Numerical Relativity | High energy physics

Madeira Island

31 Aug-3 Sep 2011

Approximation theory in GR

overview and selected topics

$E \approx mc^2$

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Physics is approximation

From *Feynman's Lectures on Physics* (1965):

[…] the whole nature is always an approximation to the complete truth […]. In fact, everything we know is only some kind of approximation, because we know that we do not know all the laws as yet.

Therefore, things must be learned only to be unlearned again or, more likely, to be corrected […]. The test of all knowledge is experiment. Experiment is the sole judge of scientific "truth".

Outline:

- Introduction: approximation methods in gravity
- One step back (and forward) Numerical Relativity
- Selected topics in astrophysics/HEP:
  - EMRIs
  - Stability analysis
  - Perturbations of AdS BHs and AdS/CFT

Approximating gravity...

- GR is difficult: nonlinear, no superposition
  - Few analytical solutions → Two body problem?
  - Multiscale problems (S-dS BHs, EMRIs,...)
- Approximate methods are very powerful and ubiquitous
  - PN methods, perturbation theory, WKB methods, ...

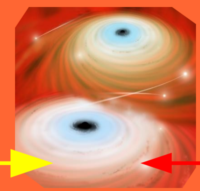
- Oscillations of BHs and stars [Ferrari's talk]
- GW emission [Many talks]
- Imprints of alternative theories [Yunes' talk]
- ...

- Gauge/gravity duality [Hubeny's talk]
- Linear stability analysis [Dias' talk]
- Inhomogeneities in cosmology
- ...

Approx. methods VS Numerical Relativity

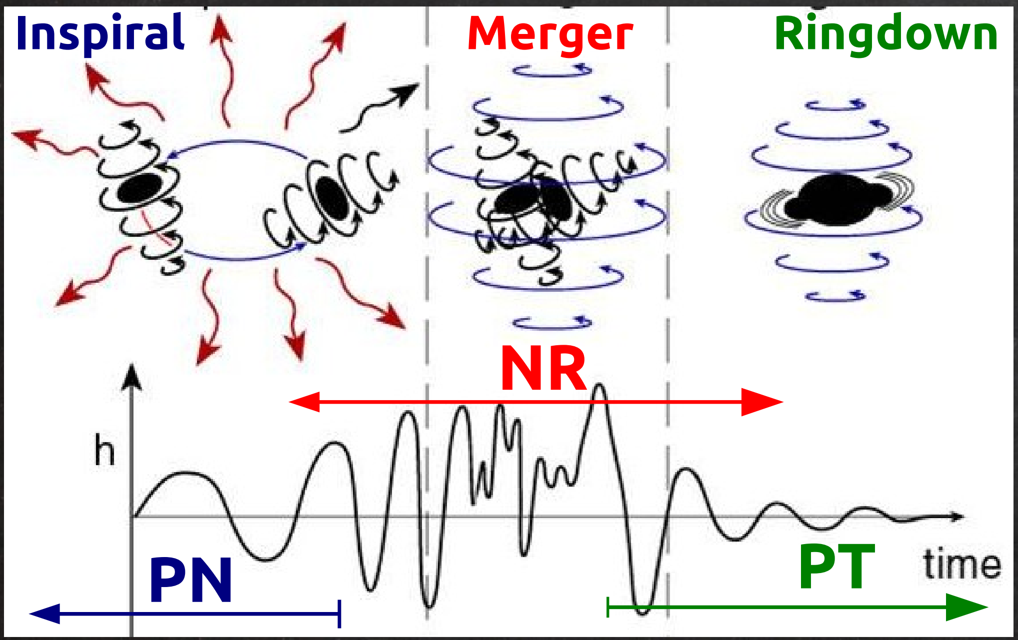
Idealized situations
Physical insights
"Easy" to perform

Approximate methods



Numerical Relativity

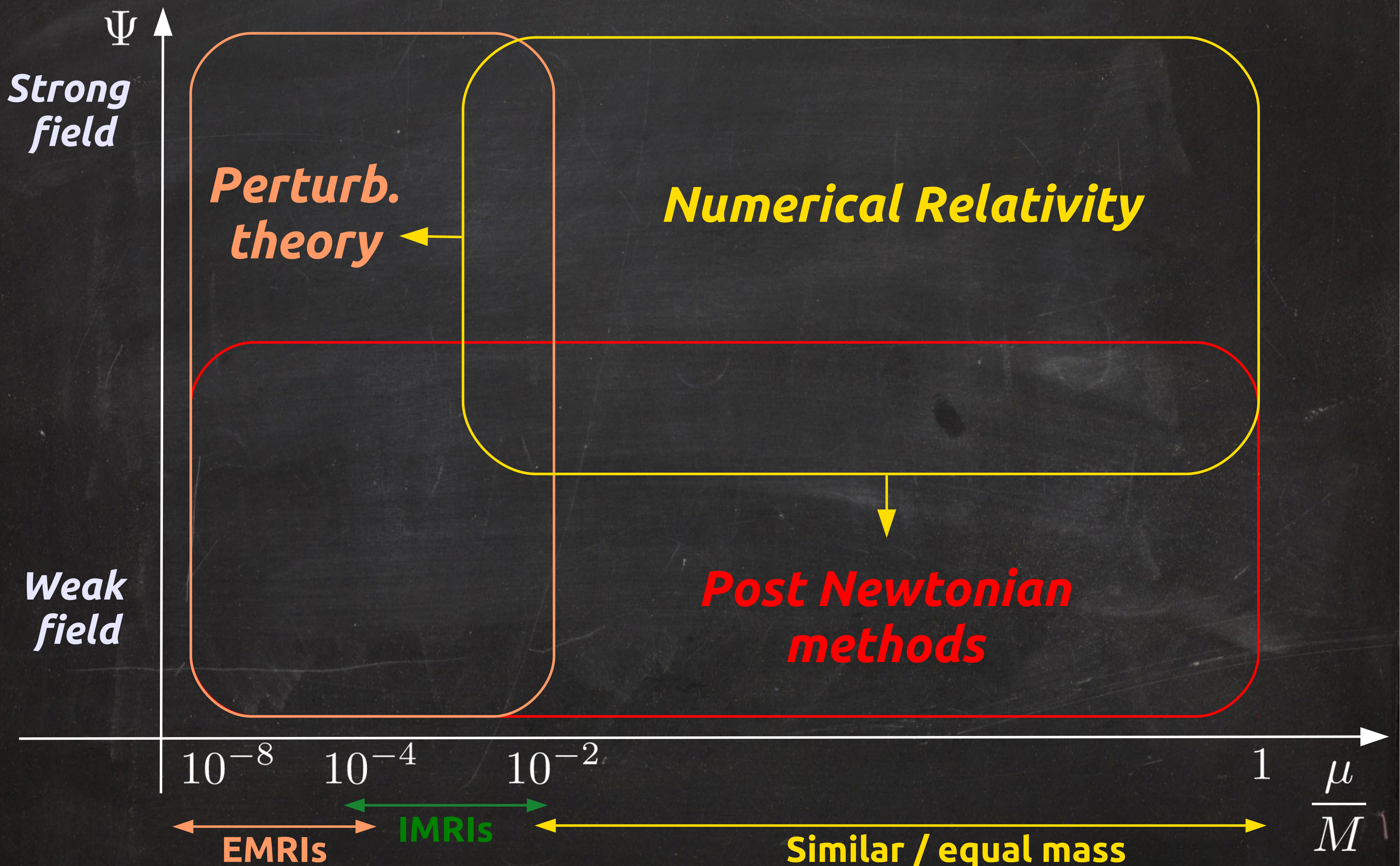
Realistic situations
Numerics → Physics
Supercomputers



Adapted from Thorne

- Approximate doesn't mean worst!
- Complementary approach
- Synergy between approx. and NR

Two body problem in General Relativity

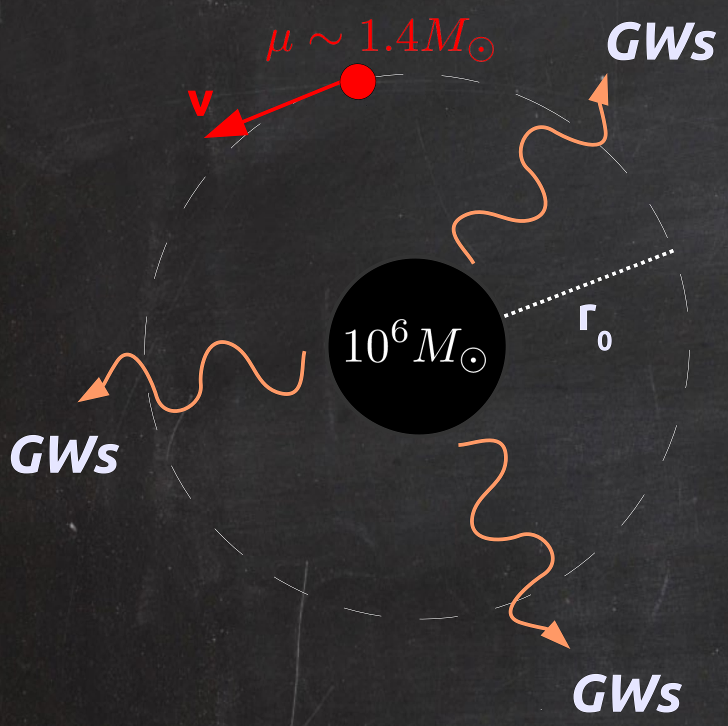


Topic I

Extreme mass ratio inspirals

EMRIs: Extreme Mass Ratio Inspirals

Emission due to "point-like" particles:



- Evidences of supermassive BHs
- $M \sim 10^6 - 10^9\ M_{sun}$
- Two scales: M and μ
- Mass ratio: $10^{-8} - 10^{-4} \rightarrow$ NO NR!
- $v \sim c \rightarrow$ NO PN!
- Perfect regime for perturbation theory

- Source for GW space detectors
- $\sim 10^5$ cycles during last year within LISA band
- SNR~μ/M → matched filtering needs extremely accurate templates!

EMRIs: GW signal? how does the system evolve?

- **Geodesic motion** **[Tanaka, Cutler, Poisson,Hughes, ...]**
  - Point-particle with (E,L,Q)
  - Solve Teukolsky eq. with source
  - Adiabatic evolution $dE_p/dt = -\dot{E}_{\rm total}$

  - **No conservative of the force**
  - **Difficult for generic orbits in Kerr**
  - **Templates?**

- **Self-force** **[CAPRA Meetings]**
  - Backreaction on the particle

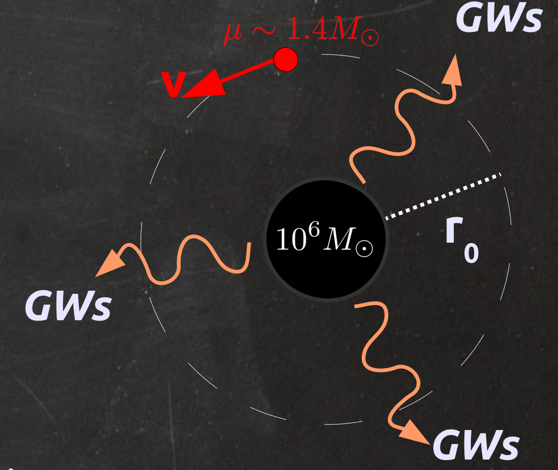
$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h^{R}_{\mu\nu}$$

  - **Needs regularization**
  - **Time consuming**
  - **Templates?**

- **Effective One Body approach** **[Buonanno & Damour, Nagar, Yunes...]**
  - Hamiltonian formulation
  - Resummation of PN series

  - **Good for templates**
  - **Needs calibration**

EMRIs: 1-slide computation



Inhomogeneous Bardeen-Press-Teukolski (BPT) equation:

$$\frac{d^2\Psi_{\ell m}(\omega,r)}{dr_*^2}+V(\omega,r)\Psi_{\ell m}(\omega,r)=\mathcal{T}_{\ell m}(\omega,r)$$

$$\mathcal{T}_{\ell m}(\omega,r)=\delta(\omega-m\Omega_p)\left[\mathcal{A}(\omega,r)\delta(r-r_0)+\mathcal{B}(\omega,r)\delta'(r-r_0)+...\right]$$

Green function techniques:

$$\Psi_{\ell m}=\frac{\Psi_\infty}{W}\int_{r_+}^{r}dr'\mathcal{T}_{\ell m}(r')\Psi_{r_+}+\frac{\Psi_{r_+}}{W}\int_{r}^{\infty}dr'\mathcal{T}_{\ell m}(r')\Psi_\infty$$

Energy flux at infinity:

$$\dot{E}_\infty^{\ell m}\sim\Omega_p^2\left(\frac{\mu}{M}\right)^2|\Psi_{lm}(r\to\infty)|^2$$

Energy flux at the horizon:

$$\dot{E}_{r_+}^{\ell m}\sim\Omega_p\left(\Omega_p-\Omega_H\right)\left(\frac{\mu}{M}\right)^2|\Psi_{lm}(r\to r_+)|^2$$

Linearization doesn't mean no interesting effects!

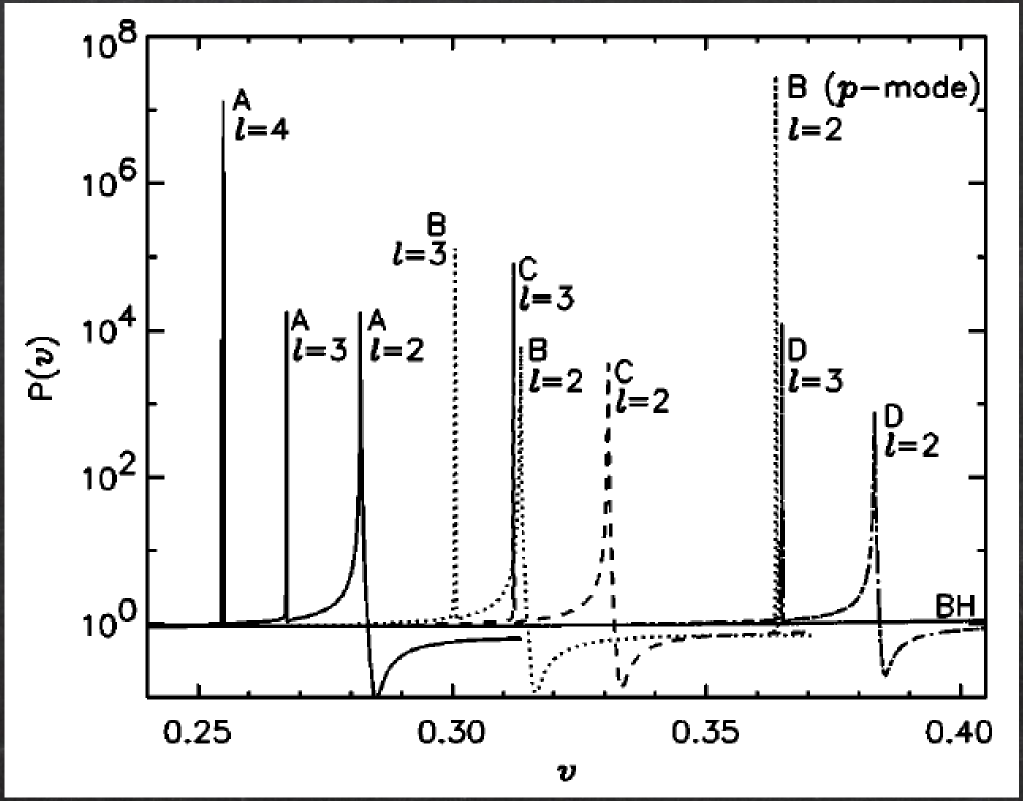
EMRIs: *resonances in GW from neutron stars*

[Pons, Berti, Gualtieri, Miniutti, Ferrari (2002)]

$$\frac{d^2\Psi_{\ell m}(\omega,r)}{dr_*^2}+V(\omega,r)\Psi_{\ell m}(\omega,r)=\mathcal{T}_{\ell m}(\omega,r)$$

$$\mathcal{T}_{\ell m}(\omega,r)\propto\delta(\omega-m\Omega_p)$$

~ forced oscillator



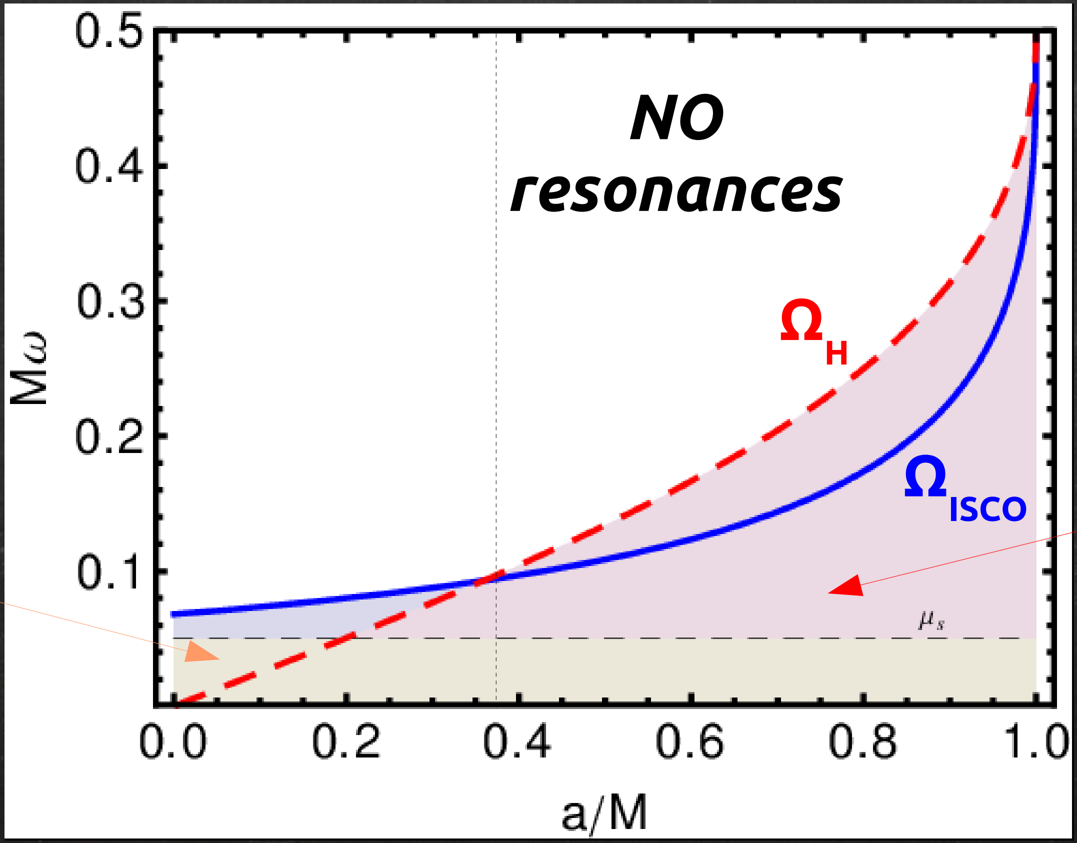
- **QNMs of perfect fluid stars can be excited when** $\Omega_p=\omega_{\rm QNM}/m$
- **Signature of different EOS**

EMRIs: *imprints of light scalars*

[Cardoso, Chakrabarti, Pani, Berti, Gualtieri (work in progress)]

- **BH QNMs cannot be excited by orbiting particles, $\omega_{QNM} > m\,\Omega_{ISCO}$**
- **Light scalars introduce a new scale → $\omega_{QNM} \sim \mu_s$**
- **Excitations of QNMs of Kerr BHs in astrophysical situations**
- **If $\omega < m\,\Omega_H$ → Superradiance, the flux at the horizon can be negative**

Positive resonance:
Sinking orbits



Negative resonance:
Floating orbits

Misner [1972], Press & Teukolski [1973]

$$\dot{E}_H \propto \omega - m\Omega_H$$

EMRIs: imprints of light scalars

[Cardoso, Chakrabarti, Pani, Berti, Gualtieri (work in progress)]

$$\left[\Box-\mu_s^2\right]\varphi=\alpha\mathcal{T}\Longrightarrow\frac{d^2\Psi_{\ell m}(\omega,r)}{dr_*^2}+V(\omega,r)\Psi_{\ell m}(\omega,r)=\mathcal{T}_{\ell m}(\omega,r)$$

Suitable for analytical computation in the large distance approx.

$$\dot{E}_H^{\rm scalar}\sim -r_0^{2\ell-3/2}$$

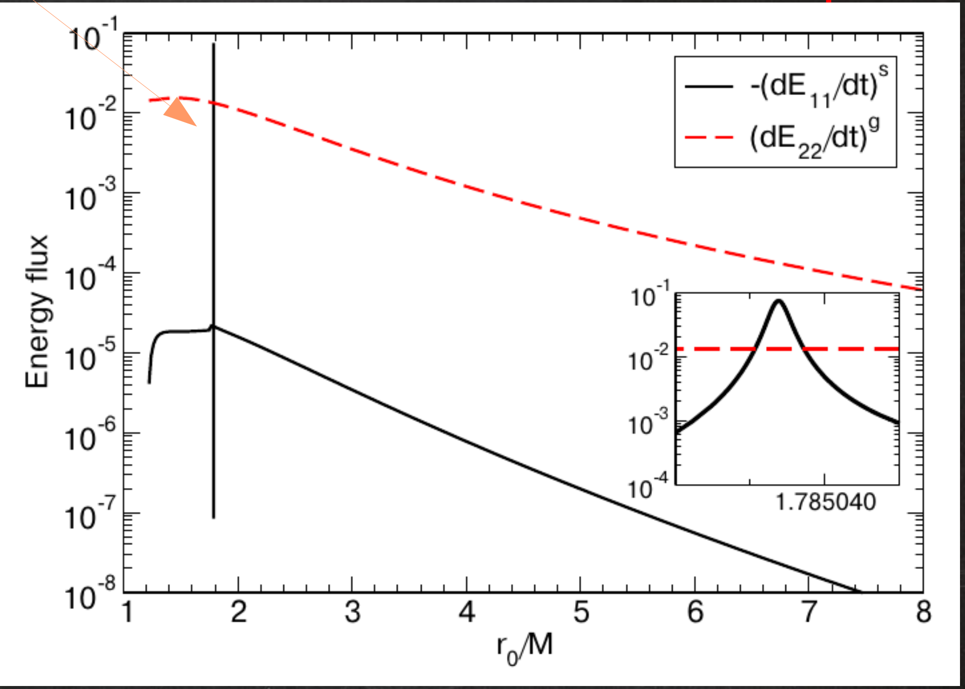
Floating orbit at $\Omega_p \sim \mu_s$

$$dE_p/dt=-\dot{E}_{\rm total}=-(\dot{E}_S+\dot{E}_G)$$

$$\text{if } \dot{E}_S=-\dot{E}_G\Longrightarrow\dot{E}_p=0$$

$$\dot{\Omega}_{\rm floating}/\Omega_p^2\sim(m_p/M)^2(M\Omega_p)^{7/3}$$

$$\dot{\Omega}_{\rm GR}/\Omega_p^2\sim(m_p/M)(M\Omega_p)^{5/3}$$



$$\omega_{\rm BD}<5\cdot10^8$$

Very general effect: only needs an ergoregion and a light scalar

EMRIs: *signatures of alternative theories of gravity*

- **Detection requires extremely precise theoretical templates**
- **Bias-free analysis: is GR correct at strong curvature?**

$$S = \underbrace{\kappa \int d^4x\sqrt{-g}R}_{\text{General Relativity}} \underbrace{-\frac{1}{2}\int d^4x\sqrt{-g}\left[g^{ab}\nabla_a\phi\nabla_b\phi + V(\phi)\right]}_{\text{free scalar field}} \underbrace{+\alpha\int d^4x\sqrt{-g}\phi\,{}^*RR}_{\text{Chern-Simons term}}$$

$$\underbrace{+\gamma\int d^4x\sqrt{-g}\,e^{\eta\phi}\left(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}\right)}_{\text{Gauss-Bonnet term}}$$

- **Quadratic curvature corrections (from HEP, string theory, etc..)**
- **Tiny corrections over one cycle, but 10^5 cycles in 1 yr!**

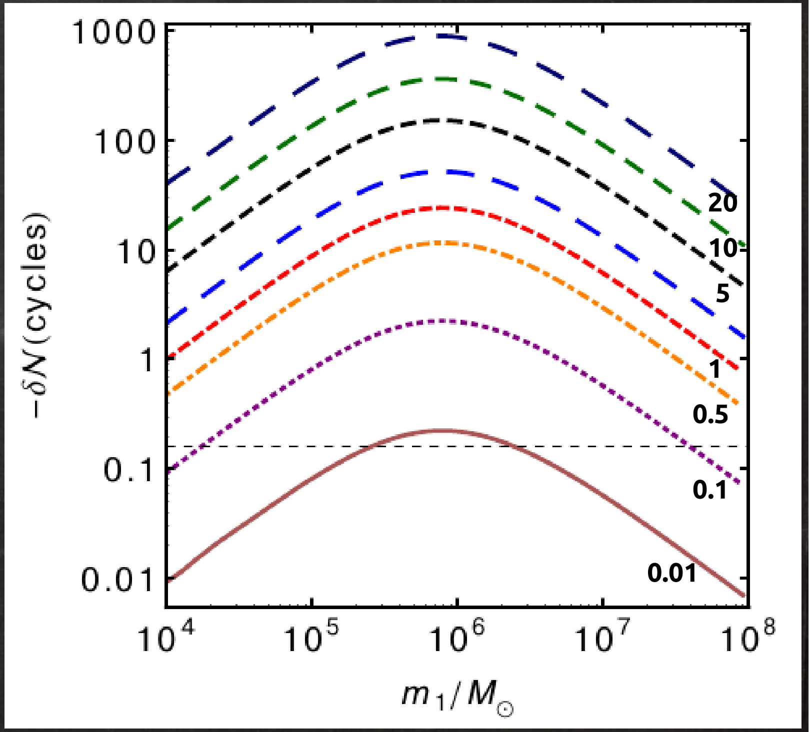
EMRIs: signatures of alternative theories of gravity

- Case study: Chern-Simons gravity [Alexander & Yunes, Sopuerta & Yunes, ...]
  - Parity violation
  - Schwarzschild BHs persist as background solutions
  - Kerr BHs are NOT solutions [Yunes & Pretorius (2009)]
  - GW emission is different! [Cardoso & Gualtieri 2009] [Molina, Pani, Cardoso & Gualtieri 2010]

$$\begin{cases} \dfrac{d^2}{dr_\star^2}\Psi + V_{\rm grav}(r)\Psi = \dfrac{96\pi\, M f}{r^5}\alpha\Theta + S_{\rm grav}(r) \\[2ex] \dfrac{d^2}{dr_\star^2}\Theta + V_{\rm scal}(r)\Theta = f\,\dfrac{(\ell+2)!}{(\ell-2)!}\dfrac{6M\alpha}{r^5}\Psi + S_{\rm scal}(r) \end{cases}$$

- Axial and scalar modes are coupled
- Larger flux at the horizon → faster inspiral
- The horizon contribution is dominant

[Pani, Cardoso, Gualtieri '11]



$$\zeta = \frac{\alpha^2}{M^4}$$

Conclusion: *approximate doesn't mean rough!*

- Physically enlightening approach
- Complementary to exact methods
- Synergy between approximate and fully numerical methods
- Can solve many interesting problems, including:
  - EMRIs (in GR and in alternative theories)
  - Stability issue of BHs and other spacetimes
  - Linear response of BHs and stars

No matter how powerful your computer is,

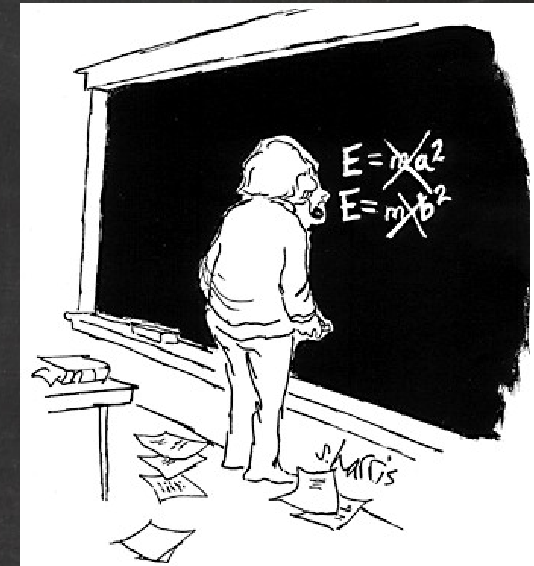
we must live with (and learn from) approximate methods

Keep approximating?

Engineers think that the equations are an approximation to reality.

Physicists think reality is an approximation to the equations.

Mathematicians don't care.



Thanks!

Backup slides

“Nothing is More Necessary than the Unnecessary”

Topic II

Stability analysis

Linear stability analysis

- **Realistic solutions must be (quasi) stable**
  - **BH mimickers**
  - **BHs in higher dimensions** [Dias' talk]
  - **Cosmological models**
- **Instabilities may signal dramatic effects**
  - **Pathologies of the theory/solution**
  - **Phase transitions**
  - **End point?**

Eigeinvalue problem for ω:

Proper modes of vibration:

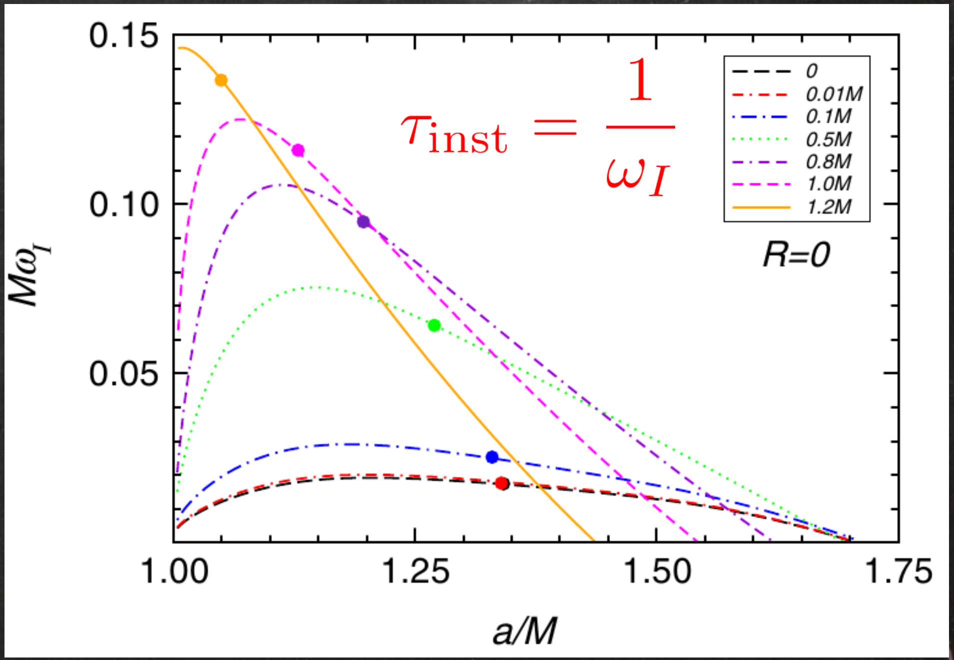
$$\frac{d^2\Psi_{\ell m}(\omega,r)}{dr_*^2}+V(\omega,r)\Psi_{\ell m}(\omega,r)=0$$

$$\text{perturbation}=\int_{-\infty}^{\infty}d\omega\sum_{\ell m}\frac{\Psi_{\ell m}(\omega,r)}{r}Y^{\ell m}(\theta,\varphi)e^{-i\omega t}$$

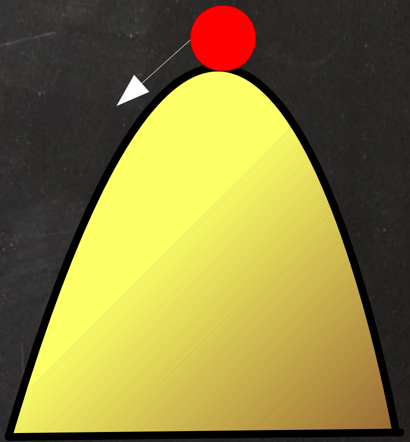
Complex eigenvalue: $\omega=\omega_R+i\omega_I\Longrightarrow\text{perturbation}\propto e^{\omega_I t}$

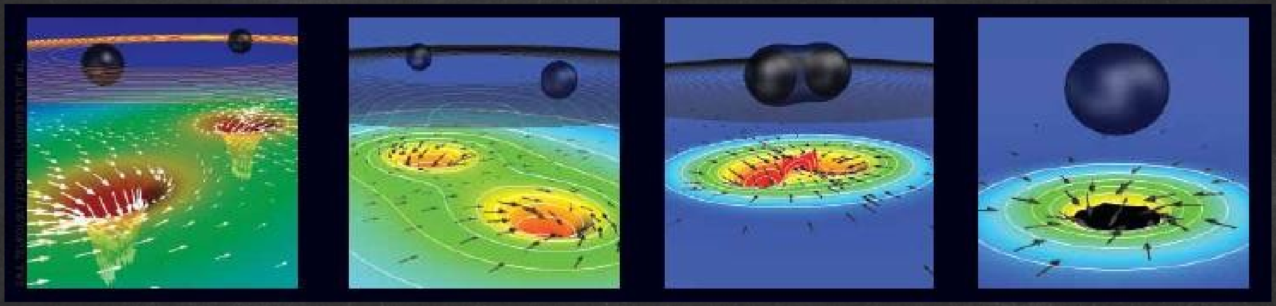
Linear stability. *Black hole mimickers*

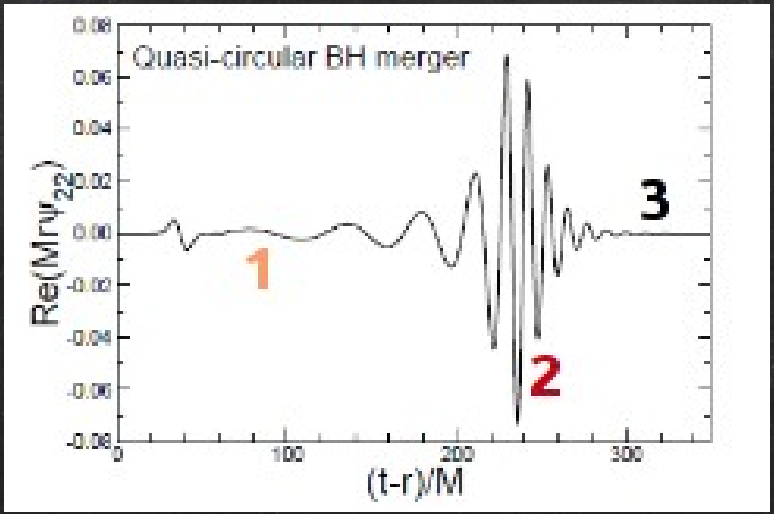
- **Straw-men for astrophysical BHs**
  - Extremely compact
  - Horizonless
  - Boson stars, gravastars, superspinars...
- **Ergoregion instability**
  - Superradiant scattering of waves
  - No horizon $\rightarrow$ instability [Friedman]
- **Thousands of HEP applications:**
  - Black string: Gregory-Laflamme
  - Charged BHs in AdS: Gubser-Mitra
  - Holographic superconductors
  - Ultra-spinning BHs
  - Spontaneous scalarization in stars



[Pani, Barausse, Berti, Cardoso, 2010]







Quasi-circular BH merger
Re($Mr\Psi_{22}$)
0.08
0.06
0.04
0.02
0.00
-0.02
-0.04
-0.06
-0.08
0
100
200
300
(t-r)/M
1
2
3

Topic III

QNMs of Anti de Sitter black holes

QNM spectrum of BHs

- The QNM spectrum can be extremely rich
  - Weakly damped modes
  - Highly damped modes
  - Eikonal limit

$$\tau = 1/|\omega_I|$$

- Complementary (approximate) methods are needed:
  - **Exact solutions** (Poschl-Teller potential, BTZ BHs,...)
  - **WKB approximation** [Mashhoon, Schutz & Will]
  - **Continued fraction method** [Leaver]
  - **Monodromy techniques** (highly damped modes)
  - **AdS BHs: Series solutions** [Horowitz & Hubeny]
  - **AdS BHs: resonance method** [Ferrari XXXX, Berti et al. 2009]

Perturbations of AdS BHs and AdS/CFT

- In the Fourier space, the asymptotic behavior of the perturbations reads

$$\delta\Phi(\omega, r, |\mathbf{k}|) \to \frac{\mathcal{A}(\omega, |\mathbf{k}|)}{r^{\Delta_-}} + \frac{\mathcal{B}(\omega, |\mathbf{k}|)}{r^{\Delta_+}}$$

Δ_+ Δ_- are related to the **conformal dimensions** of the boundary operators

- QNMs are defined by **Dirichlet boundary conditions**: $\mathcal{A}(\omega_{\rm QNM}, |\mathbf{k}|) = 0$

- Retarded Green function $\quad G^R_{\mu\nu,\lambda\sigma}(\omega, , |\mathbf{k}|) = -i \int d^4x e^{ik\cdot x} \theta(t) \langle [T_{\mu\nu}(x), T_{\lambda\sigma}(0)] \rangle_T$

$$G^R(\omega, |\mathbf{k}|) = \frac{\mathcal{B}}{\mathcal{A}} + \text{contact terms}$$

[Son and Starinets (2002)]

BH QNMs are the **poles of retarded correlators** in the holographic QFT

Encode informations about **near-equilibrium behavior** of the QFT

Transport coefficients, Viscosity, conductivities, thermalization timescale, Excitation spectrum

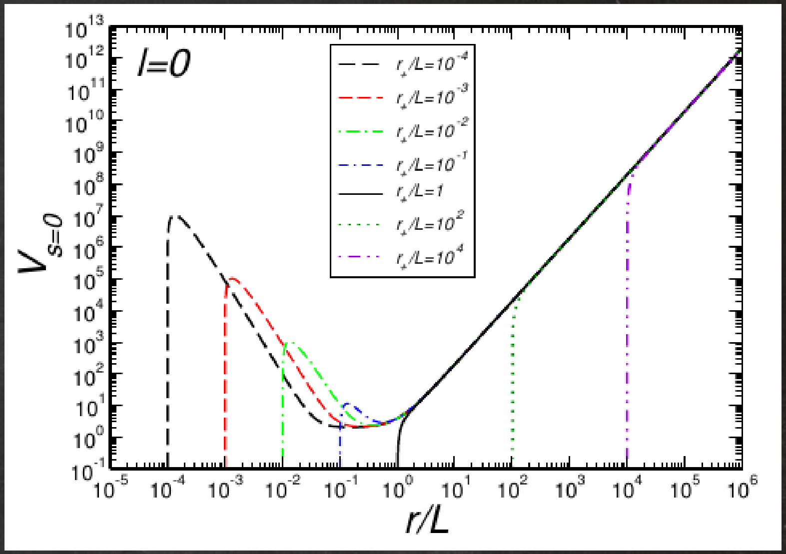
QNM spectrum of AdS-Schwarzschild BHs

- **Series expansion for large BHs**

$$\Psi = (x - x_+)^{\alpha} \sum_{n=0}^{\infty} a_n(\omega)\,(x - x_+)^n$$

Recurrence relation for a_n

$$\Psi(r \to \infty) = 0 \Longrightarrow \sum_{n=0}^{\infty} a_n(\omega)(-x_+)^n = 0$$



Credits: Berti, Cardoso, Pani (2009)

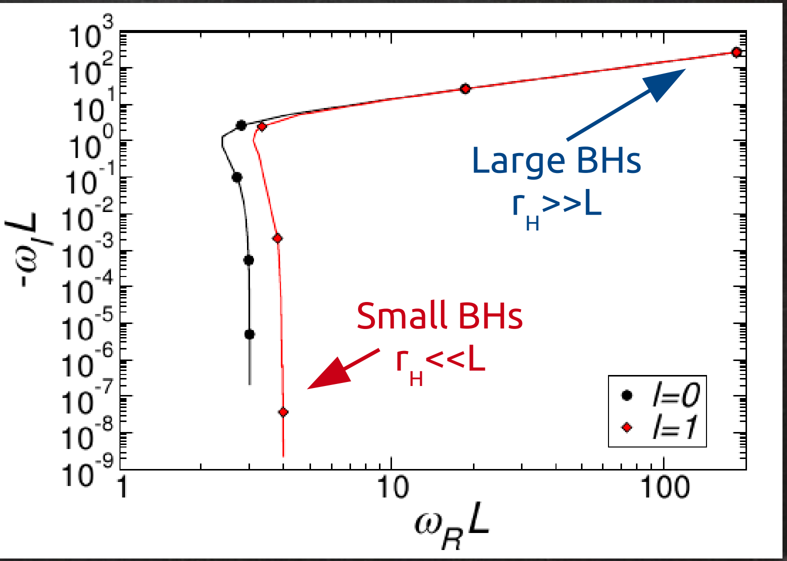
- **Resonance method for small BHs**

$$\omega_I \ll \omega_R$$

Breit-Wegner form

$$\alpha^2 + \beta^2 \approx (\omega - \omega_R)^2 + \omega_I^2$$

for real frequency!!



Boundary conditions at the horizon

$$\frac{d^2}{dr_*^2}\Psi + \left[\omega^2 - V(r)\right]\Psi = 0$$

Intuitively, nothing can come out of the event horizon

Tortoise coordinates:

$$\frac{dr}{dr_*} = f(r)$$

For non-extremal spacetime:

$$r_* = \int dr f^{-1} \sim \frac{\log(r-r_H)}{f'(r_H)} \qquad r \sim r_H$$

Ingoing modes $\rightarrow$

$$\Psi \sim e^{-i\omega(t+r_*)} = e^{-i\omega v} \qquad v = t + r_*$$

Outgoing modes:

$$\Psi \sim e^{-i\omega(t-r_*)} = e^{-i\omega v}e^{2i\omega r_*} \sim e^{-i\omega v}(r-r_H)^{2i\omega/f'(r_H)}$$

$\Rightarrow$ *Outgoing modes cannot be smooth, i.e.* C^∞

Topic 0

BH perturbations

Characteristic modes of vibration

- Seismology
- Spectroscopy

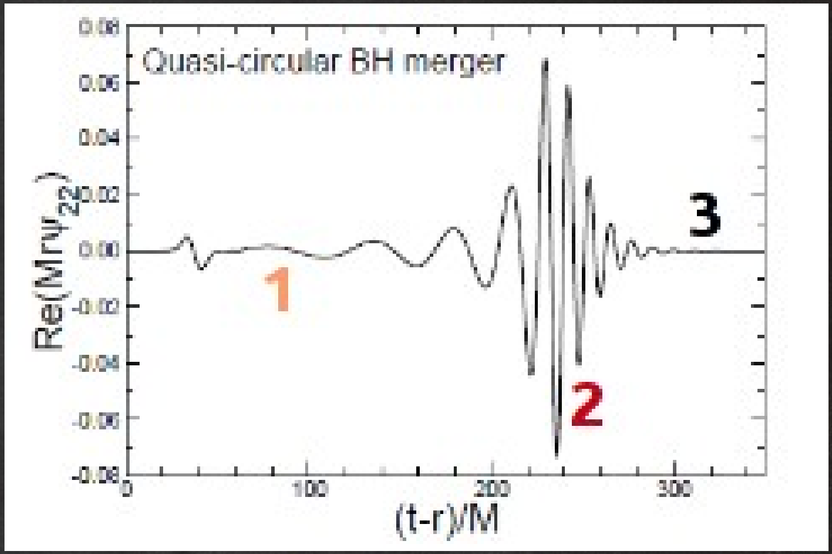
- Atmospheric science
- Civil engineering

Black hole quasinormal modes:

"Hearing" the shape of the spacetime...

The many faces of BH QNMs
"Hearing the shape of the spacetime"

- **Astrophysics**
  - gravitational-wave astronomy
  - no-hair theorems
- **AdS/CFT**
  - Poles of retarded correlators
  - Near-equilibrium properties, quasiparticle spectrum
- **Numerical Relativity**
- **Other developments**



BH perturbations in a nutshell

$$ds^2 = \underbrace{-f(r)dt^2 + h(r)^{-1}dr^2 + r^2 d\Omega_2}_{\text{background}} + \underbrace{(\delta_{\rm RW} g_{\mu\nu})e^{-i\omega t}dx^\mu dx^\nu}_{\text{perturbations}}$$

1) Regge-Wheeler gauge:

Polar ***Axial***

$$\|\delta_{\rm RW} g_{\mu\nu}\| = \begin{bmatrix} f(r)H_0(r)Y_{lm} & H_1(r)Y_{lm} & -h_0(r)\frac{1}{\sin\theta}\frac{\partial Y_{lm}}{\partial\varphi} & h_0(r)\sin\theta\frac{\partial Y_{lm}}{\partial\theta} \\ * & \frac{H_2(r)Y_{lm}}{h(r)} & -h_1(r)\frac{1}{\sin\theta}\frac{\partial Y_{lm}}{\partial\varphi} & h_1(r)\sin\theta\frac{\partial Y_{lm}}{\partial\theta} \\ * & * & r^2K(r)Y_{lm} & 0 \\ * & * & * & r^2\sin^2\theta K(r)Y_{lm} \end{bmatrix}$$

2) Insert into Einstein eqs: 10 linearized coupled eqs

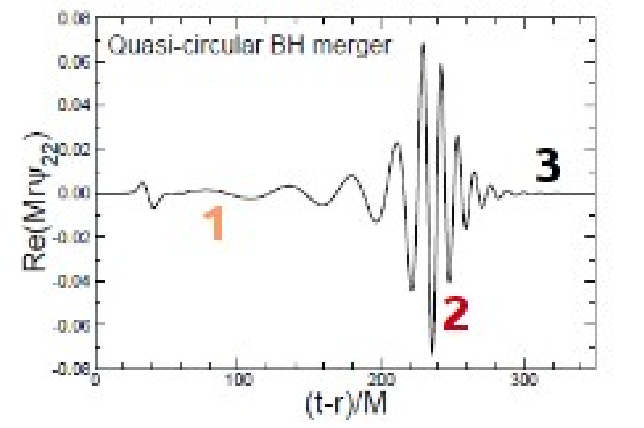
3) Fields redefinition and new “tortoise” coordinates: Schroedinger-like equation:

$$\frac{d^2\Psi}{dr_*^2} + \left[\omega^2 - V(r)\right]\Psi = 0$$

4) Solved with suitable boundary conditions (quasinormal modes, instabilities)

5) Any spherically symmetric background, any theory, any field

Green's function techniques



Inhomogeneous wave equation

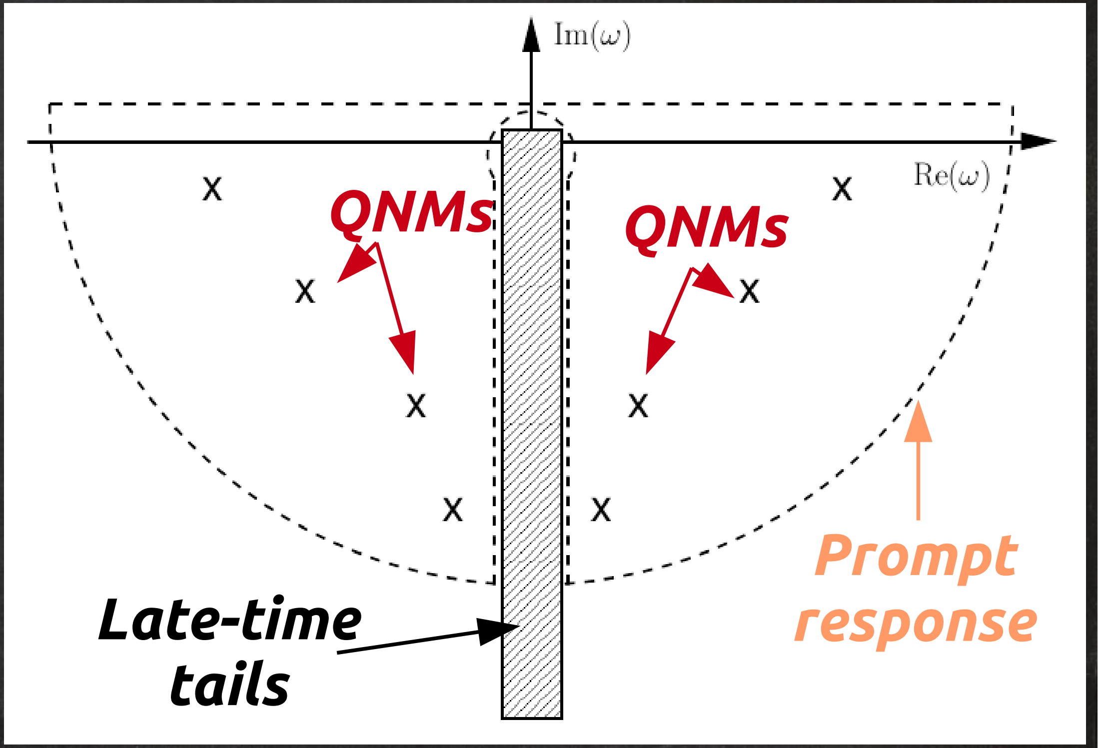
$$\frac{d^2}{dr_*^2}\Psi + \left[\omega^2 - V(r)\right]\Psi = S(\omega, r)$$

Independent solutions of the homogeneous problem:

$$\Psi_H \sim e^{-i\omega r_*}$$

$$\Psi_\infty \sim A_{in}e^{-i\omega r_*} + A_{out}e^{i\omega r_*}$$

$$W = 2i\omega A_{in}$$



General solution:

$$\Psi(\omega, r) = \Psi_\infty \int_{-\infty}^{r_*} \frac{S(\omega, r_*)\Psi_H}{W} + \Psi_H \int_{r_*}^{\infty} \frac{S(\omega, r_*)\Psi_\infty}{W}$$

Computing BH QNMs

$$\omega = \omega_R + i\omega_I \qquad \Psi \sim e^{-i\omega t}$$

$$\Psi \sim A_{out} e^{i\omega r_*} + A_{in} e^{-i\omega r_*} \rightarrow A_{out} e^{-\omega_I r_*} + A_{in} e^{\omega_I r_*}$$

- **Exact solutions** (Poschl-Teller potential, BTZ BHs,...)
- **WKB approximation** [Mashhoon, Schutz & Will]
- **Continued fraction method** [Leaver]
- **Monodromy techniques** (highly damped modes)
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- **AdS BHs: resonance method** [Ferrari XXXX, Berti et al. 2009]

Approximations is intrinsic in Physics

- Mathematics = Nature?
- Point-like particles
- Isolated objects
- T = 0
- spherically symmetric..
- XVII century: epicycles
- QM: Stark and Zeeman effect
- QED: Feynman's diagram
- Friedman cosmology
- Waves...

No matter how powerful your computer is,

we must live with approximations