The Last Three Minutes: Issues in Gravitational Wave Measurements of Coalescing Compact Binaries

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Gravitational-wave interferometers are expected to monitor the last three minutes of inspiral and final coalescence of neutron star and black hole binaries at distances approaching cosmological, where the event rate may be many per year. Because the binary's accumulated orbital phase can be measured to a fractional accuracy $\ll 10^{-3}$ and relativistic effects are large, the waveforms will be far more complex, carry more information, and be far harder to model theoretically than has been expected. Theorists must begin now to lay a foundation for extracting the waves' information.

PACS numbers: 04.30.+x, 04.80.+z, 97.60.Jd, 97.60.Lf

A network of gravitational-wave interferometers (the American LIGO [1], the French/Italian VIRGO [2] and possibly others) is expected to be operating by the end of the 1990s. The most promising waves for this network are from the inspiral and coalescence of neutron star and black hole binaries [3,4], with an estimated event rate of $\sim (3/\text{year})(\text{distance}/200\ \text{Mpc})^3$ [5].

A binary's inspiral and coalescence will produce two gravitational waveforms, one for each polarization. By cross-correlating the outputs of 3 or more interferometers, both waveforms can be monitored and the source's direction can be determined to within ~ 1 degree [4,6].

We shall divide each waveform into two parts: the *inspiral waveform*, emitted before tidal distortions become noticeable (3 or fewer orbital cycles before complete disruption or merger [7]), and the *coalescence waveform*, emitted during distortion, disruption, and/or merger.

As the binary, driven by gravitational radiation reaction, spirals together, its *inspiral waveform* sweeps upward in frequency f. The interferometers will observe the last several thousand cycles of inspiral (from $f \sim 10$ Hz to ~ 1000 Hz), followed by coalescence.

Theoretical calculations of the waveforms are generally made using the Post-Newtonian ("PN") approximation to general relativity. Previous calculations have focused on the Newtonian-order waveforms [4,3,8,1] and on PN modulations of their amplitude and frequency [9].

Two of us [11] recently realized that the PN modulations are far less important than PN contributions to the secular growth of the waves' phase $\Phi = 2\pi \int f dt$ (which arise, largely, from PN corrections to radiation reaction.) The binary's parameters are determined by integrating the observed (noisy) signal against theoretical templates [10], and if the signal and template lose phase with each other by one cycle (out of thousands) as the signal sweeps through the interferometers' band, their overlap integral will be strongly reduced. Correspondingly, one can infer each of the system's parameters λ_i to an accuracy that is roughly the change $\Delta\lambda_i$ which alters by unity the number of cycles $\mathcal{N}_{\rm cyc}$ spent in the interferometers' band.

The only parameters λ_i that can influence the inspiral template significantly are (i) the initial orbital elements, (ii) the bodies' masses and spin angular momenta, and (iii) if the spins are very large, the bodies' spin-induced quadrupole moments. (The quadrupole moments produce orbital phase shifts a little larger than unity at $f \sim 10$ Hz, but negligible phase shifts later [7]. Below we shall ignore this tiny effect.)

If (as almost always is the case) the binary's orbit has been circularized by radiation reaction, then the number of cycles spent in a logarithmic interval of frequency, $d\mathcal{N}_{\rm cyc}/d\ln f = (1/2\pi)(d\Phi/d\ln f)$, is:

$$\frac{d\mathcal{N}_{\rm cyc}}{d\ln f} = \frac{5}{96\pi}\frac{1}{\mu M^{2/3}(\pi f)^{5/3}}\left\{1 + \left(\frac{743}{336} + \frac{11}{4}\frac{\mu}{M}\right)x \right.$$
$$\left. - [4\pi + \text{S.O.}]x^{1.5} + [\text{S.S.}]x^2 + O(x^{2.5})\right\}. \quad (0.1)$$

Here M is the binary's total mass, μ its reduced mass, and $x \equiv (\pi M f)^{2/3} \simeq M/D$ the PN expansion parameter (with D the bodies' separation and $c = G = 1$). The PN correction [$O(x)$ term] is from [12]. In the P$^{1.5}$N correction, the 4π is created by the waves' interaction with the binary's monopolar gravitational field as they propagate from the near zone to the radiation zone [13], and the "S.O." denotes contributions due to spin-orbit coupling [14]. In the P^{2}N correction the "S.S." includes spin-spin coupling effects [14] plus an expression quadratic in μ/M.

Since the leading-order, Newtonian contribution to Eq. (0.1) gives several thousand cycles of oscillation in the interferometers' band, to be measureable the higher-order corrections need only be as large as a part in several thousand, even when the amplitude signal to noise ratio has only its typical value of 8. It is this that gives the waves' phasing its high potential accuracy.

The determination of the binary's parameters (masses, spins, orbital elements) is made possible by the various frequency dependences in Eq. (0.1). Analytic [15,16] and Monte Carlo [17] calculations show that (i) the "chirp mass" $M_c \equiv \mu^{3/5}M^{2/5}$ [which governs the Newtonian part of (0.1)] will typically be measured to 0.1 per cent, and (ii) *if* we somehow knew that the spins were small, then the reduced mass μ would be measured to ~ 1 per cent for NS/NS and NS/BH binaries, and ~ 3 per cent for BH/BH binaries. (Here and below NS means a $\sim 1.4 M_\odot$ neutron star and BH means a $\sim 10 M_\odot$ black hole.) Unfortunately, the frequency (x) dependences of the various terms in Eq. (0.1) are not sufficiently different to give a clean separation between μ and the spins. Preliminary estimates [17], in which the S.O. term in Eq. (0.1) was taken into account but not the S.S. term, suggest that the spin/μ correlation will worsen the typical accuracy of μ by a factor ~ 20, to ~ 20 per cent for NS/NS, ~ 25 per cent for NS/BH, and ~ 50 per cent for BH/BH.

These worsened accuracies might be improved significantly, however, by the waveform modulations [9]—most especially, modulations caused by spin-induced precession of the orbit [14]. Much additional theoretical work is needed to pin down the measurement accuracies for μ and the spins.

The highest accuracy parameter information will come from low frequencies where most of the waves' cycles are located [15–17]. (It thus is important that the interferometers achieve good low-frequency performance). When the waves have swept to higher frequencies, the binary's parameters may be fairly well known, and the subsequent, highly relativistic waveforms might be used to map out the binary's innermost spacetime geometry near the orbital plane. It would be useful to develop a parametrized, "PPN-like" multi-theory formalism for translating the measured waveforms into spacetime maps and for comparing those maps with the predictions of various gravitation theories.

To make optimal use of the interferometers' data will require general-relativity-based waveform templates whose phasing is correct to within a half cycle or so during the entire frequency sweep from ~ 10 Hz to ~ 1000 Hz. By examining an idealized limit ($\mu \ll M$ and vanishing spins), several of us [13,19] have discovered that, *to compute the templates with the desired accuracy via PN methods will be very difficult*. We calculated the waves from such a binary to high accuracy using the Teukolsky [18] black-hole perturbation formalism and then fit a PN expansion to the results, to obtain [19]

$$\frac{d\mathcal{N}_{\rm cyc}}{d\ln f} = \frac{f^2 dE/df}{dE/dt} = \frac{5}{96\pi}\frac{M/\mu}{x^{2.5}} \times$$
$$\times \frac{1 - \frac{3}{2}x - \frac{81}{8}x^2 - \frac{675}{16}x^3 + ...}{1 - \frac{1247}{336}x + 4\pi x^{1.5} - 4.9x^2 - 38x^{2.5} + 170x^3 + ...}. \quad (0.2)$$

Here dE/df is half the ratio of the changes of orbital energy dE and orbital frequency $d(f/2)$ as the orbit shrinks, and dE/dt is the power carried off by gravitational waves. The coefficients in the denominator after the 4π were obtained from the fit; all other coefficients are known analytically. The coefficient 4.9 has accuracy $\sim 2\%$; the 38, $\sim 10\%$; and the 170, $\sim 25\%$.

We can use Eq. (0.2) to estimate how the phase accuracy depends on the order to which the PN expansion is carried. If the template were computed through P^{2}N order [all $x^{2.5}$ and x^3 terms in Eq. (0.2) omitted], then the phase error typically would reach a half cycle when the frequency had swept from 10 Hz to only ~ 15 Hz, and the total phase error from 10 Hz to 1000 Hz would typically be ~ 6 cycles. For interferometers whose peak sensitivity is at 70 Hz [1], one could use such templates for a maximum range of ~ 50 Hz to ~ 90 Hz without accumulating phase errors larger than a half cycle. If the template were computed through P$^{2.5}$N order, there would be no substantial improvement. It is not at all clear how far beyond P$^{2.5}$N the template must be carried to keep the total phase error below a half cycle over the entire range from ~ 10 Hz to ~ 1000 Hz.

This slow convergence of the PN expansion might be improved by cleverness in the way one expresses the expansion (e.g., by the use of Padé approximates). However, even with great cleverness and fortitude, PN templates might never cover the entire inspiral range, from ~ 10 Hz to ~ 1000 Hz. Evidently, new techniques are needed for computing templates to higher accuracy. Two such techniques look promising: A "post-Teukolsky expansion," and a "weak-reaction expansion."

The post-Teukolsky expansion would expand in powers of μ/M and thus would be useable only for a light body orbiting a much heavier black hole. The expansion's first step would be the unperturbed hole's Kerr metric; the second, the Teukolsky formalism [18] for the light body's first-order perturbations; and the third, incorporation of radiation reaction [20]. Each of these first three steps is now in hand, though studies of the consequences of the radiation reaction are only beginning [21]. That the expansion must be carried well beyond these three steps is evident from the magnitude of the μ/M term in Eq. (0.1), and from recent studies [22] of the influence of a finite μ/M on the last stable circular orbit.

The weak-reaction expansion would be a variant of numerical relativity in which one expands in powers of $1/Q \equiv (dE/dt)(\pi f E)^{-1}$ (a measure of the effects of radiation reaction during one orbit). Because $1/Q \sim x^{2.5}$ (with x the PN parameter) is always $\ll 1$, this expansion might produce adequate inspiral templates for all binaries. The expansion's first step might be a numerical solution of the Einstein equations for the binary's metric, with standing-wave boundary conditions at infinity and at the BH horizon(s) (if any) [23]. In co-rotating coordinates, the equations would be elliptic and might be solvable to high accuracy by relaxation techniques. The second step might be to switch from standing-wave to outgoing- and downgoing-wave boundary conditions, and evaluate the resulting linearized metric perturbations, and with them the leading contributions to the waveforms. The third step might be the leading effects of radiation reaction, including the orbital inspiral. To obtain templates with the desired phase accuracy, one would have to carry the expansion beyond this third step.

Although an optimally-precise *measurement* of the binary's parameters requires templates faithful to general relativity's predictions, a near optimal *search* for binary-inspiral waves can use a family of templates that merely span the range of expected waveform behaviors, without being closely related to the true general relativistic waveforms [24]. Theorists need to develop a set of such *search templates* and optimize their efficiency for wave searches.

Turn, now, from a binary's inspiral waveforms to its *coalescence waveforms*. By the beginning of coalescence, the binary's parameters (masses, spins, geometry) will be known with fair accuracy, and from its masses the nature of its bodies (BH vs NS) should be fairly clear. The coalescence waveforms can then be used to probe issues in the physics of gravity and atomic nuclei:

If the bodies are BH's of comparable mass (and especially if their spins are large and not perpendicular to the orbital plane), then the coalescence should produce large-amplitude, highly nonlinear vibrations of spacetime curvature that may reveal aspects of gravity we have never seen. Although supercomputer simulations of such coalescences are being attempted, they are so difficult, especially for spinning holes, that they may well *not* give definitive results without observational guidance. Conversely, the observed waveforms will be very hard to interpret without the guidance of simulations.

The interferometer measurements discussed above may well be achieved in the early years of LIGO/VIRGO. Most of the measurements discussed below are more difficult, and may require more mature interferometers.

For a NS/BH binary with the BH spinning moderately fast, the NS should disrupt tidally before plunging into the BH. The NS disruption should be quick, as should the final coalescence of a NS/NS binary: within about one orbit, the NS/BH and NS/NS binaries may get smeared into roughly axially symmetric configurations, thereby shutting off their waves [25,7]. The rapid wave shutoff and contamination by laser shot noise may prevent the coalescence waveforms from revealing more than two numbers: the maximum frequency $f_{\rm max}$ reached, and the total wave energy $E_{\rm GW}$ in the last ~ 0.02 seconds (when the energy emission is maximal). Either number, however, would be valuable. It could tell us the NS radius $R_{\rm NS}$ (or a combination of the two NS radii) [27]. Since the NS masses will already be known from the inspiral waveforms, such measurements on a number of NS's could reveal the NS radius-mass relation $R_{\rm NS}(M_{\rm NS})$, from which one can infer the equation of state of matter at densities from ~ 1 to 10 times nuclear [26].

In preparation for the interferometers' measurements of $f_{\rm max}$ and/or $E_{\rm GW}$, theorists need to model tidal disruption in NS/BH binaries and coalescence of NS/NS binaries, to determine the dependence of $f_{\rm max}$ and $E_{\rm GW}$ on $R_{\rm NS}$ and the binary's other parameters. (Such modeling is also driven by the possibility that these events are the sources of observed gamma-ray bursts [28].) Simple arguments [7,27] suggest that $f_{\rm max}$ will be roughly $(1/2\pi)(M_{\rm NS}/R_{\rm NS}^3)^{1/2}$, so for a $1.4 M_\odot$ NS, a 15 km radius will lead to $f_{\rm max} \sim 1000$ Hz, while a 10 km radius will lead to $f_{\rm max} \sim 2000$ Hz. Similarly, $E_{\rm GW}$ will be proportional to $1/R_{\rm NS}$ so a 10 km NS will emit ~ 50 per cent more energy just before disruption than will a 15 km NS.

To measure the shutoff frequency $f_{\rm max} \sim 1000$ Hz in the midst of the interferometers' shot noise will probably require specially configured ("dual recycled" [29]) interferometers that operate over adjustable, narrow bands Δf around adjustable central frequencies f_o. Such interferometers, operated in concert with broad-band ones, could give simple "yes/no" answers as to whether the waves swept through their frequency bands. By collecting such data on a number of binaries, with various choices for the frequencies f_o, one might zero in on $f_{\max}$ for various types of binaries, and thence on the NS radius-mass relation. The dual-recycled sensitivities required for such a program, however, will be difficult to achieve [27].

Even more difficult, but not totally hopeless, will be the measurement of E_{GW} via the waves' "Christodoulou memory" [30,27] (a slowly changing mean of the waveforms, produced by the $1/r$ gravitational field of the emitted gravitons). When the waves' mean is monitored by an optimally filtered, broad-band interferometer, it gets averaged over a time ~ 0.02 seconds, so the maximum level it reaches is proportional to E_{GW}. Unfortunately, to measure the memory's level and thence E_{GW} with even 50 per cent accuracy for binaries at 200 Mpc distance is likely to require interferometers modestly better than LIGO's so-called "advanced detectors" [1].

Turn, next, from a binary's coalescence to the use of its waveforms as cosmological probes. Schutz [4] has pointed out that a binary's distance from Earth (more precisely, its "luminosity distance" r_L) can be inferred from its inspiral waveforms, and that such gravitationally measured distances, when combined with electromagnetically measured redshifts z, might determine the Hubble constant H_o to better precision than heretofore. However, this will require identifying electromagnetically the clusters of galaxies in which the binaries lie, at least in some statistical sense—a difficult task [4].

Remarkably, for NS/BH binaries at $z \gtrsim 1$, it may be possible to determine z as well as r_L from the gravitational waveforms, and then from $r_L(z)$ one might infer the Universe's Hubble constant H_o, mean density $\bar{\rho}$, and cosmological constant Λ [31]. (A variant of the method could also work for NS/NS binaries, but NS/BH binaries emit more strongly and thus can be seen to greater distances, making them more promising than NS/NS.)

The key to measuring z is the expectation (based on radio pulsar observations) that NS masses will cluster around $1.4M_\odot$. (If this fails, but there is a sharp cutoff in NS masses at some upper limit, then that limiting mass can be used as the key to the measurements.) For a binary at $z \gtrsim 1$, the waveforms will be measured as functions of *redshifted* time, and correspondingly they will reveal redshifted masses $(1+z)M_{\mathrm{NS}}$ and $(1+z)M_{\mathrm{BH}}$. From the measured luminosity distance and redshifted masses, it should be fairly clear whether the binary is NS/BH or not; and if so, then one can infer a "candidate redshift" for the binary, $1+z_{\mathrm{cand}} \equiv (1+z)M_{\mathrm{NS}}/1.4M_\odot$. If the NS masses cluster around $1.4M_\odot$, then a large sample of measured distances and candidate redshifts should cluster around the true distance-redshift relation.

This gravitational method of determining cosmological parameters might be much more immune to evolutionary and ill-understood systematic effects than are electromagnetic methods, because gravitational waves are immune to absorption and scattering, and neutron star masses might not be sensitive to the Universe's evolution.

A detailed analysis by one of us [31] suggests that this gravitational approach to cosmology might begin to bring useful information when broad-band interferometers reach the LIGO "advanced detector" sensitivities [1]: with one year of observational data, the one-sigma accuracies might be $\delta H_o \sim 0.01 H_o$, $\delta\bar{\rho} \sim 0.1\rho_{\rm crit}$, and $\delta\Lambda \sim 0.3 H_o^2$, where $\rho_{\rm crit}$ is the critical density to close the Universe.

We conclude by noting that some of the issues discussed in this Letter have implications for a possible future space-based interferometer called LAGOS, which would operate in the band 0.0001–0.03 Hz [32]. We shall discuss those implications elsewhere.

For helpful discussions we thank David Chernoff, Larry Kidder, Andrzej Krolak, E. Sterl Phinney, Bernard F. Schutz, Clifford M. Will, and Alan Wiseman. This research was supported in part by National Science Foundation Grant PHY-9213508, and, in view of its applications to LAGOS, by NASA grants NAGW-2897, 2920, and 2936, and by a Weingart Foundation, Lee A. Dubridge Fellowship to L. Bildsten and an Alfred P. Sloan Foundation Fellowship to L. S. Finn.

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