Amplification of Cylindrical Electromagnetic Waves Reflected from a Rotating Body

Ya. B. Zel'dovich

Institute of Applied Mathematics, USSR Academy of Sciences

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The amplification of waves reflected from a rotating body is analyzed in detail for cylindrical electromagnetic waves of varying polarization and for an infinite cylinder of low conductivity. In the general case, particularly for a gravitational interaction of waves with a collapsing rotating body (Kerr metric), it is shown that the energy criterion of the amplification requires that the rotational velocity of the body exceed the rotational velocity of the constant-phase surface of the wave.

AN effect wherein waves are amplified when reflected from a rotating body was pointed out in[1]. The purpose of the present article is to explain the physical nature of the phenomenon, to present an energy-based proof of the existence of the effect, to consider the effect for gravitational interaction of waves with a collapsing body, and to present a quantitative estimate for electromagnetic waves interacting with a weakly-conducting cylinder.

1. According to a remark by P. L. Kapitza, the effect is analogous to amplification of sound by reflection from a resting-medium boundary that moves with supersonic velocity; this reflection was considered in[2]. In the case of plasma waves, a similar effect was considered recently by Ostrovskiĭ[3]. Mention can also be made of earlier studies dealing with the motion of a conducting liquid in a resonator[4] or the motion of carriers in the interior of an elastic piezoelectric[5] or over its surface[6,7]. In all cases it is important that the linear velocity of the substance from which the energy is drawn must exceed the phase velocity of the corresponding oscillations.

For electromagnetic waves in vacuum and in plane geometry, this is impossible (it would require a superluminal velocity $u > c$). In a cylindrical or spherical geometry, however, for a wave with a large multipole moment n, the angular phase velocity is equal to ω/n. The angular velocity of the body Ω can exceed the corresponding value of ω/n. The linear velocity of the surface of the body is $u = \Omega r_0 = \beta c$, $\beta < 1$, $r_0 < c/\Omega$, and the body is entirely in the non-wave zone ($r_0 < \lambda n = nc/\omega$), i.e., in a place where the fields of the wave decrease like high powers of r^n. Thus, the amplification effect, while possible in principle, is small.

In terms of the coordinates z, r, and φ, a cylindrical wave with given multipolarity and frequency takes the form

$$e^{-i\omega t+in\varphi}f(r),$$

and for an incident wave we have asymptotically $f(r) = \mathrm{const}\cdot e^{-ikr}/\sqrt{r}$. At a given instant ($t = \mathrm{const}$), the constant-phase lines (in the plane $z = \mathrm{const}$) are the spirals $n\varphi - kr = \mathrm{const}$. The normal to such a spiral occurs at a distance $R = n/k = nc/\omega$ from the axis. In quantum language, the radiation of the photons in the induced regime is small by virtue of the need for overcoming the potential barrier of the centrifugal forces connected with the angular momentum $n\hbar$ of one photon; this angular momentum corresponds to an impact parameter $R = nc/\omega$, which characterizes the width of the barrier.

The incident wave carries a definite energy flux (W erg/sec-cm), normalized to unit length along the z axis. It carries also a definite angular-momentum flux L_z which can be calculated from the stress tensor in exactly the same manner as W is calculated from the energy-flux vector. It is easy to verify that

$$L_z = WR/c = Wn/\omega \ \ \mathrm{erg/cm}$$

Let us assume that the rotating body has absorbed the energy of the incident wave almost completely. We would then have for the total energy of the body E, for its angular momentum M_z, and for the rotation energy E_{rot} the relations

$$dE/dt = W, \quad dM_z/dt = L_z = Wn/\omega,$$

$$E_{\mathrm{rot}} = \frac{M_z^2}{2I}, \quad \frac{dE_{\mathrm{rot}}}{dt} = \frac{M_z}{I}L_z = \Omega L_z = \frac{\Omega n}{\omega}W.$$

Consequently, when $\Omega n/\omega > 1$ we have

$$dE_{\mathrm{rot}}/dt > dE/dt,$$

which calls for a decrease in the entropy of the body, and is obviously impossible.

Is partial absorption and partial scattering of the wave possible?

It is important that we are considering a cylindrical body rotating about the symmetry axis. Consequently, terms describing the interaction of the field with the body are invariant against rotation ($\varphi \to \varphi + d\varphi$) and time shift ($t \to t + dt$). Therefore the frequency and the multipolarity of the scattered waves must coincide with the frequency and multipolarity of the incident wave; it was already noted earlier that no Doppler effect takes place[1].

In such a case, however, the asymptotic form of the scattered wave is

$$\mathrm{const}\cdot\exp(-i\omega t + in\varphi + ikr)/\sqrt{r}$$

and the ratio of the energy carried away (W_1) to the momentum (L_{z1}) is the same for the scattered wave as for the incident wave. Thus, if we take reflection into account, we obtain for the total energy E of the body, for the rotation energy of the body E_{rot}, and for the thermal energy of the body E_{th}

$$dE/dt = W - W_1, \quad dE_{\mathrm{rot}}/dt = L_z - L_{z1} = (W - W_1)\Omega n/\omega,$$
$$dE_{\mathrm{th}}/dt = d(E - E_{\mathrm{rot}})/dt = (W - W_1)(1 - \Omega n/\omega).$$

From the condition that the entropy $dS = dE_{\mathrm{th}}/T$ must grow, it follows that $dE_{\mathrm{th}}/dt > 0$. We thus arrive at the conclusion that when $n\Omega > \omega$, when the second factor of the last formula is negative, we should have $W_1 > W$, i.e., only scattering with amplification of the cylindrical wave with given multipolarity is possible energy wise.

2. It can be assumed that the effect in question appears also when the interaction of the waves with the rotating body is connected with the gravitational field of this body and does not depend on dissipated processes in the material of the body.

Such a situation is encountered when the body is in the the state of relativistic collapse; for an introduction to the theory of this phenomenon, cf., e.g.,[8]. If the body does not rotate, a Schwarzschild metric is produced. It is known that in this case a converging wave of any frequency and multipolarity is reflected with a decreased amplitude.

In the case of collapse of a rotating body, a Kerr metric is produced on the outside (see[8] and the references contained therein), which depends not only on the mass but also on the angular momentum of the body. Recent investigations have shown that for such a body there exists a curve of reversible change of mass with changing angular momentum[8-10].

The derivative of the energy ($E = mc^2$) with respect to the momentum M is obviously the effective rotation velocity Ω, and the energy considerations developed above are fully applicable to this case. The effect depends on the change of the sign of g_{00} in the ergosphere.

A Kerr solution exists apparently only at an angular momentum $M \leq m^2G/c$, corresponding to an effective angular velocity $\Omega = c/2r_g$. The amplification condition is $\omega \lesssim \Omega_n$, corresponding to a radius $R \gtrsim cn/\omega \sim r_g$ of the transition from the near zone to the wave zone of the amplified wave. Consequently, in the limiting Kerr solution, at the optimal choice of wavelength $\lambda \sim r_g/n$, the gain does not contain any small parameter, but only numbers such as 2, 3, π, and the multipolarity n. At large n, the effect undoubtedly vanishes, since the transition to geometrical optics takes place. At n = 1 or 2, the effect may be of the order of unity.

We note one possibility that should be investigated. A rotating conducting body amplifies the reflected wave. If this body is surrounded by a semitransparent mirror with a surface having axial symmetry, then generation, or exponential amplification (depending on the time) of the wave field is possible in such a system in the presence of a diverging outgoing wave, without an incident wave.

It is not excluded, in principle, that in a Kerr metric, for certain values of the multipolarity and frequency, a gravitational field would not only amplify the wave, but also produce generation conditions. This phenomenon, if it exists, could be compared with the result of Press[11], who states that the Kerr metric is unstable. The instability corresponds to generation of gravitational waves superimposed on the unperturbed Kerr metric.

3. We turn to a concrete calculation of the phenomenon for electromagnetic waves and a conducting body, without gravitation.

To prove the existence of the effect, a scalar field φ was considered in[1]. Here we consider the amplification of an electromagnetic cylindrical wave for two different polarizations, with the electric field along the axis and along the tangent to the circle z = const, r = const. The calculation is made in the limit of low electric conductivity σ; we assume $\epsilon = \mu = 1$ and obtain the field in vacuum in the zeroth approximation.

Let us verify the presence of the effect for a cylindrical wave (coordinates r, φ, z) with a polarization such that **E** is parallel to z everywhere, and **H** is directed along φ at $r \gg \lambda$. Near the axis, the solution has the following form for vacuum ($c \equiv 1$)

$$E_z = A\omega r^n \cos(n\varphi - \omega t),$$
$$H_\varphi = -Anr^{n-1} \sin(n\varphi - wt), \quad H_r = Anr^{n-1} \cos(n\varphi - \omega t);$$

the solution far from the axis is

$$H_\varphi = E_z = [\cos(-\omega r + n\varphi - \omega t + \delta_{in}) + \cos(\omega r + n\varphi - \omega t + \delta_{out})]/\sqrt{r}.$$

In the near solution we have discarded terms of higher order (r^{n+1} ... in **E** and r^n ... in **H**). In the far solution we have discarded small quantities ($r^{-3/2}$ in **E** and **H**; in particular, H_r is of the same order of smallness). The coefficient A and the phase shifts δ_{in} and δ_{out} follow from an analysis of the Bessel functions. There is no need to write them out, since they are not required for the qualitative study of the problem. We calculate the current from the formula $j = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{H})$ using the field in vacuum at $r \ll \lambda$. This is an expansion in terms of the small σ: the difference between the field and the vacuum field is $\sim j \sim \sigma$, the change of j is in this case $\sim \sigma^2$ and is discarded. Obviously, **v** has only one component $v_\varphi = \Omega r$; hence

$$j_z = \sigma A r^n \cos(n\varphi - \omega t)(\omega - n\Omega).$$

This means that the sign of the work of the field changes together with the characteristic quantity $\omega - n\Omega$.

We shall not calculate the amplitude of the reflected wave explicitly. However, from the conservation of the electromagnetic-field energy it is clear that in the next approximation, taking into account the radiation of the current j, the answer in the far zone is

$$[\cos(-\omega r + n\varphi - \omega t + \delta_{in}') + c\cos(\omega r + n\varphi - \omega t + \delta_{out})]/\sqrt{r},$$

with

$$c^2 = 1 - \text{const}\overline{\int \mathbf{jE}\, dV}.$$

This means that the amplitude of the reflected wave is

$$1 - \sigma(\omega - n\Omega) r_0^{2n+2} \cdot \text{const.}$$

and if $\omega - n\Omega < 0$ we have amplification.

We consider the case of the other polarization at the same multipolarity. The far field ($r \gg \lambda$) is

$$E_\varphi = H_z = [\cos(-\omega r + n\varphi - \omega t + \delta_{in}) + \cos(\omega r + n\varphi - \omega t + \delta_{out})]/\sqrt{r},$$

and the near field ($r \ll \lambda$):

$$H_z = A\omega r^n \cos(n\varphi - \omega t),$$
$$E_\varphi = Anr^{n-1} \sin(n\varphi - \omega t), \quad E_r = -Anr^{n-1} \cos(n\varphi - \omega t).$$

All the remaining components, which have not been written out, are small or are equal to zero. In this case we can assume H_z also to be small (of higher order in the small r near the axis[1)]), and all the more the contribution of $\mathbf{v} \times \mathbf{H}$ to the current, since $|\mathbf{v}| \sim r$ and $\mathbf{v} \times \mathbf{H}$

[1)]This means that one considers at each instant an electrostatic field div E = curl E having a given multipolarity and rotating slowly around the axis.

$\sim r^{n+1}$. Thus, we are left with $\mathbf{j} = \sigma \mathbf{E}$. But in fact the important factor here is the formation of free charges and the contribution made to the current by the dragging of the charges by the moving body (cf.[12]). In fact, in the region $r < r_0$, where $\sigma = \text{const} \neq 0$, we have $\operatorname{div} \mathbf{j} \equiv \sigma \operatorname{div} \mathbf{E} = 0$. On the boundary, however, at $r = r_0$, there is a discontinuity in $(\sigma(r_0 - 0) = \sigma, \sigma(r_0 + 0) = 0$, and here we get a surface charge density

$$\rho = \mu\delta(r - r_0),$$

$$d\mu/dt = j_r(r_0 - 0) = -A\sigma n r_0^{n-1} \cos(n\varphi - \omega t).$$

A. A. Starobinskiĭ has made the following important remark that d/dt is taken here over a moving surface:

$$d/dt = \partial/\partial t + \Omega\partial/\partial\varphi.$$

As a result, the answer takes the unusual form

$$\mu = \frac{A\sigma n r_0^{n-1}}{\omega - n\Omega} \sin(n\varphi - \omega t).$$

Obviously, the low-conductivity approximation is not valid at sufficiently small $\omega - n\Omega$. The case $\omega = n\Omega$ can be considered in elementary fashion: the field is constant in the system rotating with the body, meaning that at any conductivity σ the field will become screened by the surface charges after a sufficient time:

$$r \leqslant r_0, \quad E_r = E_\varphi = 0, \quad \mu_0 = -(An/2\pi) r_0^{n-1} \cos(n\varphi - \omega t),$$
$$r > r_0, \quad E_\varphi = An \sin(n\varphi - \omega t)(r^{n-1} - r_0^{2n} r^{-n-1}),$$
$$E_r = -An \cos(n\varphi - \omega t)(r^{n-1} + r_0^{2n} r^{-n-1}).$$

In this situation, since $E_\varphi(r_0) = 0$, the work of the current is equal to zero. The cylinder changes the phase but not the amplitude of the reflected wave, just as an ideally conducting ($\sigma = \infty$) cylinder at rest.

From the condition $|\mu| < |\mu_0|$ we can easily obtain the condition for the applicability of the preceding approximation of small σ: we obtain $|\omega - n\Omega| > 4\pi\sigma$. On the whole, we obtain for the change $\delta A/A$ of the amplitude of the reflected wave a relation in the form

$$\delta A / A = \text{const} \cdot x / (1 + x^2), \quad x = (n\Omega - \omega) / 4\pi\sigma$$

with a maximum that is closer to the critical velocity the smaller σ.

The expression for the current in the moving body contains a convective term

$$\mathbf{j} = \sigma(\mathbf{E} + [\mathbf{v} \times \mathbf{H}]) + \mathbf{v}\rho.$$

In this case we have

$$j_\varphi = An\sigma r^{n-1} \sin(n\varphi - \omega t) + \frac{\Omega A n \sigma r_0^n}{\omega - n\Omega} \sin(n\varphi - \omega t)\delta(r - r_0).$$

In calculating the work of the current

$$W = 2\pi \int \overline{\mathbf{jE}} r\, dr$$

It is precisely this last convective term which gives the effect: at $4\pi\sigma < |\omega - n\Omega|$ we have

$$W = \pi A^2 n \sigma r_0^{2n} \left[1 + \frac{n\Omega}{\omega - n\Omega}\right] = \pi A^2 n \sigma r_0^{2n} \frac{\omega}{\omega - n\Omega}.$$

Thus, the sign of W depends on $\omega - n\Omega$, or on the relation between the angle of velocity Ω of the body and the angular velocity ω/n of the field. In essence, $\omega < n\Omega$ (the amplification condition) corresponds to "superluminal velocity." Here "superluminal" rotation is realized at a linear velocity smaller than c, $\Omega r_0 > 1$ in our notation ($c = 1$), but at the expense of the fact that the body is in a non-wave zone, where $\mathbf{E}, \mathbf{H} \sim r^n, r^{n-1}$, and not $1/\sqrt{r}$ (see Sec. 1).

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