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SLOWLY ROTATING RELATIVISTIC STARS

I. EQUATIONS OF STRUCTURE

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ABSTRACT

Equations are given for the calculation of the equilibrium configurations of slowly rotating stars in the framework of general relativity. The rotation is treated to second order in the angular velocity, but no other approximation is made.

I. INTRODUCTION

Zee and Wheeler (1967) (see also Thorne and Meltzer 1966; Wheeler 1966) have suggested that the rotation of a neutron star may play an important role in the damping of its radial oscillations. In their picture the deformation of the figure of equilibrium which results from the rotation will couple the radial and quadrupole modes of oscillation permitting the energy stored in the radial mode to be radiated away gravitationally. Zee and Wheeler have estimated that such quadrupole radiation is a very efficient mechanism for damping vibrations. This view has been challenged by Tsuruta and Cameron (1967), who propose several mechanisms which may dissipate the angular momentum of a neutron star before the pulsations have been fully damped.

The rotation of neutron stars has also been proposed (see Wheeler 1966) as a possible source of power in supernova remnants, although the mechanism for the release of this energy has not been spelled out.

The rotation of supermassive stars has been investigated by Fowler and others as a possible mechanism to prevent their gravitational collapse before the onset of nuclear burning (see Thorne [1967] for a review of this subject). A non-rotating supermassive star cannot be gravitationally stable at the onset of nuclear burning if its mass is much greater than $10^6 M_\odot$ (see Fowler 1966). However, calculations of the effects of rotation in the post-Newtonian approximation give a limit of at most about $10^8 M_\odot$ for a star which burns its nuclear fuel before undergoing gravitational collapse. The question naturally arises as to whether these limits involve the post-Newtonian approximation in any crucial way or whether the inclusion of higher-order relativistic corrections could result in a supermassive star of higher mass whose collapse is delayed beyond the time necessary for nuclear burning.

To pursue all these issues in more detail it is, therefore, of interest to determine what *are* the equilibrium configurations of a rotating star in general relativity. The equilibrium configurations of a *cold* star in general relativity have been calculated for cases in which the star is *not* rotating by a number of authors. They obtained their results by numerical integration of the general relativity equation of hydrostatic equilibrium using equations of state for "cold, catalyzed matter." (See Harrison, Thorne, Wakano, and Wheeler [1965] for a review of this work to 1964 and Thorne [1967] for a review of subsequent work.) Models of non-rotating supermassive stars have been calculated by Fowler (1964, 1966) and Tooper (1966) by integrating the equation of hydrostatic equilibrium using a polytropic equation of state of index 3.

To extend these results on neutron and supermassive star models to arbitrarily relativistic stars rotating with arbitrary angular velocity is a problem which, while presenting no difficulties of principle, is numerically complicated. Instead of one radial dimension, one now has two or three dimensions. Instead of two ordinary differential equations to solve, one has the equivalent of an infinite system of ordinary differential equations—one for each coefficient of an expansion of all relevant quantities in spherical harmonics. Approximate solutions of this problem may be obtained from the variational principle suggested by Hartle and Sharp (1967), but an exact numerical solution for arbitrary angular velocities seems formidable.

If, however, the star is rotating slowly, the calculation of its equilibrium properties is much simpler, because then the rotation can be considered as a small perturbation on an already-known non-rotating configuration. It is appropriate to begin a discussion of rotation in massive stars by a treatment of this case. We therefore consider in this paper the following problem:

a) A one-parameter equation of state is specified: (pressure) = (known function of the density of mass-energy).

b) A non-rotating equilibrium configuration is calculated using this equation of state and the general relativity equation of hydrostatic equilibrium for spherical symmetry. The distribution of pressure, energy density, and gravitational field are thereby known.

c) This configuration is given a uniform angular velocity sufficiently slow so that the changes in pressure, energy density, and gravitational field are small.

d) These small changes are considered as perturbations on the known non-rotating solution. The field equations are expanded in powers of the angular velocity and the perturbations calculated by retaining only the first- and second-order terms.

In this paper the equations necessary to solve this problem are obtained. Their numerical solution for particular equations of state and the analysis of the stability of the resulting configurations will be discussed in subsequent papers of this series.

In § II the solution of the above problem in Newtonian gravitational theory is briefly reviewed. In Newtonian theory the presence of a massive body does not affect the determination of an inertial frame. In general relativity, however, a rotating massive body tends to drag the inertial frames along with it (Brill and Cohen 1966). The rate of rotation of the inertial frames inside a massive body (the "dragging rate") is calculated in § IV. The difference between the angular velocity and this dragging rate governs the centrifugal forces acting on the star. In §§ V and VI the perturbation of the field equations and their expansion into spherical harmonics is given. In §§ VII and VIII the equations are obtained which determine the entire structure of a slowly rotating relativistic star. A prescription to find the relation between mass and central density from these equations is given in § VII. In § VIII we find the general relativistic generalization of Clairaut's equation, which determines the ellipticity of the surfaces of constant density and hence the shape of the fluid.

At the outset we wish to discuss some of the assumptions made in this work.

a) *One-parameter equation of state.*—The matter in the equilibrium configuration is assumed to satisfy a one-parameter equation of state, $\mathcal{P} = \mathcal{P}(\mathcal{E})$, where $\mathcal{P}$ is the pressure and $\mathcal{E}$ is the density of total mass-energy. In general situations the pressure is also a function of temperature. This restricted form of the equation of state is appropriate when the temperature is a known function of the density inside the star. For example, this is the case when (1) all the matter is cold at the end point of thermonuclear evolution (see Harrison *et al.* 1965) or (2) when the star is in convective equilibrium so that changes in state are adiabatic (see, e.g., Chandrasekhar 1939).

b) *Axial and reflection symmetry.*—Attention is limited here to configurations which are axially symmetric. A configuration can be in equilibrium in general relativity only if it is not radiating gravitational waves. A sufficient condition for the absence of radiation is the absence of time-dependent moments of the mass distribution (see, e.g., Landau and Lifshitz 1962). This is guaranteed by axial symmetry. We also assume that the configuration is symmetric about a plane perpendicular to the axis of rotation. From one's experience with the Newtonian theory of figures of equilibrium it is plausible that both of these assumptions are really consequences of the slow rotation of the configuration and not restrictions at all.

c) Uniform rotation.—Only uniformly rotating configurations are considered here. It has been shown previously that configurations which minimize the total mass-energy (e.g., all stable configurations) must rotate uniformly (see Hartle and Sharp 1967).

d) Slow rotation.—By slow rotation we mean angular velocities Ω small enough so that the fractional changes in pressure, energy density, and gravitational field due to the rotation are all much less than unity. From simple dimensional consideration this requirement implies

$$\Omega^2 \ll \left(\frac{c}{R}\right)^2 \frac{GM}{Rc^2}, \tag{1}$$

where M is the mass of the unperturbed configuration and R its radius. The expression on the right is the only multiplicative combination of M, R, G, and c which goes over into the Newtonian expression for the critical angular velocity as c becomes large. For the unperturbed configuration the factor GM/Rc^2 is less than unity (Buchdahl 1959; Bondi 1964). Consequently the condition in equation (1) also implies

$$R\Omega \ll c\,. \tag{2}$$

In other words, every particle must move at non-relativistic velocities if the perturbation of the geometry is to be small in terms of percentage.

II. SLOWLY ROTATING STARS IN NEWTONIAN GRAVITATIONAL THEORY

The theory of the equilibrium configurations of slowly rotating self-gravitating bodies has long been known in Newtonian gravitational theory (see, e.g., Jeffreys 1959; Chandrasekhar and Roberts 1963). It is reviewed here briefly as a guide to formulating the same problem in general relativity.

In Newtonian gravitational theory the equilibrium values of pressure, p, density, ρ, and gravitational potential, Φ, of a fluid mass rotating with a uniform angular velocity $\boldsymbol{\Omega}$ are determined by the solution of the three equations of Newtonian hydrostatic equilibrium. These are (1) the Newtonian field equation:

$$\nabla^2\Phi = 4\pi G\rho\,; \tag{3}$$

(2) the equation of state which, following our discussion in § I, we have assumed to have a one-parameter form

$$p = p(\rho)\,; \tag{4}$$

(3) the equation of hydrostatic equilibrium which can be summarized in the case of uniform rotation and a one-parameter equation of state by its first integral

$$\text{constant} = \mu = \int_0^p \frac{dp}{\rho} - \tfrac{1}{2}(\boldsymbol{\Omega}\times\boldsymbol{r})^2 + \Phi\,. \tag{5}$$

The problem posed in § I to find the properties of a configuration of given central density and angular velocity can be phrased here as follows. A solution $\Phi^{(0)}$, $p^{(0)}$, and $\rho^{(0)}$ of the Newtonian equations in the absence of rotation is known. This solution is the leading term in an expansion of the solution including rotation in powers of the angular velocity Ω. It is clear from the symmetry of the configuration under reversal of the direction of rotation that only even powers of the angular velocity will appear in this expansion. The equations of Newtonian hydrostatics are now expanded in powers of Ω^2, and the equations which govern the second-order terms in the solution determined.

Care must be exercised in choosing the coordinate system in which these expansions are carried out. For example, an expansion of the density as a function of the ordinary polar coordinates r, θ is not valid throughout the star. Such an expansion could be valid only if the fractional change in density *at each point in space* were small. This condition cannot be met near the surface of the star as the surface of the configuration will be displaced from its non-rotating position and the perturbation in the density may be finite where the unperturbed density vanishes.

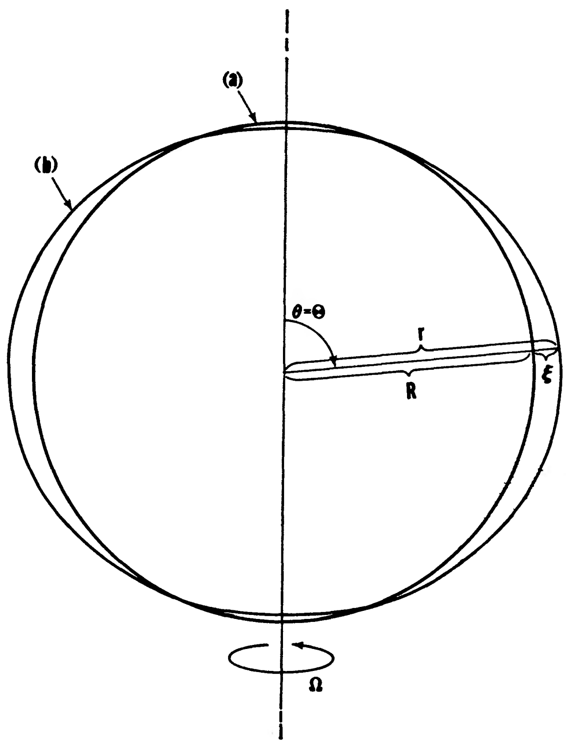

FIG. 1.—Definition of the coordinates R, Θ, and the displacement ξ. The surface (a) is the surface of constant density ρ (R) in the non-rotating configuration. The surface (b) is the surface of constant density $\rho(R)$ in the rotating configuration.

To avoid this difficulty the points of space in the rotating configuration will not be labeled by the usual coordinates r and θ. Instead two coordinates R and Θ defined as follows will be used: Consider a point inside the rotating configuration. This point lies on a certain surface of constant density. Ask for the radius of the surface in the non-rotating configuration which has precisely the same constant density. This radius is defined to be the coordinate R. The coordinate Θ is defined to be identical with the usual polar angle θ. These definitions are given pictorially in Figure 1 and mathematically by the following equations:

$$\Theta = \theta , \qquad \rho[r(R,\Theta),\Theta] = \rho(R) = \rho^{(0)}(R) . \tag{6}$$

The function $r(R,\Theta)$ then replaces the density as a function to be calculated in the rotating configuration. The expansion of $r(R,\Theta)$ in powers of the angular velocity will be written

$$r = R + \xi(R,\Theta) + O(\Omega^4) . \tag{7}$$

The quantity $\xi(R,\Theta)$ is the difference in radial coordinate, r, between a point located by polar angle Θ on the surface of constant density $\rho(R)$ in the rotating configuration and the point located by the same polar angle on the surface of the same constant density in the non-rotating configuration (see Fig. 1).

For small angular velocities, the fractional displacement of the surfaces of constant density due to the rotation is small at the surface and in the middle of the star,

$$\xi(R,\Theta)/R \ll 1 . \tag{8}$$

It will also be small at the center of the star if the rotating configuration is chosen to have the same central density as the non-rotating configuration so that ξ vanishes at $R = 0$. We are always free to consider the rotating configuration as a perturbation on a non-rotating configuration of the same central density; so that equation (8) can be satisfied throughout the star.

In the R,Θ coordinate system the two functions which characterize the rotating star are $r(R,\Theta)$ and the gravitational potential $\Phi(R,\Theta)$. The density and pressure are known functions of R related by the equation of state

$$\rho(R) = \rho^{(0)}(R), \qquad p(R) = p^{(0)}(R) . \tag{9}$$

The expansion of r to terms in Ω^2 is given by equation (7) and the expansion of Φ is denoted by

$$\Phi(R,\Theta) = \Phi^{(0)}(R) + \Phi^{(2)}(R,\Theta) + O(\Omega^4) . \tag{10}$$

These expansions are to be inserted in equations (3) and (5) written in the coordinates R,Θ with only terms of order Ω^2 retained. The calculation of ξ and $\Phi^{(2)}$ from the resulting equations is greatly simplified if these functions are first expanded in spherical harmonics since only a few terms in this series will remain in the final result. The reflection symmetry of the configuration implies that only spherical harmonics of even order will appear in this expansion if the polar axis is taken to be the axis of rotation.

$$\xi(R,\Theta) = \Sigma_l \xi_l(R) P_l(\Theta) , \tag{11}$$

$$\Phi^{(2)}(R,\Theta) = \Sigma_l \Phi_l^{(2)}(R) P_l(\Theta) . \tag{12}$$

These expansions are to be substituted into the three equations of Newtonian hydrostatic equilibrium (eqs. [3]–[5]) and the equations governing $\xi_l(R)$ and $\Phi_l^{(2)}(R)$ derived.

When the expansions contained in equations (7), (10), (11), and (12) are substituted into the integral of the equation of hydrostatic equilibrium (eq. [5]), only those equations corresponding to the l values 0 and 2 are found to contain the angular velocity, Ω, in any way. This is because the centrifugal potential term in equation (5) has the angular dependence $\sin^2 \Theta$. The Newtonian field equation when expanded in this way couples together only quantities with the same value of l. The equations for $\xi_l(R)$, $\Phi_l^{(2)}(R)$, with $l \geq 4$ are thus independent of Ω and their solution is

$$\xi_l = 0, \qquad \Phi_l^{(2)} = 0 , \qquad l \geq 4 . \tag{13}$$

There remain only the quantities with $l = 0$ and $l = 2$ to be determined. This reduction in the number of l values from infinity to 2 is the central simplification of the slow rotation approximation. In place of a system of partial differential equations one now only has ordinary differential equations for the four unknown functions $\Phi_0(R)$, $\Phi_2(R)$, $\xi_0(R)$, and $\xi_2(R)$.

Two problems of major interest are to determine (1) the relation between mass and central density for a rotating star, and (2) the shape of the star. The differential equations for $\Phi_0(R)$, $\Phi_2(R)$, $\xi_0(R)$, and $\xi_2(R)$, which completely determine the equilibrium configuration, will now be given in forms suitable for solving these problems.

a) Relation between Mass and Central Density

The relation between mass and central density may be determined from the $l = 0$ equations alone. The mass can be found from the term in Φ which is proportional to $1/r$ at large distances. All components except $l = 0$ vanish more strongly than this. Similarly near the origin all components of the density except $l = 0$ vanish, so only the $l = 0$ component contributes to the central density. It is convenient to display these $l = 0$ equations in a form in which they resemble the equation of hydrostatic equilibrium. To do this we make the definitions

$$p^*(R) = \frac{GM(R)}{R^2}\,\xi_0(R), \qquad M(R) = \int_0^R 4\pi R^2 \rho(R)\, dR. \tag{14}$$

The $l = 0$ equations of structure are then

$$M^{[2]} = \int_0^R dR 4\pi R^2 \frac{d\rho}{dp}\,\rho p^*, \qquad -\frac{dp^*}{dR} + \tfrac{2}{3}\Omega^2 R = \frac{GM^{[2]}(R)}{R^2}. \tag{15}$$

These equations show the balance between the pressure, centrifugal, and gravitational forces per unit mass in the rotating star. The correspondence with an equation of hydrostatic equilibrium may be made more explicit by considering a region of the star where an Eulerian expansion of the pressure and density are legitimate:

$$\rho(r,\theta) = \rho^{[0]}(r) + \rho^{[2]}(r,\theta) + O(\Omega^4), \qquad p(r,\theta) = p^{[0]}(r) + p^{[2]}(r,\theta) + O(\Omega^4). \tag{16}$$

The quantities $p*$ and $M^{[2]}$ may then be written

$$p* = p_0{}^{[2]}/\rho^{[0]}, \qquad M^{[2]} = \int_0^r dr 4\pi r^2 \rho_0{}^{[2]}, \tag{17}$$

where a subscript zero denotes an $l = 0$ component. It is irrelevant here whether the quantities are written as function of r, θ or R, Θ since they are already second order in Ω. Equations (15) are then seen to be the equations of hydrostatic equilibrium which govern the Eulerian changes in pressure, density, and mass. Near the surface where the Eulerian expansion is no longer valid the equations remain formally the same, but the identification of $p*$ with the ratio of the change in pressure to the unperturbed density can no longer be made. This circumstance in no way impairs the usefulness of the equations for $p*$ and $M^{[2]}$ as employed in the rest of this section for calculating ξ_0.

The total mass of the rotating configuration is given by the integral of the density over the volume. Writing this out in the coordinates R,Θ, expanding to order Ω^2, and performing an integration by parts, one finds for the change in mass δM of the rotating from the non-rotating configuration,

$$\delta M = 4\pi \int_0^a \left(-\xi_0(R)\,\frac{d\rho}{dR}\right) R^2 dR = M^{[2]}(a). \tag{18}$$

To calculate the relation between mass and central density for the rotating star one now proceeds as follows: (1) Pick a value of the central density. Calculate the non-rotating configuration with this central density. (2) Integrate equations (15) outward from the origin starting with the boundary condition which guarantees that the central density of the rotating configuration will have the same value.

$$p^* \to \tfrac{1}{3}\Omega^2 R^2, \qquad R \to 0. \tag{19}$$

(3) The value of $M^{[2]}$ at the radius of the unperturbed star gives the change in mass of the rotating star over its non-rotating value for the same central density.

b) *The Shape of the Star*

The calculation of the shape of the rotating star involves the $l = 2$ equations as well as those with $l = 0$. If the surface of the non-rotating star has radius a, equations (7) and (11) show that the equation for the surface of the rotating star has the form

$$r = a + \xi_0(a) + \xi_2(a)P_2(\theta) . \tag{20}$$

The value of $\xi_0(a)$ is already determined in the $l = 0$ calculation

$$\xi_0(a) = \frac{a^2}{GM}\,p^*(a) , \tag{21}$$

where M is the mass of the non-rotating configuration. The quantity $-3\xi_2(R)/2R$ is the ellipticity of the surface of constant density labeled by R. This will be denoted by $\epsilon(R)$. It may be calculated by the $l = 2$ equations obtained by substituting equations (7), (10), (11), and (12) in equations (3) and (5). These are equivalent to Clairaut's equation:

$$\frac{d}{dR}\frac{1}{R^4}\frac{d}{dR}[\epsilon(R)M(R)R^2] = 4\pi\epsilon(R)\frac{d\rho}{dR}. \tag{22}$$

Here both M and ρ are known functions of R. The ellipticity must be regular at small R, and equation (22) shows that it approaches a constant at $R = 0$. With this boundary condition equation (22) may be integrated to find the shape of $\epsilon(R)$ but not its magnitude. To find the magnitude of $\epsilon(r)$ the as-yet-unused boundary condition $\Phi \to 0$ as $r \to \infty$ must be incorporated. The interior solution of the field equation for Φ must be matched on to that exterior solution which vanished at large r. If we denote by $\Phi_2^{[2]}$ the Eulerian change in the $l = 2$ component of Φ, that exterior solution will be

$$\Phi_2^{[2]} = \text{const.}/r^3 , \qquad r > a . \tag{23}$$

This boundary condition at the surface, together with the condition of regularity at the origin and the differential equation (22) uniquely determine the ellipticity of the surfaces of constant density as a function of the coordinate R. Equivalently it determines the ellipticity as a function of the coordinate r since $\epsilon(r)$ and $\epsilon(R)$ are identical in the limit of slow rotation considered here. To calculate the parameters in the equation (20) for the surface of the rotating star one proceeds as follows.

1. Solve the $l = 0$ equations for $p^*(r)$ and calculate $\xi_0(a)$ through equation (21).
2. Solve Clairaut's equation with the boundary conditions discussed above to find the remaining parameter $\xi_2(a) = -\frac{2}{3}a\epsilon(a)$.

III. NON-ROTATING RELATIVISTIC STELLAR MODELS

The equilibrium configurations of non-rotating fluid masses in general relativity have been investigated by many authors beginning with K. Schwarzschild in 1916. Some of their results are briefly summarized here (for details see Landau and Lifshitz [1962] or Harrison *et al.* [1965]). This summary contains those properties of the non-rotating configuration which we will need in calculating how the configuration is changed by a small rotation.

The stress-energy tensor of a perfect fluid will be written as[1]

$$T_\mu{}^\nu = (\mathcal{E} + \mathcal{P})u^\nu u_\mu + \mathcal{P}\delta_\mu{}^\nu . \tag{24}$$

[1] With slight changes in the labeling and $c = G = 1$ we follow the notational conventions of Landau and Lifshitz (1962) in general relativity, except that Greek tensor indices range over space and time variables while Latin indices run over the space variables.

Here, u^μ is the 4-velocity of the fluid; $\mathcal{P}$ is the total pressure and $\mathcal{E}$ is the total density of mass-energy. The pressure and energy within a *non-rotating* equilibrium configuration will be denoted by P and E, respectively.

A non-rotating equilibrium configuration is spherically symmetric. The metric which describes its gravitational field has the form originally written down by Schwarzschild

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) . \tag{25}$$

Here, ν and λ are functions of the radius variable, r, alone, and we shall often write for $e^{\lambda(r)}$,

$$e^{-\lambda(r)} = 1 - \frac{2M(r)}{r} . \tag{26}$$

The results of Bondi (1964) show that $M(r)/r \leq 0.485$ for any physically realizable equilibrium configuration so that e^λ is never singular. If the surface of the fluid sphere is at $r = a$, then outside of this radius the functions ν and λ are given by

$$e^\nu = e^{-\lambda} = 1 - \frac{2M}{r} , \tag{27}$$

where the constant M is the total mass of the fluid in units of length. Inside the fluid, the distributions of $\nu(r)$, $\lambda(r)$, $E(r)$, and $P(r)$ are determined by the four equations of general-relativistic hydrostatics. For a non-rotating configuration these may be taken to be: (1) the equation of hydrostatic equilibrium

$$-\frac{dP}{dr} = \frac{(E+P)(M+4\pi r^3 P)}{r(r-2M)} ; \tag{28}$$

(2) the equation of state in the form discussed in the introduction; and (3) and (4) two field equations to determine $M(r)$ and $\nu(r)$:

$$\frac{dM(r)}{dr} = 4\pi r^2 E, \qquad \frac{d\nu}{dr} = -\frac{2}{E+P}\frac{dP}{dr} . \tag{29}$$

These equations completely determine the equilibrium configuration once, say, the central density has been given. We now turn to the investigation of the effects of a slow rotation on these configurations.

IV. ROTATIONAL DRAGGING OF INERTIAL FRAMES

In general relativity as in Newtonian gravitational theory the magnitude of the centrifugal forces acting on a fluid element is governed by the rate of rotation of the fluid element relative to a local inertial frame. In contrast to Newtonian theory, however, the inertial frames inside a general-relativistic fluid are not at rest with respect to the distant stars. Rather, the local inertial frames are dragged along by the rotating fluid. This general-relativistic effect was first analyzed by Thirring (1918) and recently in more detail by Brill and Cohen (1966), who clarify its connection with Mach's principle. The calculation of the rate of rotation is then essential for the determination of the equilibrium between gravitational, pressure, and centrifugal forces. It is considered in this section.

How is the dragging of inertial frames manifested in the metric which describes the gravitational field of the slowly rotating equilibrium configuration? The metric of a stationary, axially symmetric system may be written in the form (see, e.g., Hartle and Sharp 1967, § V)

$$ds^2 = -H^2dt^2 + Q^2dr^2 + r^2K^2[d\theta^2 + \sin^2\theta(d\varphi - Ldt)^2] , \tag{30}$$

where H, Q, K, and L are functions of r and θ alone. The quantity $L(r,\theta)$ is the angular velocity $(d\varphi/dt)$ acquired by an observer who falls freely from infinity to the point (r,θ). We will, therefore, call $L(r,\theta)$ the rate of rotation of the inertial frame at (r,θ) relative to the distant stars.[2] The dragging of the inertial frames thus appears in the metric as the non-vanishing of the $g_{t\varphi}$ metric component for a rotating configuration.

The density and metric of a stationary, axially symmetric system will behave in the same way under a reversal in the direction of rotation as under a reversal in the direction of time. An expansion of the density or of H, Q, and K in powers of the angular velocity Ω can, therefore, contain only even powers; while an expansion of L will have only odd ones. If one is interested in calculating all effects up to order Ω^2 it is then sufficient to include only terms of order Ω in L. To first order in the angular velocity only the coefficient L changes from its value zero at $\Omega = 0$. This first-order term is denoted by $\omega(r,\theta)$,

$$L(r,\theta) = \omega(r,\theta) + O(\Omega^3) . \tag{31}$$

The calculation of this first-order term then requires only one field equation which is conveniently taken to be

$$R_\varphi{}^t = 8\pi T_\varphi{}^t . \tag{32}$$

To find the first-order rate of rotation of the inertial frames, ω, one expands both sides of equation (32) and retains only the lowest order term in the angular velocity, Ω. The left-hand side of this equation may be expressed by the identity[3]

$$(-g)^{1/2} R_\varphi{}^t = [(-g)^{1/2} g^{t\alpha} \Gamma_{\varphi\alpha}{}^\nu]_{,\nu} . \tag{33}$$

With the metric of equation (30) this can be written

$$(-g)^{1/2} R_\varphi{}^t = -\tfrac{1}{2}[(-g)^{1/2} g^{ij}(g^{t\varphi} g_{\varphi\varphi,j} - g^{tt} g_{t\varphi,j})]_{,i} . \tag{34}$$

The coefficients $g^{t\varphi}$ and $g_{t\varphi}$ are at least first order in the angular velocity so that the other terms may be replaced by their zero-order values

$$-2r^2 \sin\theta\, e^{(\nu+\lambda)/2} R_\varphi{}^t = [e^{-(\nu+\lambda)/2} r^4 \sin^3\theta\, \omega_r]_r + [e^{(\lambda-\nu)/2} r^2 \sin^3\theta\, \omega_\theta]_\theta + O(\Omega^3) , \tag{35}$$

where we have introduced the convention that df/dx^α may be written f_α.

If Ω is the angular velocity of the fluid, the 4-velocity which satisfies the normalization condition $u^\mu u_\mu = -1$ is

$$u^r = u^\theta = 0, \qquad u^\varphi = \Omega u^t, \qquad u^t = [-(g_{tt} + 2\Omega g_{t\varphi} + \Omega^2 g_{\varphi\varphi})]^{1/2} . \tag{36}$$

For the uniformly rotating configuration considered here Ω is a constant throughout the fluid. With this 4-velocity the component of the stress-energy tensor on the left of equation (32) may be expanded as

$$\begin{aligned} T_\varphi{}^t &= (\mathcal{E} + \mathcal{P}) u^t u_\varphi = (\mathcal{E} + \mathcal{P})(u^t)^2 (g_{t\varphi} + \Omega g_{\varphi\varphi}) \\ &= (E + P) e^{-\nu} (\Omega - \omega) r^2 \sin^2\theta + O(\Omega^3) . \end{aligned} \tag{37}$$

Now, Ω is the angular velocity of the fluid as seen by an observer at rest at some point (t,r,θ,φ) in the fluid. The quantity $\omega(r,\theta)$ is the angular velocity acquired by an observer falling freely from infinity calculated to first order in Ω. Their difference, $\Omega - \omega$, is, there-

[2] This definition of the rate of rotation of inertial frames is to be distinguished from that given in terms of the Coriolis force (see, e g., Landau and Lifshitz 1962, p. 295). Appreciation is expressed to Kip S Thorne for a discussion of this point.

[3] For the proof of an analogous identity see Landau and Lifshitz (1962), p 348.

fore, the coordinate angular velocity of the fluid element at (r,θ) seen by the freely falling observer to this order. This is the quantity of interest and will be denoted by $\bar{\omega}(r,\theta)$:

$$\bar{\omega}(r,\theta) = \Omega - \omega(r,\theta) . \tag{38}$$

Retaining only first-order terms in Ω the field equation (eq. [32]) is now

$$\frac{1}{r^4}\frac{\partial}{\partial r}\left[r^4 e^{-(\nu+\lambda)/2}\frac{\partial\bar{\omega}}{\partial r}\right]+\frac{e^{(\lambda-\nu)/2}}{r^2\sin^3\theta}\frac{\partial}{\partial\theta}\left(\sin^3\theta\frac{\partial\bar{\omega}}{\partial\theta}\right)-16\pi(E+P)e^{(\lambda-\nu)/2}\bar{\omega}=0 . \tag{39}$$

Use may be made of the zero-order field equations (28)–(29) and the definition

$$j(r) = \exp\left[-(\nu+\lambda)/2\right] \tag{40}$$

to express the coefficients of $\bar{\omega}$ in equation (39) completely in terms of the unperturbed metric

$$\frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4 j\frac{\partial\bar{\omega}}{\partial r}\right)+\frac{4}{r}\frac{dj}{dr}\bar{\omega}+\frac{e^{(\lambda-\nu)/2}}{r^2}\frac{1}{\sin^3\theta}\frac{\partial}{\partial\theta}\left(\sin^3\theta\frac{\partial\bar{\omega}}{\partial\theta}\right)=0 . \tag{41}$$

An expansion of $\bar{\omega}(r,\theta)$ in Legendre polynomials will not separate equation (41) because $g_{t\varphi}$, and hence ω, transforms under rotations not like a scalar but like a component of a vector. To solve equation (41) by separation of variables, an expansion in vector spherical harmonics must be used. The relevant angular functions may be found directly or from the group-theoretic arguments of Regge and Wheeler (1957):

$$\bar{\omega}(r,\theta)=\sum_{l=1}^{\infty}\bar{\omega}_l(r)\left(-\frac{1}{\sin\theta}\frac{dP_l}{d\theta}\right). \tag{42}$$

The radial functions $\bar{\omega}_l(r)$ then satisfy the equation

$$\frac{1}{r^4}\frac{d}{dr}\left[r^4 j(r)\frac{d\bar{\omega}_l}{dr}\right]+\left[\frac{4}{r}\frac{dj}{dr}-e^{(\lambda-\nu)/2}\frac{l(l+1)-2}{r^2}\right]\bar{\omega}_l=0 . \tag{43}$$

We now examine the behavior of the solution to equation (43) at small r where we demand that the geometry be regular and at large r where we demand it be flat. At small r, $j(r)$ is a regular function so that the differential equation admits a small r behavior of the form

$$\bar{\omega}_l(r)\to \text{const. } r^{S+} + \text{const. } r^{S-} , \qquad r\to 0 ,$$

$$S_{\pm} = -\tfrac{3}{2}\pm\left[\tfrac{9}{4}+\frac{l(l+1)-2}{j(0)}\right]^{1/2} . \tag{44}$$

The function $j(r)$ is positive everywhere since e^{ν} and e^{λ} are both positive. If the geometry is to be regular at the origin we must demand that the coefficient of r^{S-} vanish.

At large r, $j(r)$ becomes unity and $\bar{\omega}_l$ has the form

$$\bar{\omega}_l(r)\to \text{const. } r^{-l-2} + \text{const. } r^{l-1} . \tag{45}$$

The behavior has already been fixed at the origin. Consequently the ratio of the two constants in equation (45) is determined and neither will vanish unless they both do. If space is to be flat at large r, ω must decrease faster than $1/r^3$, so that $\bar{\omega}(r) = \Omega - \omega$ approaches Ω. From equation (45) it is then clear that all coefficients in the Legendre expansion of $\bar{\omega}$ vanish except $l = 1$. Consequently *$\bar{\omega}$ is a function of r alone. It obeys the differential equation*

$$\frac{1}{r^4}\frac{d}{dr}\left(r^4 j \frac{d\bar{\omega}}{dr}\right)+\frac{4}{r}\frac{dj}{dr}\bar{\omega}=0\,. \tag{46}$$

This equation is to be integrated outward starting with $\bar{\omega}(0) = \text{const}$. Outside the star $j(r) = 1$ and the solution has the form

$$\bar{\omega}(r)=\Omega-\frac{2J}{r^3}\,, \qquad r>a\,. \tag{47}$$

The constant J can be identified with the total angular momentum of the star (see Papapetrou 1948).

One consequence of the linearity of equation (46) for $\bar{\omega}$ is that the angular momentum is linearly related to the angular velocity for slow rotation:

$$J = I(M,\nu)\Omega\,. \tag{48}$$

The constant of proportionality $I(M,\nu)$ defines the relativistic generalization for slowly rotating systems of the usual Newtonian concept of moment of inertia. As defined here, the moment of inertia arises not only from the particulate content of the star, but also from the mass associated with the energy required to compress the matter to its given density and the effective energy of the long-range gravitational interaction of different parts of the star with one another.

A single integration of equation (46) is enough to calculate I. The moment of inertia, as defined here, is a functional of the metric of the non-rotating star and of this metric alone. It is remarkable that a quantity so complex in its origin can be calculated so simply.

It is instructive to see how the Newtonian expression for the moment of inertia arises from the definitions in equations (47) and (48). Multiplying the differential equation for $\bar{\omega}$ (eq. [46]) by r^3 and integrating out to the radius of the star the moment of inertia may be expressed (reinserting the correct factors of G and c) as

$$\frac{GI}{c^2}=-\tfrac{2}{3}\int_0^a dr\; r^3\left(\frac{dj}{dr}\right)\left(\frac{\bar{\omega}}{\Omega}\right). \tag{49}$$

The Newtonian limit of this expression is calculated by expanding the right-hand side in powers of $(1/c)$ and retaining only the lowest-order term. In that limit the non-rotating configuration may be described by the line element

$$ds^2 = -[1+2\Phi(r')/c^2]dt^2 + [1-2\Phi(r')/c^2]\,[dr'^2 + r'^2(d\theta^2+\sin^2\theta d\varphi^2)]\,, \tag{50}$$

where Φ is the Newtonian potential (see, e.g., Landau and Lifshitz 1962). This line element is not of the form given in equation (30), but may be brought to that form by the coordinate transformation

$$r' = r(1+\Phi/c^2)\,. \tag{51}$$

One finds for the line element to order $1/c^2$:

$$ds^2=-\left(1+\frac{2\Phi}{c^2}\right)dt^2+\left(1-\frac{2r}{c^2}\frac{d\Phi}{dr}\right)dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2)\,. \tag{52}$$

The function $j(r)$ may then be expanded as

$$j(r)=1-\frac{1}{c^2}\left(\Phi+r\frac{d\Phi}{dr}\right)+O\left(\frac{1}{c^4}\right). \tag{53}$$

To lowest order in $1/c$ the angular velocity of the fluid relative to a local inertial frame $\bar{\omega}$ is just Ω. The Newtonian limit of equation (49) is, therefore,

$$I_{\text{Newtonian}} = \frac{2}{3G}\int_0^a dr\; r^4 \left(\frac{d^2\Phi}{dr^2} + \frac{2}{r}\frac{d\Phi}{dr}\right). \tag{54}$$

Use of the Newtonian field equation (eq. [3]) then yields

$$I_{\text{Newtonian}} = \frac{8\pi}{3}\int_0^a dr\; r^4 \rho(r), \tag{55}$$

in agreement with the classical result.

Some of the general properties of $\bar{\omega}(r)$ may be obtained from the following integral equation which is equivalent to equation (46) and its boundary condition

$$\bar{\omega}(r) = \bar{\omega}(0) + \int_0^r G(r,r')\bar{\omega}(r')dr'. \tag{56}$$

Here,

$$G(r,r') = r'^3 \frac{dj}{dr'}[q(r) - q(r')], \tag{57}$$

with

$$q(r) = \int_r^\infty dr'/r'^4 j(r). \tag{58}$$

The integral equation (eq. [56]), is of Volterra type so that its solution may always be found by iteration if $G(r,r')$ is bounded[4] (see, e.g., Smithies 1958). We write it as

$$\bar{\omega}(r) = \bar{\omega}(0) + \sum_{n=1}^{\infty} \bar{\omega}_n(r), \tag{59}$$

where

$$\bar{\omega}_0(r) = \bar{\omega}(0), \qquad \bar{\omega}_n(r) = \int_0^r dr' G(r,r')\bar{\omega}_{n-1}(r'). \tag{60}$$

We can now show that the kernel $G(r,r')$ is positive for $r > r'$. To see this we note first that, since the unperturbed configuration is assumed to have $e^\nu > 0$, $e^\lambda > 0$, we have

$$j(r) = \exp\left[-\left(\frac{\nu+\lambda}{2}\right)\right] > 0. \tag{61}$$

Further from equations (27)–(29) we may write

$$-dj/dr = 4\pi r e^{-\nu}(E+P)/j > 0. \tag{62}$$

Finally as a consequence of equation (61) we have

$$q(r) - q(r') = -\int_{r'}^r dx/x^4 j(x) < 0, \qquad r' < r. \tag{63}$$

Equations (62) and (63) lead directly to the conclusion that $G(r,r') > 0$ for $r' < r$. This property of the kernel is reflected directly in the solution through equations (60).

$$\bar{\omega}(r) = \bar{\omega}(0)[1 + (\text{sum of positive terms})]. \tag{64}$$

[4] We exclude thereby the case of where all of the matter is concentrated in an infinitely thin spherical shell already treated by Brill and Cohen (1966).

If $\bar{\omega}(0) > 0$ then $\bar{\omega}(r) > 0$; if $\bar{\omega}(0) < 0$ then $\bar{\omega}(r) < 0$; and if $\bar{\omega}(0) = 0$ then $\bar{\omega}(r) = 0$. At large r, $\bar{\omega}(r) \rightarrow \Omega$ so that the sign of $\bar{\omega}(r)$ is the same as the sign of the angular velocity. We then have the following conclusions:

1. The local rate of rotation, $\bar{\omega}(r)$, is in the direction of the angular velocity and never vanishes. thus *at no point does the local rate of rotation of an inertial frame equal or exceed the angular velocity of rotation.*
2. The rate of change of $\bar{\omega}(r)$ is given by

$$\frac{d\bar{\omega}}{dr} = -4\int_0^r r'^3 dr' \frac{dj}{dr'} \frac{1}{r^4 j} \bar{\omega}(r') \tag{65}$$

and always has the same sign as Ω. Therefore, $|\bar{\omega}(r)|$ is an increasing function of r. Correspondingly, $|\omega(r)|$ *is a decreasing function so that the largest rate of dragging of inertial frames always occurs at the heart of the fluid.*

V. ROTATIONAL PERTURBATIONS IN GENERAL RELATIVITY AND THEIR EXPANSION IN SPHERICAL HARMONICS

In the next four sections we seek the general-relativistic equations needed to calculate the shape of a slowly rotating relativistic star, the relation between its mass and central density, and its internal structure. These equations are the generalizations in Einstein's theory of the equations of Newtonian theory discussed in § II.

It has already been argued in § IV that the lowest-order contributions to the changes in density and pressure brought about by the rotation are second order in the angular velocity. They are determined by solving the Einstein equations to this order. As input information for this computation not only the parameters characterizing the non-rotating configuration are needed but also those which characterize the *first*-order changes in the metric. How to obtain this additional input information—rate of rotational dragging of the inertial frame inside the fluid—has already been spelled out in § IV. Consequently it is assumed in the following that $\bar{\omega}(r)$ is known as well as the functions which describe the non-rotating configuration.

The metric for the rotating configuration is given in equation (30). The second-order terms we will compute are denoted by $h(r,\theta)$, $m(r,\theta)$, and $k(r,\theta)$. They are defined by the following expansion:

$$ds^2 = -e^{\nu}(1+2h)dt^2 + e^{\lambda}[1+2m/(r-2M)]dr^2 + r^2(1+2k)[d\theta^2 + \sin^2\theta(d\varphi - \omega dt)^2] + O(\Omega^3)\,. \tag{66}$$

If an expansion of the metric in spherical harmonics is also made, it takes the following form because H, K, and Q all transform like scalars under rotations (see Regge and Wheeler 1957)

$$h(r,\theta) = h_0(r) + h_2(r)P_2(\theta) + \ldots\,, \tag{67}$$

$$m(r,\theta) = m_0(r) + m_2(r)P_2(\theta) + \ldots\,, \tag{68}$$

$$k(r,\theta) = k_0(r) + k_2(r)P_2(\theta) + \ldots\,. \tag{69}$$

One further and convenient simplification of the metric may be made here. Transformations of the type $r \rightarrow f(r)$ do not change the form of the metric given in equation (30). Such a coordinate transformation may, therefore, be used to guarantee the additional condition

$$k_0(r) = 0\,. \tag{70}$$

This will be assumed in the following.

An expansion of the metric as a function of r and θ in powers of Ω^2 is legitimate because the fractional change in any metric coefficient caused by the rotation will be much less than unity throughout space. In contrast, this would not be true for an expansion of the pressure or density as a function of r and θ for the reasons given in the discussion of the analogous expansion in the Newtonian case. As in the Newtonian theory we, therefore, introduce new variables R, Θ defined by

$$\Theta = \theta, \qquad \mathcal{E}[r(R,\Theta),\Theta] = E(R)\,, \tag{71}$$

and put

$$r = R + \xi(R,\Theta) + O(\Omega^4)\,. \tag{72}$$

The pressure and density are now known functions of R common to both the rotating and non-rotating configurations.

To calculate the Einstein's equations to order Ω^2 we must evaluate the $G_\mu{}^\nu = R_\mu{}^\nu - \frac{1}{2}\delta_\mu{}^\nu R$ to this approximation using the metric given in equation (66). This calculation is carried out in the Appendix. The corrections to $G_\mu{}^\gamma$ of order Ω^2 are functions of h, k, m, and ω and will be denoted by $\delta G_\mu{}^\gamma$. To obtain these corrections in the new coordinate system we transform the tensor $G_\mu{}^\nu + \delta G_\mu{}^\nu$. Denote the corrections of order Ω^2 in the R, Θ coordinate system by $\Delta G_\mu{}^\gamma$. One then has, for example,

$$\Delta G_t{}^t(R,\Theta) = \delta G_t{}^t(R,\Theta) + \xi(R,\Theta)\frac{\partial}{\partial R}[G_t{}^t(R,\Theta)^{(0)}]\,. \tag{73}$$

Here $(G_t{}^t)^{(0)}$ is $G_t{}^t$ calculated with the unperturbed metric. This latter quantity may be expressed using the zero-order field equations. One finds

$$\Delta G_t{}^t(R,\Theta) = \delta G_t{}^t(R,\Theta) - 8\pi\xi(R,\Theta)dE(R)/dR\,, \tag{74}$$

$$\Delta G_R{}^R(R,\Theta) = \delta G_r{}^r(R,\Theta) + 8\pi\xi(R,\Theta)dP(R)/dR\,, \tag{75}$$

$$\Delta G_\Theta{}^\Theta(R,\Theta) = \delta G_\theta{}^\theta(R,\Theta) + 8\pi\xi(R,\Theta)dP(R)/dR\,, \tag{76}$$

$$\Delta G_\varphi{}^\varphi(R,\Theta) = \delta G_\varphi{}^\varphi(R,\Theta) + 8\pi\xi(R,\Theta)dP(R)/dR\,, \tag{77}$$

$$\Delta G_R{}^\Theta(R,\Theta) = \delta G_r{}^\theta(R,\Theta)\,. \tag{78}$$

The space-time cross terms are of odd order in Ω. It can be seen using the identities of equations (74)–(78) that the equations obtained by this perturbation method are formally the same as those which would be found by working in the coordinates r, θ and expanding $\mathcal{E}$ and $\mathcal{P}$ in powers of Ω^2 with the identifications.

$$(\text{2d-order term in } \mathcal{E}) = -\xi\frac{dE}{dR}, \tag{79}$$

$$(\text{2d-order term in } \mathcal{P}) = -\xi\frac{dP}{dR}. \tag{80}$$

In general relativity, as in Newtonian theory, the calculation of rotational perturbations is greatly simplified if the equations are expanded in spherical harmonics. The expansion of the metric coefficients is given in equations (67)–(69). From its definition in equation (72) ξ transforms like a scalar under rotations. The expansion of ξ thus has the form

$$\xi = \xi_0(R) + \xi_2(R)P_2(\Theta) + \ldots\,. \tag{81}$$

Expanded to order Ω^2 the Einstein equations will have the general form

$$\mathcal{H}_\mu{}^\nu(h,m,k,\xi) = \Omega_\mu{}^\nu(\bar{\omega})\,, \tag{82}$$

where $\mathcal{H}_\mu{}^\nu$ is linear in the second-order quantities h, m, k, and ξ and $\Omega_\mu{}^\nu$ is quadratic in $\bar{\omega}$. The different l values in the spherical harmonic expansion of h, m, k, and ξ are not coupled together by the left-hand side of equation (82) because that expression is linear in these components. The right-hand side is quadratic in the quantity $\bar{\omega}$ which transforms like $l = 1$. This side can, therefore, contain at most $l = 0, 1, 2$. The value $l = 1$ is ruled out from the assumed reflection symmetry of the configuration. Thus, only the $l = 0$ and $l = 2$ equations involve the angular velocity Ω in any way. The coefficients in the expansion of h, m, k, and ξ possessing other l values must, therefore, vanish since they vanish when the star is not rotating. The reduction in the number of l values from infinity to 2 in the slow rotation approximation greatly simplifies the subsequent numerical computations in complete analogy to the Newtonian analysis.

VI. INTEGRAL OF THE EQUATIONS OF HYDROSTATIC EQUILIBRIUM

Not all of the six independent Einstein equations need be written down in order to solve for the rotational perturbations when the rotation is uniform. A first integral of these equations is contained in the integral of the equation of hydrostatic equilibrium for a uniformly rotating configuration with a one parameter equation of state (Tauber and Weinberg 1961; Boyer 1965; Hartle and Sharp 1967)

$$\text{constant} = \mu_c = \frac{\mathcal{E}+\mathcal{P}}{u^t} \exp\left(-\int \frac{d\mathcal{E}}{\mathcal{E}+\mathcal{P}}\right). \tag{83}$$

If the configuration is isentropic then μ_c is the chemical potential. Both sides of equation (83) are now expanded in powers of Ω^2. The constant injection energy μ_c we write as

$$\mu_c = \mu[1 + \gamma + O(\Omega^4)] , \tag{84}$$

where μ is the non-rotating injection energy and γ a constant of order Ω^2. In the expansion of the right-hand side of equation (83) we use equation (36) to express u^t in terms of the metric and angular velocity Ω. We find for the non-rotating configuration

$$\mu = (E+P)e^{\nu/2} \exp\left(-\int \frac{dE}{E+P}\right), \tag{85}$$

and for the terms of order Ω^2

$$\gamma = p^*(R,\Theta) + h(R,\Theta) - \tfrac{1}{2}e^{-\nu(R)}\bar{\omega}(R)^2 R^2 \sin^2\Theta , \tag{86}$$

where we have defined the relativistic "pressure perturbation factor"

$$p^*(R,\Theta) = \tfrac{1}{2}\nu_R\,\xi(R,\Theta) = -\,\xi\left(\frac{1}{E+P}\frac{dP}{dR}\right), \tag{87}$$

and the last equality follows from the second of equations (29). If we define

$$p_0{}^*(R) = -\,\xi_0(R)\left(\frac{1}{E+P}\frac{dP}{dR}\right), \tag{88}$$

$$p_2{}^*(R) = -\,\xi_2(R)\left(\frac{1}{E+P}\frac{dP}{dR}\right), \tag{89}$$

then both sides of equation (86) may be expanded in spherical harmonics to find

$$l = 0, \qquad \gamma = p_0{}^*(R) + h_0(R) - \tfrac{1}{3}e^{-\nu(R)}R^2\bar{\omega}(R)^2 , \tag{90}$$

$$l = 2, \qquad 0 = p_2{}^*(R) + h_2(R) + \tfrac{1}{3}e^{-\nu(R)}R^2\bar{\omega}(R)^2 . \tag{91}$$

In this way we have an integral of the second-order Einstein's equations for each relevant value of l.

The equations for different values of l are not coupled together and may be considered separately. In the next two sections we will consider the equations for $l = 0$ and $l = 2$ in turn with special emphasis on putting them into a form to calculate the relation between mass and central density and the shape of the star.

VII. THE $l = 0$ EQUATIONS—RELATION BETWEEN MASS AND CENTRAL DENSITY—BINDING ENERGY

How does the mass of a rotating star differ from that of the non-rotating one possessing the same central density? This question can be answered by examining the $l = 0$ equations alone because the mass, M, of any configuration in general relativity is determined by the spherically symmetric part of the metric at large distances,

$$g_{tt} \to -(1 - 2M/r), \qquad r \to \infty . \tag{92}$$

In this section the $l = 0$ equations are organized to solve this problem. By considering the rotating star as a perturbation on the non-rotating star with the same central density we guarantee, as in the Newtonian theory, that the fractional displacement $\xi(R,\Theta)/R$ is small throughout the star as is necessary for a valid perturbation analysis.

Three functions h_0, m_0, and ξ_0 suffice to specify the $l = 0$ parts of the configuration completely. As an integral of the $l = 0$ field equations is already in hand in equation (90) only two additional field equations need be solved. These are chosen to be $G_t{}^t = 8\pi T_t{}^t$ and $G_R{}^R = 8\pi T_R{}^R$, where $G_\mu{}^\nu = R_\mu{}^\nu - \frac{1}{2}\delta_\mu{}^\nu R$. The second-order contributions to the $G_\mu{}^\nu$ are calculated in the Appendix. Combining equations (A.9) and (74) and (A.11) and (75) one has

$$(\Delta G_t{}^t)_{l=0} = -\frac{2}{R^2}\frac{dm_0}{dR} + \frac{j}{6R^2}(jR^4\bar{\omega}_R{}^2 + 8R^3 j_R \omega\bar{\omega}) - 8\pi\xi\frac{dE}{dR} \tag{93}$$

and

$$\begin{aligned}(\Delta G_R{}^R)_{l=0} &= -\frac{2m_0}{R^2}\left(\nu_R + \frac{1}{R}\right) + \left(1 - \frac{2M}{R}\right)\frac{2}{R}\frac{dh_0}{dR} + \tfrac{1}{6}R^2 j^2(\bar{\omega}_R)^2 + 8\pi\xi_0\frac{dP}{dR} \\ &= -\frac{2m_0}{(R-2M)}\left(8\pi P + \frac{1}{R^2}\right) + \left(1 - \frac{2M}{R}\right)\frac{2}{R}\frac{dh_0}{dR} + \tfrac{1}{6}R^2 j^2(\bar{\omega}_R)^2 + 8\pi\xi_0\frac{dP}{dR}.\end{aligned} \tag{94}$$

Here equation (28) has been used in obtaining the last line. The second-order terms on the right-hand side of Einstein's equation are

$$(\Delta T_t{}^t)_{l=0} = -\frac{16\pi}{3}(E+P)\Omega\bar{\omega}R^2 e^{-\nu(R)} = \tfrac{2}{3}(j^2)_R\Omega\bar{\omega}R, \tag{95}$$

and

$$(\Delta T_R{}^R)_{l=0} = 0 . \tag{96}$$

Thus the two Einstein's equations considered here become

$$\frac{dm_0}{dR} = 4\pi R^2\frac{dE}{dP}(E+P)p_0{}^* + \tfrac{1}{12}j^2R^4(\bar{\omega}_R)^2 - \tfrac{1}{3}R^3(j^2)_R\bar{\omega}^2, \tag{97}$$

and

$$\frac{dh_0}{dR} - \frac{m_0R^2}{(R-2M)^2}\left(8\pi P + \frac{1}{R^2}\right) = \frac{4\pi(E+P)R^2}{(R-2M)}p_0{}^* - \tfrac{1}{12}\frac{R^4}{(R-2M)}j^2(\bar{\omega}_R)^2, \tag{98}$$

where

$$p_0^* = -\xi_0\left(\frac{1}{E+P}\frac{dP}{dR}\right). \tag{99}$$

Differentiation of the integral of the equation of hydrostatic equilibrium yields an identity which may be used to eliminate dh_0/dR from equation (98).

$$\begin{aligned} -\frac{dp_0^*}{dR}+\tfrac{1}{12}\frac{R^4}{(R-2M)}\,j^2(\bar{\omega}_R)^2+\tfrac{1}{3}\frac{d}{dR}\left(\frac{R^3 j^2\bar{\omega}^2}{R-2M}\right) \\ = 4\pi\frac{(E+P)R^2}{(R-2M)}\,p_0^*+\frac{m_0R^2}{(R-2M)^2}\left(8\pi P+\frac{1}{R^2}\right). \end{aligned} \tag{100}$$

Equation (100) is the relativistic generalization of the Newtonian equation (second of eqs. [15]) of hydrostatic equilibrium for slowly rotating configurations. It shows the balance of pressure, gravitational, and centrifugal forces. Together with equation (97) it completely determines the $l = 0$ problem. The linearity of these equations and of those which determine $\bar{\omega}$ means that from the solution for one value of Ω the solution for other values may be obtained by scaling.

It is of interest to see how the Newtonian balance of pressure, centrifugal, and gravitational forces shows up in equations (97) and (100) when all quantities are expanded in powers of $(1/c)$. To make this expansion we reinsert the factors of G and c and write

$$\begin{aligned} E &= G\rho/c^2+O(1/c^4), & P &= Gp/c^4+O(1/c^6)\,, \\ M &= GM/c^2+O(1/c^4)\,, & \bar{\omega} &= \Omega/c+O(1/c^3)\,, \end{aligned} \tag{101}$$

$$j_R/jR = -4\pi(E+P)e^{-\nu} = -4\pi G\rho/c^2+O(1/c^4), \qquad \nu_R = 2GM/Rc^2+O(1/c^4)\,.$$

The equations (97) and (100) become

$$\frac{dm_0}{dR} = 4\pi GR^2\frac{d\rho}{dP}\,\rho p_0^*\,, \tag{102}$$

and

$$-\frac{dp_0^*}{dR}+\tfrac{2}{3}\Omega^2R = \frac{m_0}{R^2}\,, \tag{103}$$

where

$$p_0^* = -\xi\,\frac{1}{\rho}\frac{dp}{d\rho}. \tag{104}$$

These equations are identical with the Newtonian equations discussed in § II.

From its definition in equation (66) one sees that the value of $m_0(r)$ at large r is the change in mass, δM, of the perturbed configuration from its non-perturbed value. To relate the change in mass δM to the value of m_0 at the surface of the star we solve equations (97) and (100) in the exterior region using the known form for $\bar{\omega}(r)$ given in equation (47). We find outside the star

$$m_0(r) = \delta M - J^2/r^3\,, \tag{105}$$

$$h_0(r) = -\frac{\delta M}{(r-2M)}+\frac{J^2}{r^3(r-2M)}\,, \tag{106}$$

so that

$$\delta M = m_0(a)+J^2/a^3\,. \tag{107}$$

To determine the relation between mass and central density one now proceeds as follows. (1) Choose a value of the central density. Calculate from the general relativity equation of hydrostatic equilibrium the non-rotating configuration with this value of the central density. At the same time calculate $e^{\nu(r)}$ for this configuration. (2) Solve equation (46) to determine $\bar{\omega}(r)$ for this configuration and its moment of inertia I. (3) Integrate the coupled equations (97) and (100) out from the origin with the boundary conditions that as $R \to 0$,

$$p_0^*(R) \to \tfrac{1}{3}(j_c\bar{\omega}_c)^2R^2, \qquad m_0^*(R) \to \frac{4\pi}{15}(E_c+P_c)\left[\left(\frac{dE}{dP}\right)_c+2\right](j_c\bar{\omega}_c)^2R^5. \tag{108}$$

Here a subscript c indicates that the quantity is to be evaluated at the center of the star. These boundary conditions guarantee that the central density of the rotating and non-rotating configurations are the same. (4) Determine the change in mass of the rotating configuration with the chosen value of central density over that of the non-rotating configuration with the same central density from equation (107).

The total baryon number of a neutron star and the binding energy of a supermassive star are two other quantities of interest which can be computed from the $l = 0$ equations. The expression for the total baryon number, A, as an integral over the number density of baryons, $\mathfrak{N}$, follows directly from the differential form of the baryon conservation law $[(-g)^{1/2}\mathfrak{N}u^\mu]_{,\mu} = 0$ (see Misner and Sharp 1964). The expression is

$$A = \int d^3x(-g)^{1/2}u^t\mathfrak{N}\,. \tag{109}$$

For an isentropic configuration the number density of baryons can be related to the pressure and energy density by the first law of thermodynamics

$$\frac{d\mathfrak{N}}{d\mathcal{E}} = \frac{\mathfrak{N}}{\mathcal{E}+\mathcal{P}}. \tag{110}$$

The negative of the binding energy, $-E_B$, of a relativistic star is defined to be the difference between its mass and the mass of all its matter when cold and dispersed.

$$-E_B = (\text{total mass-energy}) - (\text{rest mass-energy}) = M - \mu A\,, \tag{111}$$

where μ is the rest mass-energy associated with a single baryon. A calculation of the binding energy is, therefore, equivalent to a calculation of the total baryon number.

The quantities of interest for the slowly rotating stars will be the *change* in binding energy δE_B or change in total baryon number δA over their values in the non-rotating configuration,

$$-\delta E_B = \delta M - \mu\delta A\,. \tag{112}$$

When calculating the change in binding energy greater accuracy is obtained by specifying not the number density of baryons directly but rather the difference, ϵ, between the density of mass-energy and the density of rest mass-energy,

$$\epsilon = \mathcal{E} - \mu\mathfrak{N}\,. \tag{113}$$

An expression for δE_B in terms of ϵ may be obtained from equation (112) as follows: Express equation (109) in the coordinates R, Θ. Expand the resulting equation to order Ω^2 and integrate all derivatives of ξ by parts to find δA. Only $l = 0$ perturbations will survive in this expression. Subtract this result from δM found through equation (107). One finds:

$$\delta E_B = -\frac{J^2}{a^3} + \int_0^a 4\pi R^2 B(R)\,dR, \tag{114}$$

where

$$B(R) = (E+P)p_0^*\left\{\frac{dE}{dP}\left[\left(1-\frac{2M}{R}\right)^{-1/2}-1\right]+\frac{d\epsilon}{dP}\left(1-\frac{2M}{R}\right)^{-1/2}\right\}$$
$$+(E-\epsilon)\left(1-\frac{2M}{R}\right)^{-3/2}\left(\frac{m_0}{R}+\tfrac{1}{3}j^2\bar{\omega}^2\right)-\frac{1}{4\pi R^2}\left[\tfrac{1}{12}j^2R^4(\bar{\omega}_R)^2-\tfrac{1}{3}(j^2)_R R^3\bar{\omega}^2\right]. \tag{115}$$

To find the binding energy or equivalently the total baryon number, one now proceeds as in the calculation of the relation between mass and central density described above with the following additional steps: (5) Determine the difference, ϵ, between the energy density and the rest mass-energy density from the equation of state. (6) Use the already calculated functions p_0*, m_0, E, P, M, and $\bar{\omega}$ to calculate $B(R)$ from equation (114). (7) Integrate equation (114) to find the change in binding energy of the rotating star over the value for the non-rotating configuration. Determine the change in total baryon number from equation (112).

VIII. THE $l = 2$ EQUATIONS—THE SHAPE OF THE STAR

How is the surface of a rotating star of given central density deformed from the spherical surface of a non-rotating star with the same central density?

The surface of the rotating star is given according to equations (72) and (81) by the equation

$$r = r(\theta) = a + \xi_0(a) + \xi_2(a)P_2(\theta)\,, \tag{116}$$

where a is the radius of the non-rotating configuration. The parameter $\xi_0(a)$ has already been determined from the solution to the $l = 0$ equations

$$\xi_0(a) = p_0*(a)a(a-M)/M\,, \tag{117}$$

where M is the mass of the non-rotating configuration. To calculate the other parameter, $\xi_2(a)$, the field equations with $l = 2$ must be used. In this section the generalization to general relativity of Clairaut's equation in Newtonian gravitational theory for determining $\xi_2(a)$ will be derived.

In addition to $\xi_2(R)$ there are three unknown functions $h_2(R)$, $k_2(R)$, and $m_2(R)$ which completely determine the $l = 2$ solution. As the integral of the equation of hydrostatic equilibrium (eq. [91]) is already in hand, only three field equations need be written down. The equations are chosen with a view to yielding the simplest non-trivial expressions. The $G_\mu{}^\nu$ necessary to write these equations have been computed in the Appendix.

The first equation we consider is one which vanishes identically in the case of no rotation

$$R_\Theta{}^\Theta - R_\varphi{}^\varphi = 8\pi(T_\Theta{}^\Theta - T_\varphi{}^\varphi)\,. \tag{118}$$

The right-hand side to order Ω^2 is

$$T_\Theta{}^\Theta - T_\varphi{}^\varphi = -(\mathcal{E}+\mathcal{P})u_\varphi u^\varphi = -(E+P)e^{-\nu}\Omega\bar{\omega}R^2\sin^2\Theta + O(\Omega^4)\,. \tag{119}$$

Equating this with the result for the left-hand side from equations (A.15), (76), and (77), one has

$$h_2 + \frac{m_2}{r-2M} = -\tfrac{1}{3}R^3(j^2)_R\bar{\omega}^2 + \tfrac{1}{6}j^2R^4(\bar{\omega}_R)^2\,. \tag{120}$$

This relation provides a useful first integral of the field equations.

The next equation is taken to be

$$R_R{}^\Theta = 0\,, \tag{121}$$

which from equations (A.14) and (78) has the form

$$\frac{d}{dR}(h_2+k_2)=h_2\left(\frac{1}{R}-\frac{\nu_R}{2}\right)+\frac{m_2}{R-2M}\left(\frac{1}{R}+\frac{\nu_R}{2}\right). \tag{122}$$

Finally, we use the $l=2$ component of the field equation

$$G_R{}^R=8\pi T_R{}^R=8\pi\mathcal{P}\,. \tag{123}$$

We rewrite this employing equations (A.12) and (75) as

$$\begin{aligned}\frac{2}{R}\left(1-\frac{2M}{R}\right)\frac{dh_2}{dR}+\left(1-\frac{2M}{R}\right)\left(\nu_R+\frac{2}{R}\right)\frac{dk_2}{dR}-\frac{2m_2}{R}\left(\nu_R+\frac{1}{R}\right)\\ -\frac{6h_2}{R}-\frac{4k_2}{R}+8\pi\xi_2\frac{dP}{dR}-\tfrac{1}{6}R^2j^2(\bar{\omega}_R)^2=0\,.\end{aligned} \tag{124}$$

Equations (120), (122), and (124) and the integral of the equation of hydrostatic equilibrium (eq. [91]) are the four equations we will use to determine the $l=2$ terms in the deformation of the star.

To find the general relativity analogue of Clairaut's equation we use the two integrals (eqs. [91] and [120]) to eliminate the functions m_2 and ξ_2 leaving two coupled first-order equations for h_2 and k_2. For the purpose of numerical computation it is convenient to introduce $v=h_2+k_2$ and write the resulting equations in the form

$$\frac{dv}{dR}=-\nu_Rh_2+\left(\frac{1}{R}+\frac{\nu_R}{2}\right)\left[-\tfrac{1}{3}R^3(j^2)_R\bar{\omega}^2+\tfrac{1}{6}j^2R^4(\bar{\omega}_R)^2\right], \tag{125}$$

$$\begin{aligned}\frac{dh_2}{dR}=\left\{-\nu_R+\frac{R}{(R-2M)\nu_R}\left[8\pi(E+P)-\frac{4M}{R^3}\right]\right\}h_2-\frac{4v}{R\nu_R(R-2M)}\\ +\tfrac{1}{6}\left[\tfrac{1}{2}\nu_RR-\frac{1}{(R-2M)\nu_R}\right]R^3j^2(\bar{\omega}_R)^2-\tfrac{1}{3}\left[\tfrac{1}{2}\nu_RR+\frac{1}{(R-2M)\nu_R}\right]R^2(j^2)_R(\bar{\omega})^2.\end{aligned} \tag{126}$$

The quantity ν_R which appears in these equations may be expressed in terms of the energy and pressure by

$$\nu_R=\frac{8\pi PR^3+2M}{R(R-2M)}. \tag{127}$$

The two equations (125) and (126) are solved for the derivatives so that they are in a form where their solutions can be computed numerically by integrating outward from the origin.

At the origin the solutions must be regular. An examination of the equations (125) and (126) shows that, as $R\to 0$,

$$h_2(R)\to AR^2 \tag{128}$$

$$v(R)\to BR^4\,, \tag{129}$$

where A and B are any constants related by

$$B+2\pi(P_c+\tfrac{1}{3}E_c)A=-\frac{4\pi}{3}(E_c+P_c)(j_c\bar{\omega}_c)^2, \tag{130}$$

and where E_c and P_c are the values of the energy and pressure at the center of the star. The remaining constant in the solution is determined by the boundary condition that

$h_2(r) \to 0$ at large values of r. The constant is thus determined by joining the interior solution to that exterior solution which satisfies this boundary condition.

In the exterior region the values of $\bar{\omega}$ and ν and λ are given through equations (47) and (27). In terms of the mass of the unperturbed star, M, and the angular momentum, J, the $l = 2$ equations in the exterior region become

$$\frac{dv}{dr} = -\frac{2Mh_2}{r(r-2M)} + \frac{6J^2}{r^5}\frac{(r-M)}{(r-2M)}, \tag{131}$$

$$\frac{dh_2}{dr} = -\frac{2v}{M} - \frac{2h_2(r-M)}{r(r-2M)} - \frac{3J^2}{M}\frac{(r^2-2Mr-2M^2)}{r^5(r-2M)}. \tag{132}$$

The general solution to these equations is the sum of any particular solution plus a solution to the homogeneous equations,

$$\frac{dv}{dr} = -\frac{2Mh_2}{r(r-2M)}, \qquad \frac{dh_2}{dr} = -\frac{2v}{M} - \frac{2h_2(r-M)}{r(r-2M)}. \tag{133}$$

A particular solution may readily be found by making the ansatz $v = ar^{-3} + br^{-4} + cr^{-5}$ and a similar form for h_2. One finds

$$v = -\frac{J^2}{r^4}, \qquad h_2 = J^2\left(\frac{1}{Mr^3} + \frac{1}{r^4}\right). \tag{134}$$

To find the solutions of the homogeneous equation we write equations (133) as one second-order equation for h_2.

$$\frac{d^2h_2}{dr^2} + \frac{2(r-M)}{r(r-2M)}\frac{dh_2}{dr} - \frac{6r^2-12Mr+4M^2}{r^2(r-2M)^2}h_2 = 0. \tag{135}$$

At large r this has solutions which behave as r^{-3} and r^2. Only the one which vanishes at infinity is relevant here. To find it we introduce the new variable $\zeta = (r/M) - 1$:

$$(1-\zeta^2)\frac{d^2h_2}{d\zeta^2} - 2\zeta\frac{dh_2}{d\zeta} + \left(6 - \frac{4}{1-\zeta^2}\right)h_2 = 0. \tag{136}$$

This is a form of Legendre's equation so that the solution with the desired asymptotic properties may be written

$$h_2(\zeta) = AQ_2^2(\zeta) = A\left[\tfrac{3}{2}(\zeta^2-1)\log\left(\frac{\zeta+1}{\zeta-1}\right) - \frac{3\zeta^3-5\zeta}{\zeta^2-1}\right], \tag{137}$$

where Q_l^m is the associated Legendre function of the second kind. The constant A is related to the mass quadrupole moment of the configuration by the relation

$$(\text{quadrupole moment}) = Q = J^2/M + 16AM^3/5. \tag{138}$$

The general $l = 2$ exterior solution is then[5]

$$h_2(r) = AQ_2^2\left(\frac{r}{M}-1\right) + J^2\left(\frac{1}{Mr^3} + \frac{1}{r^4}\right), \tag{139}$$

$$v(r) = 2AM[r(r-2M)]^{-1/2}Q_2^1\left(\frac{r}{M}-1\right) - \frac{J^2}{r^4}, \tag{140}$$

[5] The author has been unable to make the solutions given by Bach (1922) for the case of vanishing quadrupole moment agree with those given here.

where

$$Q_2^1(\zeta) = (\zeta^2 - 1)^{1/2}\left[\frac{3\zeta^2 - 2}{\zeta^2 - 1} - \tfrac{3}{2}\zeta \log\left(\frac{\zeta + 1}{\zeta - 1}\right)\right]. \tag{141}$$

The interior solution like the exterior solution may be written as the sum of a particular solution and a homogeneous solution. The particular solution may be obtained by integrating equations (125) and (126) outward from the center with any values of A and B which satisfy equation (130). The homogeneous solution is then obtained by integrating the equations

$$\frac{dv}{dR} = -\nu_R h_2, \tag{142}$$

$$\frac{dh_2}{dR} = \left\{-\nu_R + \frac{R}{(R-2M)\nu_R}\left[8\pi(E+P) - \frac{4M}{R^3}\right]\right\} h_2 - \frac{4v}{R\nu_R(R-2M)}, \tag{143}$$

with A and B as in equation (128) but related by

$$B + 2\pi(P_c + \tfrac{1}{3}E_c)A = 0 . \tag{144}$$

The general solution may then be written

$$h_2 = A' h_2^H + h_2^P , \qquad v = A' v^H + v^P , \tag{145}$$

where a superscript H denotes the homogeneous solution and P the particular solution. The constants A and A' are then obtained by joining h_2 and its first derivative at $r = a$ with the exterior solution given in equation (139).

To summarize the calculation of the shape of the surface one proceeds as follows: (1) Write the equation of the surface in the form given in equation (116). The parameter $\xi_0(a)$ is determined from the $l = 0$ equation by the relation in equation (117). (2) Integrate equations (125) and (126) outward from the center with arbitrary initial conditions satisfying equations (128), (129), and (130). This determines particular solutions h_2^P and v^P. (3) Integrate the homogeneous equations (142) and (143) outward from the center with arbitrary initial conditions satisfying equations (128), (129), and (144). (4) Match the general solutions of equations (145) with the exterior solutions in equations (139) and (140). All constants are now determined and h_2 and v are known as functions of r. (5) Calculate the ellipticity, $\epsilon(a)$, of the surface with the relation derived from equations (89) and (91).

$$\epsilon(R) = -\frac{3}{2a}\,\xi_2(a) = \frac{3(a-2M)}{2M}\left[h_2(a) + \tfrac{1}{3}\,\frac{a^3}{(a-2M)}\left(\Omega - \frac{2J}{a^3}\right)^2\right]. \tag{146}$$

It is instructive to see how the procedure summarized above yields Clairaut's equation in the Newtonian limit. To do this we must reinsert factors of G and c and make an expansion of the relativistic equations in powers of $(1/c)$. Here we use the already known expansions given in § VIII and,

$$h_2 = \varphi_2/c^2 + O(1/c^4), \qquad v = \psi/c^2 + \chi/c^4 + O(1/c^6) . \tag{147}$$

Equations (125) and (126) then reduce to

$$\frac{d\varphi_2}{dr} = \varphi_2\left(\frac{4\pi R^2\rho}{M} - \frac{2}{R}\right) - \frac{2\chi}{GM} + \frac{4\pi}{3M}\,\rho\Omega^2 R^4, \tag{148}$$

$$\psi = 0 , \tag{149}$$

$$\frac{d\chi}{dR} = -\frac{2GM}{R^2}\varphi_2 + \frac{8\pi}{3}\Omega^2 R^3 G\rho . \tag{150}$$

The integral of the equation of hydrostatic equilibrium (eq. [91]) becomes

$$0 = \xi_2 \frac{GM}{R^2} + \varphi_2(R) + \tfrac{1}{3}\Omega^2 R^2 . \tag{151}$$

If we write this in terms of the ellipticity $\epsilon(R) = -\frac{3}{2}\xi_2(R)/R$ and compare with equation (5) we see that φ_2 is the Eulerian change in the Newtonian potential,

$$\varphi_2 = \Phi_2^{[2]} . \tag{152}$$

By using equation (120) to calculate m_2, the metric becomes

$$\begin{aligned} ds^2 = &-\left(1+\frac{2\Phi}{c^2}\right)dt^2 + \left[1+\frac{2r}{c^2}\frac{d\Phi_0}{dr} + \Phi_2^{[2]}(r)P_2(x)\right]dr^2 \\ &+ r^2\left[1+\frac{2\Phi_2^{[2]}}{c^2}P_2(x)\right](d\theta^2 + \sin^2\theta\, d\varphi^2) . \end{aligned} \tag{153}$$

This is in agreement with the result quoted in equation (50) taking the gauge transformation into account. Finally eliminating χ and φ_2 between equations (148), (150), and (151) one has the following equation for ϵ:

$$\frac{M}{R}\frac{d^2\epsilon}{dR^2} + \frac{2}{R}\frac{dM}{dR}\frac{d\epsilon}{dR} + \frac{2dM}{dR}\frac{\epsilon}{R^2} - \frac{6M\epsilon}{R^3} = 0 . \tag{154}$$

This is identical with Clairaut's equation (eq. [22]). The relativistic theory presented here thus gives the correct Newtonian limit.

IX. CONCLUSIONS

Equations have been developed for calculating the structures of slowly rotating general-relativistic stars in hydrostatic equilibrium. In particular, prescriptions have been given to find the relation between mass and central density, the shapes, and the binding energy of these massive stars. The equations are the generalization in general relativity of the corresponding equations in Newtonian gravitational theory.

The equation which determines the relation between mass and central density takes the form of an equation of hydrostatic equilibrium. It enforces the balance of pressure, gravitational, and centrifugal forces correctly to order Ω^2 in the angular velocity. In this order the surfaces of constant density are spheroids whose ellipticity varies from zero at the center of the star to the ellipticity which describes the shape of the star at the surface. The ellipticity, $\epsilon(R)$, as a function of radius is determined by a generalization to general relativity of Clairaut's differential equation for this quantity.

Both the equations which determine the relation between mass and central density and those which determine the ellipticity are systems of ordinary, first-order linear differential equations whose solution may be obtained by computer calculation. Work in this direction is now going forward.

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APPENDIX

The second-order contributions to $G_\mu{}^\nu$ are briefly summarized here. The $R_\mu{}^\nu$ were first calculated for the general metric

$$ds^2 = e^{2\alpha}dr^2 + e^{2\beta}d\theta^2 + \rho^2 d\varphi^2 - 2\omega\rho^2 d\varphi dt + (\omega^2\rho^2 - N^2)dt^2 , \tag{A.1}$$

where $\alpha, \beta, \rho, \omega$, and N are functions of r and θ alone. This line element is more general than that of equation (50), but the expressions which it leads to are more easily checked against other calculations (see, e.g., Bach 1922; van Stockum 1937). One finds

$$-e^{\alpha+\beta}\rho N R_t{}^t = [e^{\beta-\alpha}(\rho N_r - \rho^3(\omega^2)_r/4N)]_r + [e^{\alpha-\beta}(\rho N_\theta - \rho^3(\omega^2)_\theta/4N)]_\theta , \tag{A.2}$$

$$-e^{\alpha+\beta}\rho N R_\varphi{}^\varphi = [e^{\beta-\alpha}(N\rho_r + \rho^3(\omega^2)_r/4N)]_r + [e^{\alpha-\beta}(N\rho_\theta + \rho^3(\omega^2)_\theta/4N)]_\theta , \tag{A.3}$$

$$-e^{\alpha+\beta}\rho N R_\varphi{}^t = [e^{\beta-\alpha}\rho^3\omega_r/N]_r + [e^{\alpha-\beta}\rho^3\omega_\theta/N]_\theta , \tag{A.4}$$

$$\begin{aligned}-e^{+(\alpha+\beta)}\rho N R_r{}^r = {}& N\rho[(e^{\beta-\alpha}\beta_r)_r + (e^{\alpha-\beta}\alpha_\theta)_\theta] + e^{\alpha-\beta}(N\rho)_\theta\alpha_\theta + e^{(\beta-\alpha)}[\rho N_{rr} \\ &+ N\rho_{rr} - \alpha_r(N\rho)_r] - \tfrac{1}{2}(\omega_r)^2 e^{\beta-\alpha}\rho^3/N ,\end{aligned} \tag{A.5}$$

$$\begin{aligned}-e^{(\alpha+\beta)}\rho N R_\theta{}^\theta = {}& N\rho[(e^{\beta-\alpha}\beta_r)_r + (e^{\alpha-\beta}\alpha_\theta)_\theta] + e^{\beta-\alpha}(N\rho)_r\beta_r + e^{\alpha-\beta}[\rho N_{\theta\theta} \\ &+ N\rho_{\theta\theta} - \beta_\theta(N\rho)_\theta] - \tfrac{1}{2}(\omega_\theta)^2 e^{\alpha-\beta}\rho^3/N ,\end{aligned} \tag{A.6}$$

$$\begin{aligned}R_{r\theta} = {}& -N_{r\theta}/N - \rho_{r\theta}/\rho + \alpha_\theta(N_r/N + \rho_r/\rho) + \beta_r \\ &\times (N_\theta/N + \rho_\theta/\rho) + \tfrac{1}{2}\omega_r\omega_\theta\rho^2/N^2 .\end{aligned} \tag{A.7}$$

For the scalar curvature R one finds

$$\begin{aligned}-e^{\alpha+\beta}\rho N R = {}& [(e^{\beta-\alpha}\rho N)_r]_r + [e^{\alpha-\beta}(\rho N)_\theta]_\theta + 2N\rho[(e^{\beta-\alpha}\beta_r)_r \\ &+ (e^{\alpha-\beta}\alpha)_\theta] + \rho[(e^{\beta-\alpha}N_r)_r + (e^{\alpha-\beta}N_\theta)_\theta] + N[(e^{\beta-\alpha}\rho_r)_r \\ &+ (e^{\alpha-\beta}\rho_\theta)_\theta] - \tfrac{1}{2}(\rho^3/N)[e^{\beta-\alpha}(\omega_r)^2 + e^{\alpha-\beta}(\omega_\theta)^2] .\end{aligned} \tag{A.8}$$

The quantities $G_t{}^t$, $G_r{}^r$, $R_{r\theta}$, and $R_\theta{}^\theta - R_\varphi{}^\varphi$ are now to be expanded in powers of Ω using the form of the metric given in equation (85). The resulting equations are then to be separated into components of definite angular momentum. We also use here the result of the discussion of § IV that $\omega_\theta = 0$. We find:

$$(\text{2d order } G_t{}^t)_{l=0} = \frac{j}{6r^2}[8r^3 j_r\omega(\Omega-\omega) + jr^4(\omega_r)^2] - \frac{dm_0}{dr}\frac{2}{r^2}, \tag{A.9}$$

$$\begin{aligned}(\text{2d order } G_t{}^t)_{l=2} = {}& -\frac{j}{6r^2}[8r^3 j_r\omega(\Omega-\omega) - jr^4(\omega_r)^2] - \frac{dm_2}{dr}\frac{2}{r^2} \\ &+\left(1-\frac{2M}{r}\right)\left(2k_{rr}+\frac{6k_r}{r}\right)-\left(\frac{M}{r}\right)_r 2k_r - \frac{6m_2}{r^2(r-2M)} - \frac{4k_2}{r^2},\end{aligned} \tag{A.10}$$

$$(\text{2d order } G_r{}^r)_{l=0} = \tfrac{1}{6}r^2 j^2(\omega_r)^2 - \frac{2m_0}{r^2}\left(\nu_r+\frac{1}{r}\right)+\left(1-\frac{2M}{r}\right)\frac{2(h_0)_r}{r}, \tag{A.11}$$

$$\begin{aligned}(\text{2d order } G_r{}^r)_{l=2} = {}& -\tfrac{1}{6}r^2 j^2(\omega_r)^2 - \frac{2m_2}{r^2}\left(\nu_r+\frac{1}{r}\right)+\left(1-\frac{2M}{r}\right) \\ &\times\frac{2(h_2)_r}{r} - \frac{6h_2}{r^2} + \left(1-\frac{2M}{r}\right)(k_2)_r\left(\nu_r+\frac{2}{r}\right) - \frac{4k_2}{r^2},\end{aligned} \tag{A.12}$$

$$(\text{2d order } R_{r\theta})_{l=0} = 0 \;, \tag{A.13}$$

$$(\text{2d order } R_{r\theta})_{l=2} = -(h_2)_r + h\left(\frac{1}{r} - \frac{\nu_r}{2}\right) - (k_2)_r + \frac{m}{r-2M}\left(\frac{1}{r} + \frac{\nu_r}{2}\right), \tag{A.14}$$

$$(\text{2d order } R_\theta{}^\theta - R_\varphi{}^\varphi) = \sin^2\theta\left[-\frac{3}{r^2}\left(h_2 + \frac{m_2}{r-2M}\right) + \tfrac{1}{2}j^2 r^2(\omega_r)^2 + r(j^2)_r\omega(\Omega - \omega)\right]. \tag{A.15}$$

In some cases these expressions have been simplified by using the differential equation for ω (eq. [65]).

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REFERENCES

Bach, R. 1922, *Math. Zs.*, **13**, 119.
Bondi, H. 1964, *Proc. Roy. Soc A*, **284**, 303.
Boyer, R. 1965, *Proc. Cambridge Phil Soc.*, **61**, 527.
Brill, D., and Cohen, J. 1966, *Phys. Rev.*, **143**, 1011.
Buchdahl, H. 1959, *Phys. Rev.*, **116**, 1027.
Chandrasekhar, S. 1939, *Introduction to the Study of Stellar Structure* (Chicago: University of Chicago Press).
Chandrasekhar, S., and Roberts, P. 1963, *Ap. J.*, **138**, 809.
Fowler, W. A. 1964, *Rev. Mod. Phys.*, **36**, 545 and 1104.
———. 1966, *Ap. J.*, **144**, 180.
Harrison, B., Thorne, K., Wakano, M., and Wheeler, J. 1965, *Gravitation Theory and Gravitational Collapse* (Chicago: University of Chicago Press).
Hartle, J., and Sharp, D. 1967, *Ap. J.*, **147**, 317.
Jeffreys, H. 1959, *The Earth* (Cambridge: Cambridge University Press).
Landau, L., and Lifshitz, E. 1962, *Classical Theory of Fields* (Reading: Addison-Wesley).
Misner, C. A., and Sharp, D. 1964, *Phys. Rev.*, **136**, B571.
Papapetrou, A. 1948, *Proc. Roy. Irish Acad.*, **52**, 11.
Regge, T., and Wheeler, J. 1957, *Phys. Rev.*, **108**, 1063.
Smithies, F. 1958, *Integral Equations* (Cambridge: Cambridge University Press).
Stockum, W. J. van. 1937, *Proc. Roy. Soc. Edinburgh*, **57**, 137.
Tauber, G., and Weinberg, J. 1961, *Phys. Rev.*, **122**, 1342.
Thirring, H. 1918, *Phys. Zs.*, **19**, 33.
Thorne, K. 1967, "Relativistic Stellar Structure and Dynamics" in *Proc. 1966 École d'Été de Physique Théorique* (New York: Gordon & Breach).
Thorne, K., and Meltzer, D. 1966, *Ap. J.*, **145**, 514.
Thorne, K , and Zimmerman, B. 1967 (unpublished report).
Tooper R. F. 1966, *Ap. J.*, **143**, 465.
Tsuruta, S., and Cameron, A. 1967, "The Rotation of Neutron Stars" in *Proc. 1966 École d'Été de Physique Théorique* (New York: Gordon & Breach).
Wheeler, J. 1966, *Ann. Rev. Astr. and Ap.*, **4** (Palo Alto: Annual Reviews).
Zee, A., and Wheeler, J. 1967 (to be published).