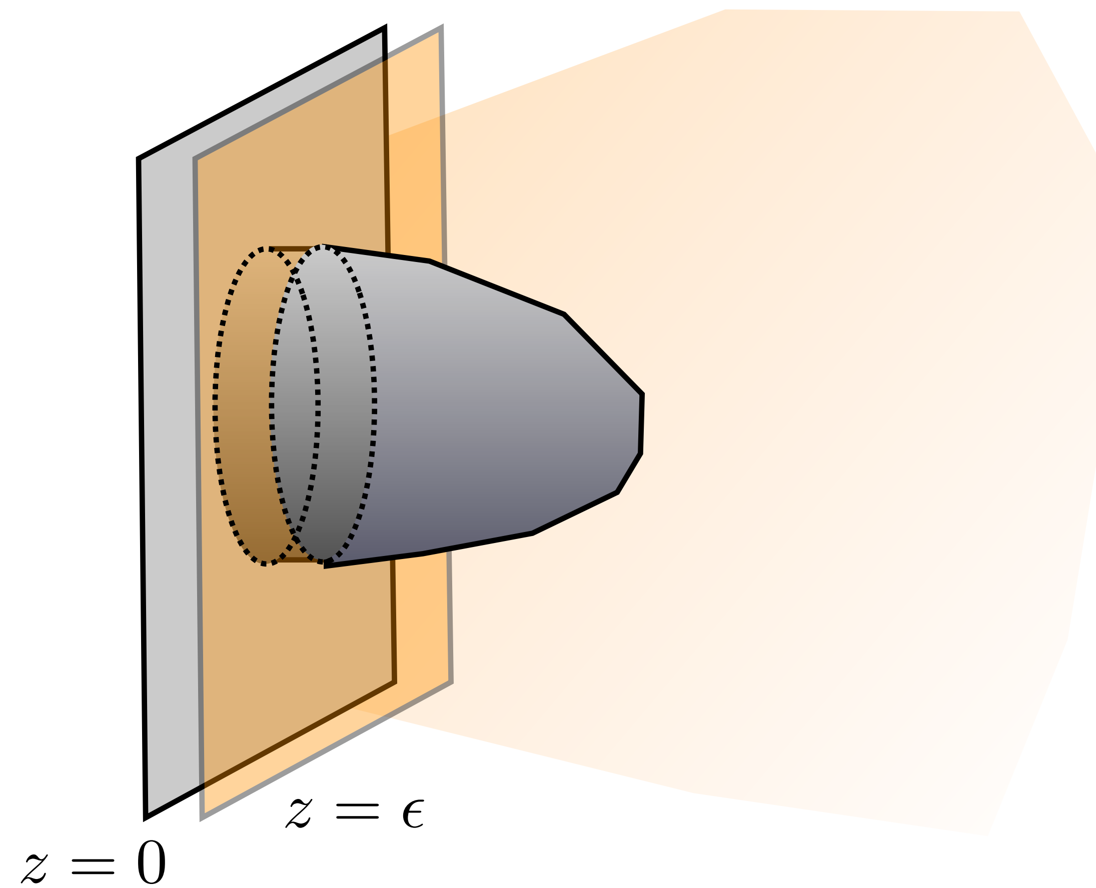


RSII black holes from AdS/CFT

- Induced metric on the brane $\gamma_{\mu\nu}$ which is approximately vacuum
- Find from F-G and Israel:

$$\gamma_{\mu\nu} = \frac{\ell^2}{\epsilon^2}\left(g^{Sch}_{\mu\nu} + \epsilon^2 \delta g_{\mu\nu}\right)$$

$$T^{brane}_{\mu\nu} = O(\epsilon^4)$$



- Correction to 4d Sch:

$$G_{\mu\nu}[g^{Sch} + \epsilon^2 \delta g] = 4\epsilon^2 t_{\mu\nu} + O(\epsilon^4)$$

- Exact solution is (static) AdS-CFT soln with perturbed boundary metric:

$$g_{\mu\nu} = g^{Sch}_{\mu\nu} + O(\epsilon^4)$$

RSII black holes

- Modify the AdS/CFT ansatz for RSII braneworld:

$$ds^2 = \Delta\left(4r^2 f^2 e^T d\tau^2 + x^2\, g\, e^S d\Omega^2_{(2)} + \frac{4}{f^2} e^{T + r^2 f A} dr^2 + \frac{4}{g} e^{S + x^2 B} dx^2 + \frac{2rx}{f} F dr dx\right)$$

$$\Delta = \frac{l^2}{1 - x^2} \quad \rightarrow \quad \Delta = \frac{l^2}{(1 - x^2) + \epsilon\,(1 - r^2)}$$

- Expect metric functions should approach AdS-CFT soln for large black holes where $\epsilon \rightarrow 0$ is the perturbation parameter.

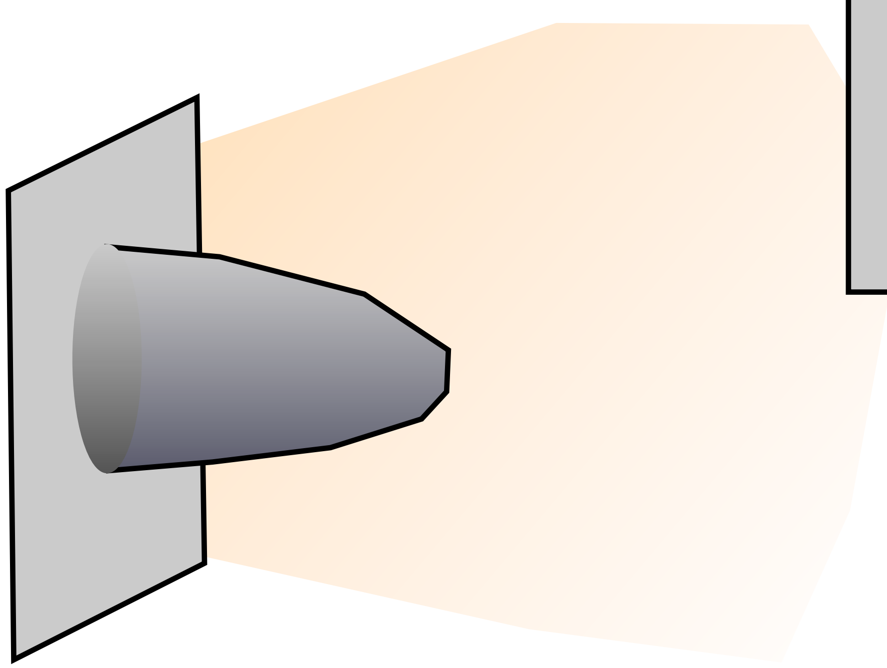
RSII black hole

AdS Poincare horizon (extremal horizon), **Dirichlet b.c. (=0)**

$r = 1$

$x = 0$

symmetry axis
(ficticious bounary)
Neumann b.c.



$x = 1$

Brane b.c.

$$K_{ij} = \kappa \, h_{ij} \qquad \left(\kappa = -\frac{3}{l}\right)$$

$$\xi_x = 0$$

$$F = 0$$

$$\Longrightarrow \qquad \partial_x \xi_r = \frac{2}{l} \xi_r$$

$r = 0$

non-extremal horizon
(ficticious bounary)
Neumann b.c.

RSII black hole

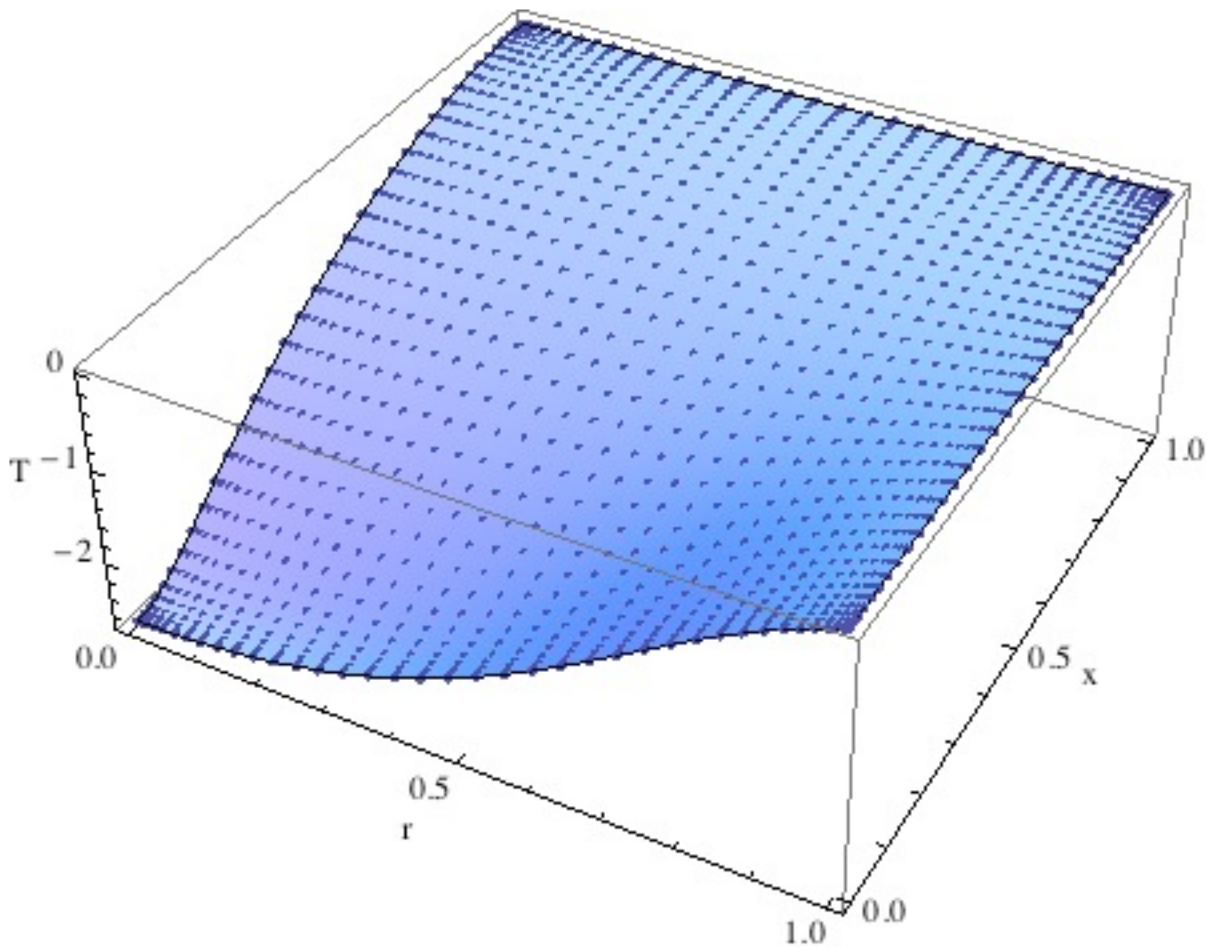
- Method works well - we can find both very small black holes ($R_4 \ll l$) and large ones ($R_4 \sim 100l$)

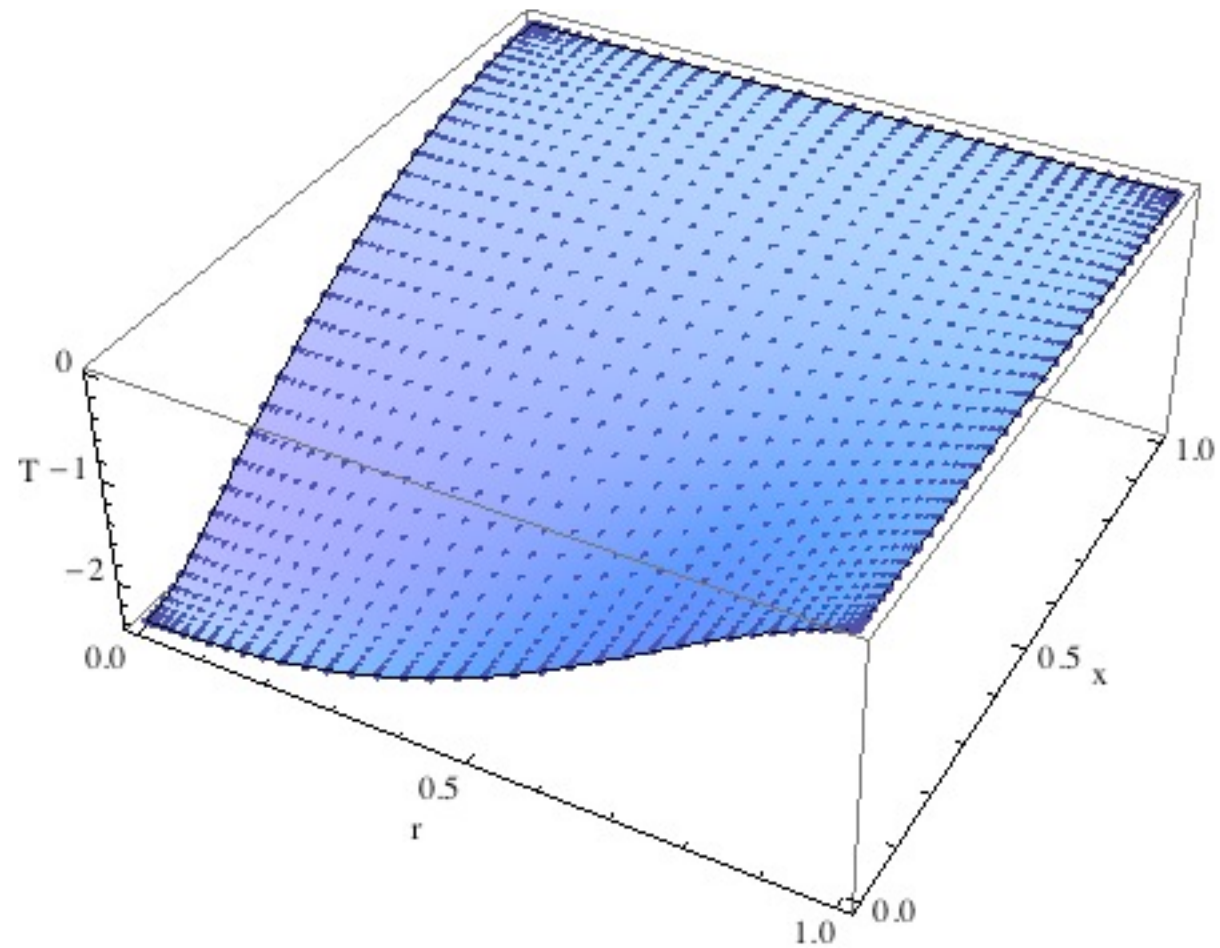
- However, unlike the AdS/CFT solution we cannot prove no solitons exist, although we don't find any.

- Also since the black hole is fully 'dynamical' the solution has a negative mode and we must use the Newton method so more complicated.

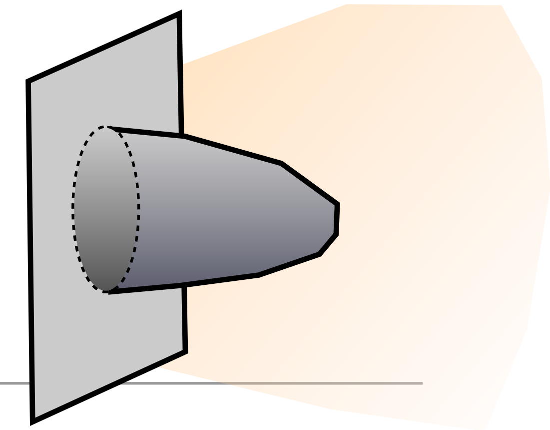
RSII results

- Metric function T:

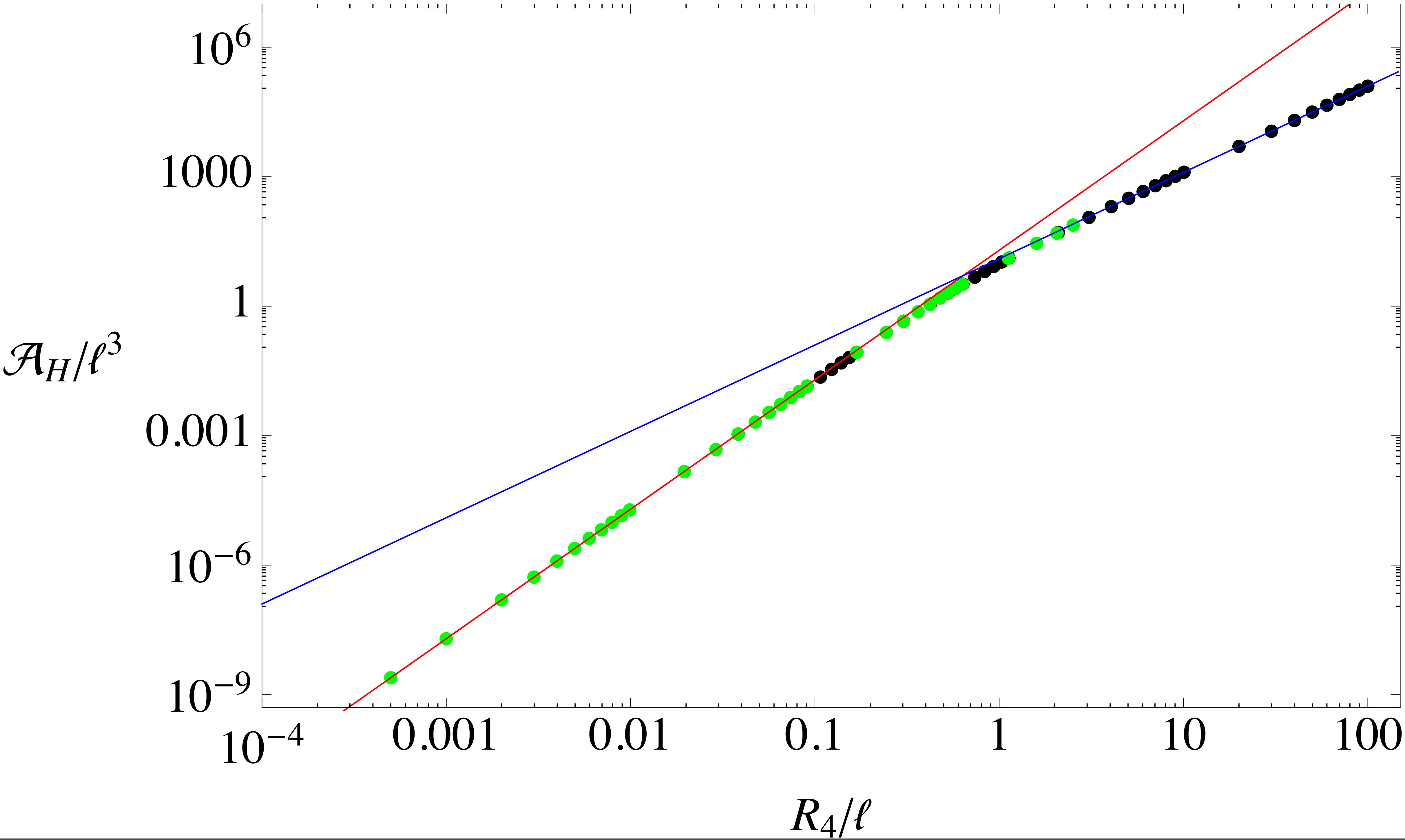




RSII results

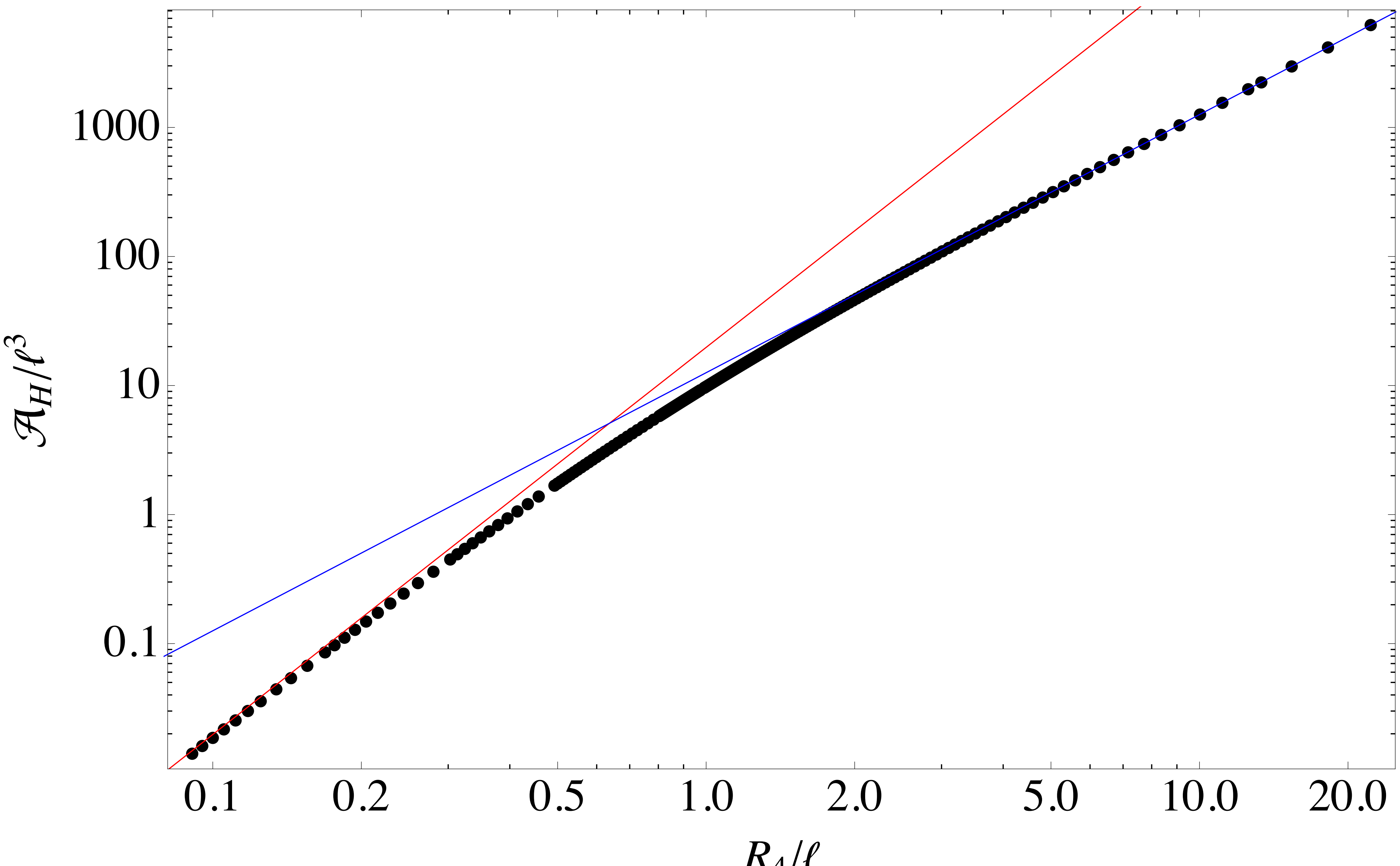


- Can find solutions from very small to very large black holes ($R_4/\ell \sim 100$).



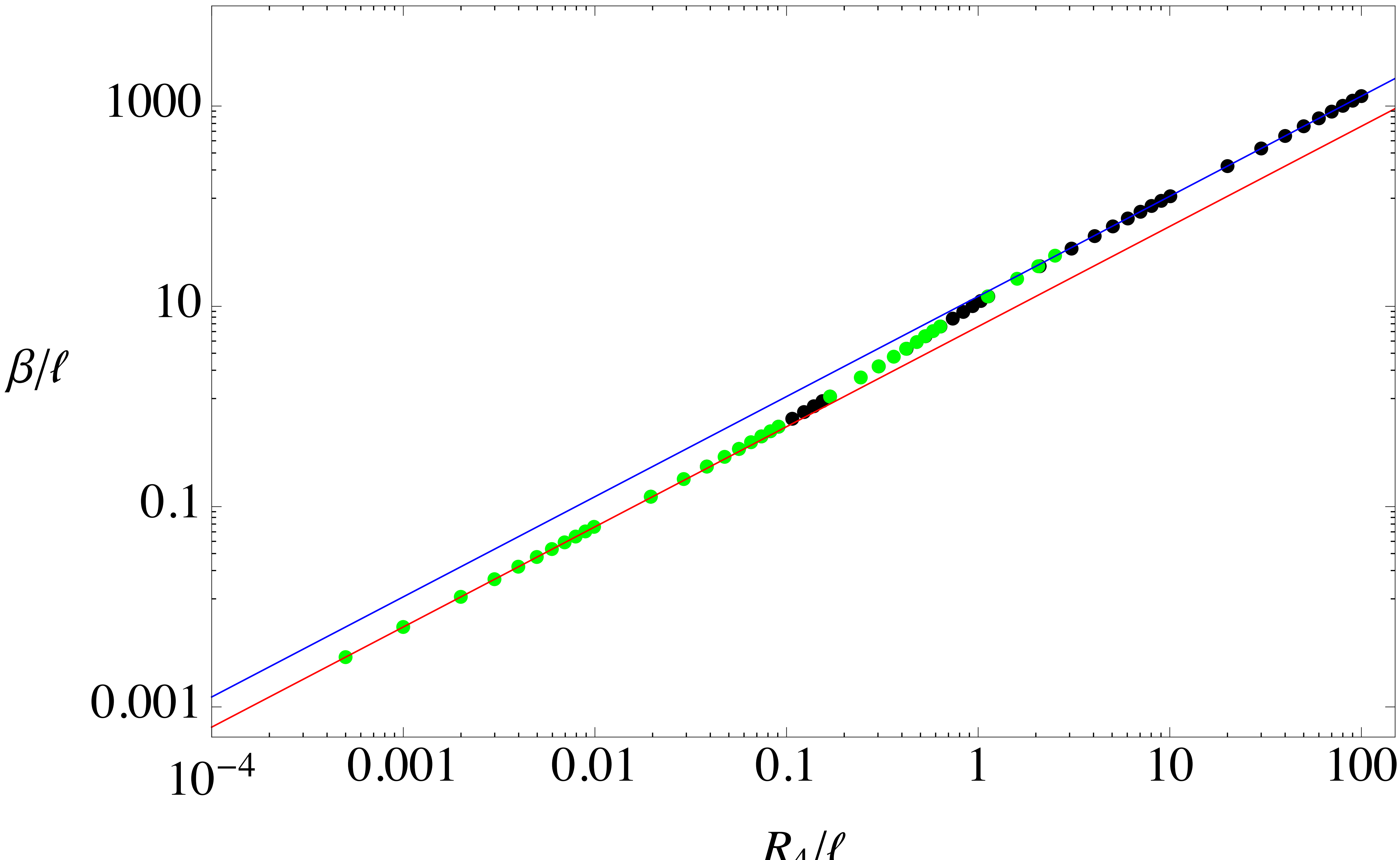
RSII results

- Smooth transition from 5-d to 4-d behaviour;



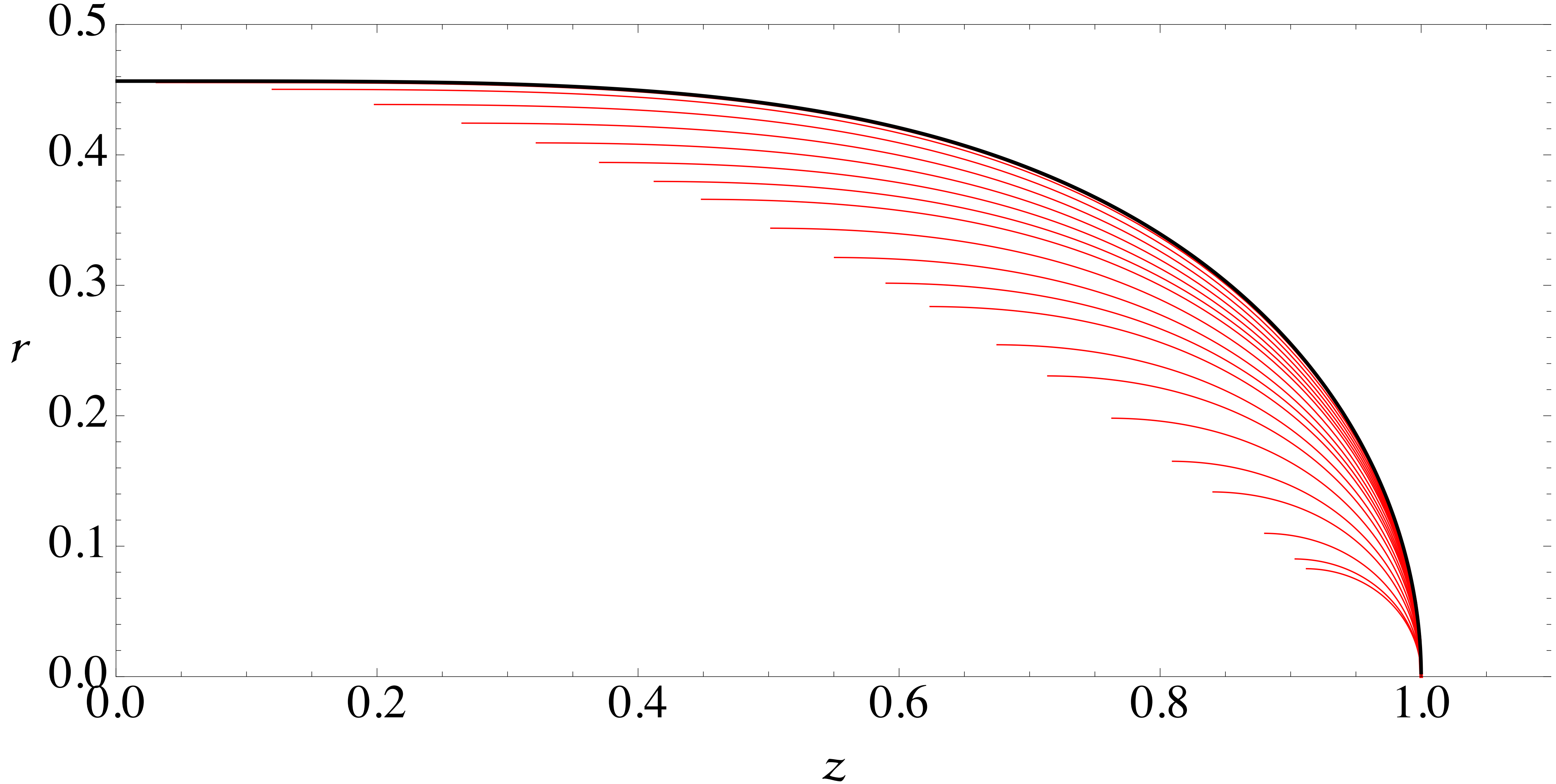
RSII results

- Inv Temperature



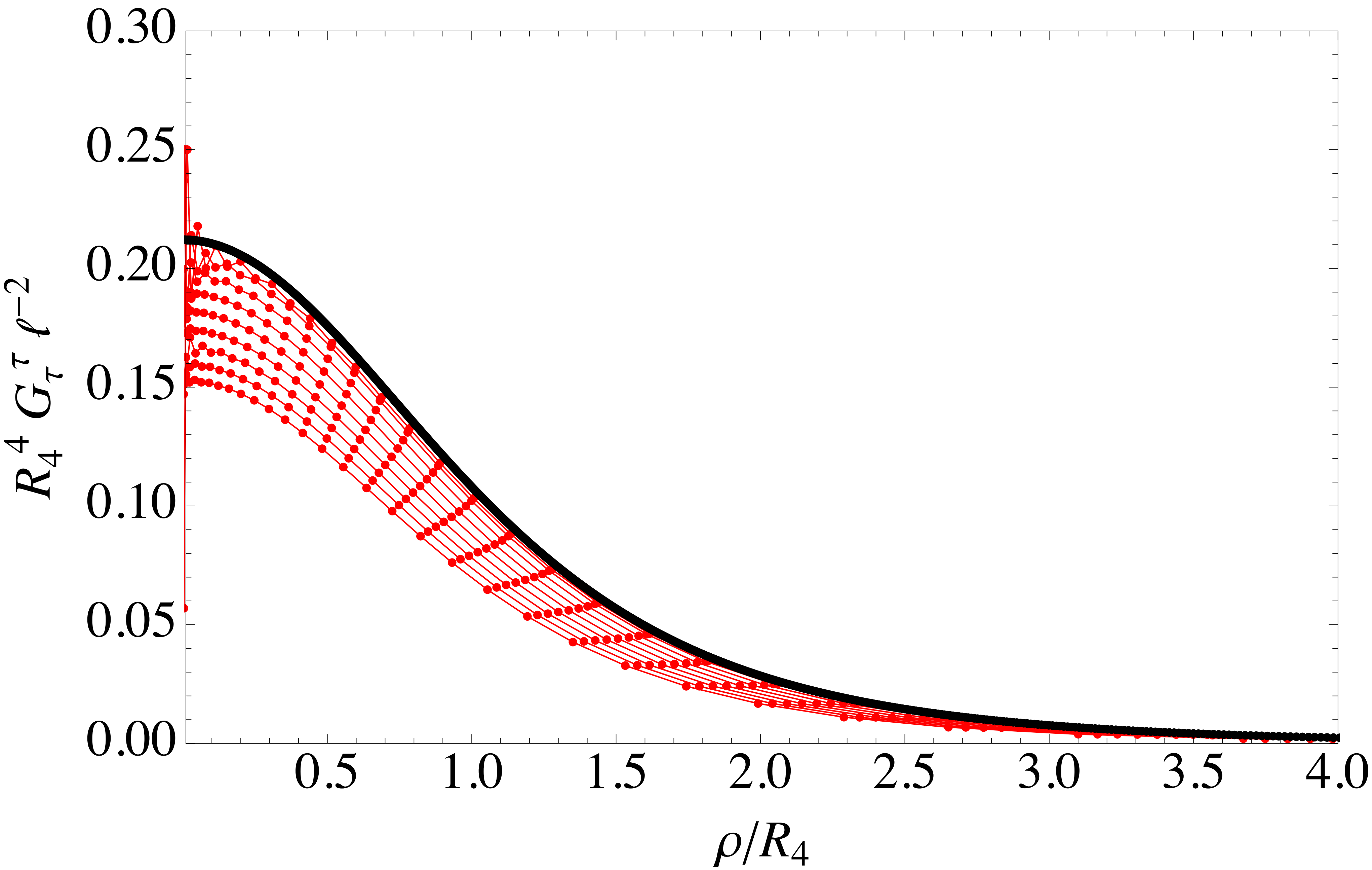
RSII results

- Embeddings of brane black hole horizons into hyperbolic space



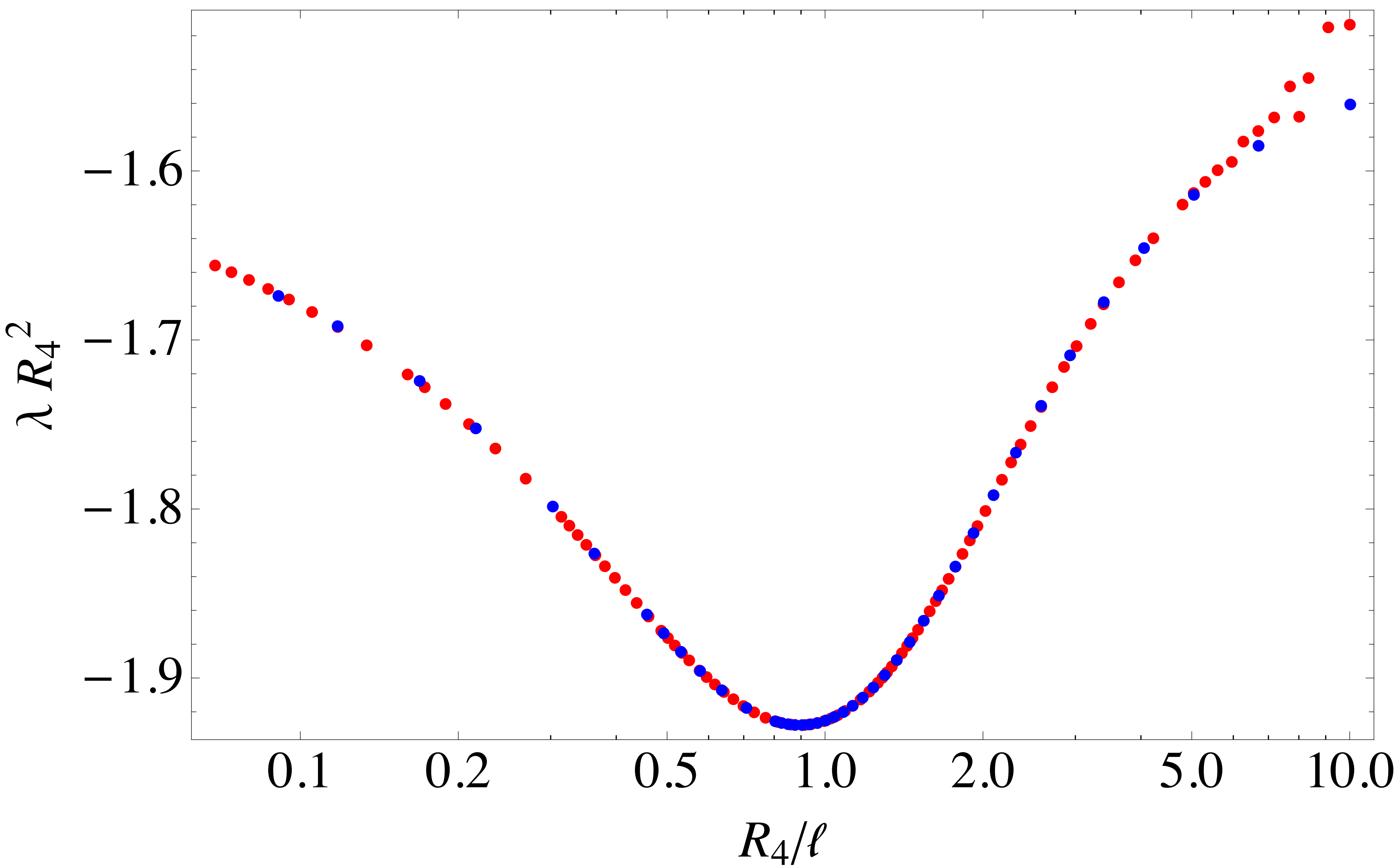
RSII results

- Brane geometry and AdS/CFT soln stress tensor



RSII stability

- Likely to be stable as we only see one Euc negative mode.



Summary

- We have effective numerical techniques to find static black holes.
- Black holes in ADD have intricate behaviour. However, at large masses there appear only to be black strings which have usual 4d behaviour. The transition from 5d to 4d behaviour is complicated.

- Large static (stable?) black holes exist in RSI and have usual 4d behaviour. The transition from 5d to 4d behaviour is apparently simple.
  - Related to existence of interesting AdS/CFT solution with Schwarzschild boundary metric.