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PERTURBATIONS OF A ROTATING BLACK HOLE. I. FUNDAMENTAL EQUATIONS FOR GRAVITATIONAL, ELECTROMAGNETIC, AND NEUTRINO-FIELD PERTURBATIONS*

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ABSTRACT

This paper derives linear equations that describe dynamical gravitational, electromagnetic, and neutrino-field perturbations of a rotating black hole. The equations decouple into a single gravitational equation, a single electromagnetic equation, and a single neutrino equation. Each of these equations is completely separable into ordinary differential equations. The paper lays the mathematical groundwork for later papers in this series, which will deal with astrophysical applications: stability of the hole, tidal friction effects, superradiant scattering of electromagnetic waves, and gravitational-wave processes.
Subject headings: black holes — gravitation — neutrinos — relativity — rotation

I. INTRODUCTION

This is the first in a series of papers which will deal with dynamical processes near a rotating black hole. The underlying mathematical technique throughout the series is to linearize the Einstein or Maxwell-Einstein equations around a known stationary black-hole solution, in this case the Kerr (1963) metric. This technique goes beyond previous work in which a rotating black hole has been treated as a fixed geometrical background for physical processes: the linearized equations give the hole the full dynamical freedom of small perturbations, including the possibility of gravitational and electromagnetic waves, secular changes in its mass and angular momentum, interaction with accreting test matter or distant massive objects, and so on.

The fundamental perturbation equations which will be used throughout the series are derived in this paper; in form, the equations are separable partial differential equations whose independent variables are certain decoupled components of the Weyl or Riemann tensor, or of the electromagnetic field tensor. Some of the applications to be treated in subsequent papers make direct use of only these decoupled components. Other applications require that one consider *all* components of the electromagnetic or gravitational field. Here, a concentration of attention on only the decoupled components would not *a priori* seem to be justified. However, for both gravitational and electromagnetic perturbations, it can be proved that the decoupled components contain complete information about all nontrivial features of the full perturbing field; this completeness will be discussed in a subsequent paper. For the electromagnetic case the result is due to Fackerell and Ipser (1972); for the gravitational case it is due to Wald (1973).

How does one obtain linearized perturbation equations, say for gravitational perturbations? A straightforward way is to start with the Einstein equations for a metric $g_{\mu\nu}$, and to let $g_{\mu\nu} = g_{\mu\nu}{}^A + h_{\mu\nu}{}^B$, where the superscripts A and B denote background and perturbation quantities, respectively. The field equations are then expanded to first order in $h_{\mu\nu}{}^B$, yielding a set of linear equations for the perturbations.

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This method has already been applied to the Schwarzschild metric (Regge and Wheeler 1957; Vishveshwara 1970; Zerilli 1970). In this case, the background metric is static and spherically symmetric, so the time and angular dependence can easily be separated out of the equations. The resulting coupled radial equations can then be reduced to two decoupled equations, one governing odd-parity perturbations (Regge and Wheeler 1957) and the other governing even-parity perturbations (Zerilli 1970).

Even in the Schwarzschild case, this procedure involves considerable algebraic complexity. In the Kerr case, where the background metric is much more complicated, nobody has carried out a similar program. Moreover, the replacement of spherical symmetry by axial symmetry means that a separation into spherical harmonics is no longer possible; one expects to end up with partial differential equations in r and θ instead of ordinary differential equations in r.

Fortunately, there is an alternative approach to the problem. This is provided by the Newman-Penrose (NP) formalism. We shall use the notation of Newman and Penrose (1962), and equations from that paper will be cited as NP 2.1 and so forth. The NP formalism arises naturally from the introduction of spinor calculus into general relativity. It can also be regarded as a special type of tetrad calculus. Four null vectors, conventionally called $\boldsymbol{l}$, $\boldsymbol{n}$, $\boldsymbol{m}$, and $\boldsymbol{m}^*$, are chosen at every point of spacetime. (An asterisk denotes complex conjugation. The vectors $\boldsymbol{l}$ and $\boldsymbol{n}$ are real.) All tensors are projected onto the null tetrad. The full set of NP equations is a system of coupled first-order differential equations linking the tetrad, the spin coefficients (essentially Ricci rotation coefficients), the Weyl tensor, the Ricci tensor, and the scalar curvature. To do perturbation theory in this formalism, one specifies the perturbed geometry by $\boldsymbol{l} = \boldsymbol{l}^A + \boldsymbol{l}^B$, $\boldsymbol{n} = \boldsymbol{n}^A + \boldsymbol{n}^B$, etc. All the NP quantities can then be written in this form: $\psi_2 = \psi_2{}^A + \psi_2{}^B$, $D = D^A + D^B$, etc. The complete set of perturbation equations is obtained from the NP equations by keeping B terms only to first order.

In the Schwarzschild case, this program has been carried out by Price (1972) and extended by Bardeen and Press (1973). The most important result of this approach is a decoupled equation for each of two components of the Weyl tensor, $\psi_0{}^B$ and $\psi_4{}^B$. As mentioned, it turns out that each of these quantities alone contains complete information about all nontrivial perturbations.

The Schwarzschild and Kerr metrics are very similar from the NP point of view. This similarity allows us, in this paper, to derive decoupled Kerr-metric equations for $\psi_0{}^B$ and $\psi_4{}^B$. Moreover, we shall demonstrate the unexpected result that these equations, like those for Schwarzschild, are separable.

Some of the results in this paper have been reported without proof in a short letter (Teukolsky 1972). The purpose of this paper is to present the results in greater detail, with full derivations, and in a form which will lay the foundation for the applications to be discussed in subsequent papers of this series.

The plan of the paper is as follows: In § II the decoupled gravitational perturbations equations are derived using the NP formalism. Section III derives decoupled equations for electromagnetic test fields. In § IV the equations are separated and written as a single master equation. Section V discusses the physical boundary conditions associated with the equations, and how to calculate the energy flux and polarization of gravitational and electromagnetic waves. Section VI previews applications of the equations to astrophysical problems. Appendix B treats the neutrino equation in the Kerr background. Readers unfamiliar with the NP formalism may skip §§ II and III and treat the first few equations of § IV as definitions of the NP quantities in terms of more familiar tensor quantities.

For reference, we give the definitions of the NP quantities on which attention will be focused in this paper. The electromagnetic field is characterized by the three complex quantities

$$\phi_0 = F_{\mu\nu} l^\mu m^\nu\,, \qquad \phi_1 = \tfrac{1}{2} F_{\mu\nu}(l^\mu n^\nu + m^{*\mu} m^\nu)\,, \qquad \phi_2 = F_{\mu\nu} m^{*\mu} n^\nu\,, \tag{1.1}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor. Equivalently,

$$F_{\mu\nu} = 2[\phi_1(n_{[\mu}l_{\nu]} + m_{[\mu}m^*{}_{\nu]}) + \phi_2 l_{[\mu}m_{\nu]} + \phi_0 m^*{}_{[\mu}n_{\nu]}] + \text{c.c.}\,, \tag{1.2}$$

where square brackets on subscripts denote antisymmetrization, and where "c.c." denotes "complex conjugate of the preceding terms." The gravitational quantities of interest will be

$$\psi_0 = -C_{\alpha\beta\gamma\delta}l^\alpha m^\beta l^\gamma m^\delta\,, \qquad \psi_4 = -C_{\alpha\beta\gamma\delta}n^\alpha m^{*\beta} n^\gamma m^{*\delta}\,, \tag{1.3}$$

where $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor, which is equal to the Riemann tensor in vacuum.

II. DECOUPLED GRAVITATIONAL EQUATIONS

The derivation in this section applies to any Type D vacuum background metric. (The Schwarzschild and Kerr solutions are both of this type.) Choose the $\boldsymbol{l}$ and $\boldsymbol{n}$ vectors of the unperturbed tetrad along the repeated principal null directions of the Weyl tensor. Then

$$\begin{gathered} \psi_0{}^A = \psi_1{}^A = \psi_3{}^A = \psi_4{}^A = 0\,, \\ \kappa^A = \sigma^A = \nu^A = \lambda^A = 0\,. \end{gathered} \tag{2.1}$$

Now consider the following three nonvacuum NP equations, taken from Pirani (1964):

$$\begin{aligned} (\delta^* - 4\alpha + \pi)\psi_0 - (D - 4\rho - 2\epsilon)\psi_1 - 3\kappa\psi_2 = (\delta + \pi^* - 2\alpha^* - 2\beta)\Phi_{00} \\ - (D - 2\epsilon - 2\rho^*)\Phi_{01} + 2\sigma\Phi_{10} - 2\kappa\Phi_{11} - \kappa^*\Phi_{02}\,, \end{aligned} \tag{2.2}$$

$$\begin{aligned} (\Delta - 4\gamma + \mu)\psi_0 - (\delta - 4\tau - 2\beta)\psi_1 - 3\sigma\psi_2 = (\delta + 2\pi^* - 2\beta)\Phi_{01} \\ - (D - 2\epsilon + 2\epsilon^* - \rho^*)\Phi_{02} - \lambda^*\Phi_{00} + 2\sigma\Phi_{11} - 2\kappa\Phi_{12}\,, \end{aligned} \tag{2.3}$$

$$(D - \rho - \rho^* - 3\epsilon + \epsilon^*)\sigma - (\delta - \tau + \pi^* - \alpha^* - 3\beta)\kappa - \psi_0 = 0\,. \tag{2.4}$$

The Ricci tensor terms on the right-hand sides of equations (2.2) and (2.3) are given by the Einstein field equations:

$$\Phi_{00} \equiv -\tfrac{1}{2}R_{\mu\nu}l^\mu l^\nu = 4\pi T_{\mu\nu}l^\mu l^\nu \equiv 4\pi T_{ll}\,, \tag{2.5}$$

and so on, where $R_{\mu\nu}$ is the Ricci tensor and $T_{\mu\nu}$ the stress-energy tensor.

Since $\psi_0{}^A$, $\psi_1{}^A$, σ^A, κ^A, and all the Φ^A vanish, the perturbation equations corresponding to equations (2.2)–(2.4) are

$$\begin{aligned} (\delta^* - 4\alpha + \pi)^A\psi_0{}^B - (D - 4\rho - 2\epsilon)^A\psi_1{}^B - 3\kappa^B\psi_2{}^A = 4\pi[(\delta + \pi^* - 2\alpha^* - 2\beta)^A T_{ll}{}^B \\ - (D - 2\epsilon - 2\rho^*)^A T_{lm}{}^B]\,, \end{aligned} \tag{2.6}$$

$$\begin{aligned} (\Delta - 4\gamma + \mu)^A\psi_0{}^B - (\delta - 4\tau - 2\beta)^A\psi_1{}^B - 3\sigma^B\psi_2{}^A = 4\pi[(\delta + 2\pi^* - 2\beta)^A T_{lm}{}^B \\ - (D - 2\epsilon + 2\epsilon^* - \rho^*)^A T_{mm}{}^B]\,, \end{aligned} \tag{2.7}$$

$$(D - \rho - \rho^* - 3\epsilon + \epsilon^*)^A\sigma^B - (\delta - \tau + \pi^* - \alpha^* - 3\beta)^A\kappa^B - \psi_0{}^B = 0\,. \tag{2.8}$$

To simplify the notation, the labels A will now be dropped from all unperturbed quantities.

The background ψ_2 satisfies

$$D\psi_2 = 3\rho\psi_2 \,, \qquad \delta\psi_2 = 3\tau\psi_2 \,. \tag{2.9}$$

Hence equation (2.8) gives

$$(D - 3\epsilon + \epsilon^* - 4\rho - \rho^*)\psi_2\sigma^B - (\delta + \pi^* - \alpha^* - 3\beta - 4\tau)\psi_2\kappa^B - \psi_0{}^B\psi_2 = 0 \,. \tag{2.10}$$

The key step in the derivation is to eliminate $\psi_1{}^B$ from equations (2.6) and (2.7). This is most easily effected by using the following commutation relation:

$$\begin{aligned}[][D - (p + 1)\epsilon + \epsilon^* + q\rho - \rho^*](\delta - p\beta + q\tau) \\ - [\delta - (p + 1)\beta - \alpha^* + \pi^* + q\tau](D - p\epsilon + q\rho) = 0 \,, \end{aligned} \tag{2.11}$$

where p and q are any two constants. This relation holds in any Type D metric, where equations (2.1) hold, and can be proved using equations (NP 4.4), (NP 4.2c), (NP 4.2e), and (NP 4.2k).

Operate with $(D - 3\epsilon + \epsilon^* - 4\rho - \rho^*)$ on equation (2.7) and with $(\delta + \pi^* - \alpha^* - 3\beta - 4\tau)$ on equation (2.6), and subtract one equation from the other. The terms in $\psi_1{}^B$ then vanish by equation (2.11) with $p = 2$ and $q = -4$. The combination of σ^B and κ^B remaining is exactly that in equation (2.10), and so both of these quantities can be eliminated in favor of $\psi_2\psi_0{}^B$. The resulting equation is:

$$\begin{aligned}[][(D - 3\epsilon + \epsilon^* - 4\rho - \rho^*)(\Delta - 4\gamma + \mu) \\ - (\delta + \pi^* - \alpha^* - 3\beta - 4\tau)(\delta^* + \pi - 4\alpha) - 3\psi_2]\psi_0{}^B = 4\pi T_0 \,, \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} T_0 = {} & (\delta + \pi^* - \alpha^* - 3\beta - 4\tau)[(D - 2\epsilon - 2\rho^*)T_{lm}{}^B - (\delta + \pi^* - 2\alpha^* - 2\beta)T_{ll}{}^B] \\ & + (D - 3\epsilon + \epsilon^* - 4\rho - \rho^*)[(\delta + 2\pi^* - 2\beta)T_{lm}{}^B - (D - 2\epsilon + 2\epsilon^* - \rho^*)T_{mm}{}^B] \,. \end{aligned} \tag{2.13}$$

This is the decoupled equation for $\psi_0{}^B$. The full set of NP equations is invariant under the interchange $\boldsymbol{l} \leftrightarrow \boldsymbol{n}$, $\boldsymbol{m} \leftrightarrow \boldsymbol{m}^*$ (Geroch, Held, and Penrose 1972). This symmetry is not destroyed by the choice of $\boldsymbol{l}$ and $\boldsymbol{n}$ which gave equations (2.1). We can therefore derive an equation for $\psi_4{}^B$ by applying this transformation to equations (2.12) and (2.13):

$$\begin{aligned}[][(\Delta + 3\gamma - \gamma^* + 4\mu + \mu^*)(D + 4\epsilon - \rho) \\ - (\delta^* - \tau^* + \beta^* + 3\alpha + 4\pi)(\delta - \tau + 4\beta) - 3\psi_2]\psi_4{}^B = 4\pi T_4 \,, \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} T_4 = {} & (\Delta + 3\gamma - \gamma^* + 4\mu + \mu^*)[(\delta^* - 2\tau^* + 2\alpha)T_{nm*} - (\Delta + 2\gamma - 2\gamma^* + \mu^*)T_{m*m*}] \\ & + (\delta^* - \tau^* + \beta^* + 3\alpha + 4\pi)[(\Delta + 2\gamma + 2\mu^*)T_{nm*} - (\delta^* - \tau^* + 2\beta^* + 2\alpha)T_{nn}] \,. \end{aligned} \tag{2.15}$$

For those readers familiar with the Geroch-Held-Penrose (1972) version of the NP formalism, the derivation in this section is even simpler in that formalism. An equivalent derivation in that formalism has been given by Stewart (1972).

Appendix A proves that $\psi_0{}^B$ and $\psi_4{}^B$ are invariant under gauge transformations and infinitesimal tetrad rotations, and are therefore completely measurable physical quantities.

III. DECOUPLED ELECTROMAGNETIC EQUATIONS

Many realistic problems involving electromagnetic interactions near uncharged black holes can be treated in the "test field" approximation. Since the amplitude of the electromagnetic stress-energy is second order in the electromagnetic field, the change in the background geometry caused by the electromagnetic perturbation is also second order. Thus in Maxwell's equations this change in the geometry can be neglected to first order.

When equations (2.1) are satisfied, Maxwell's equations are

$$(D - 2\rho)\phi_1 - (\delta^* + \pi - 2\alpha)\phi_0 = 2\pi J_l\,, \tag{3.1}$$

$$(\delta - 2\tau)\phi_1 - (\Delta + \mu - 2\gamma)\phi_0 = 2\pi J_m\,, \tag{3.2}$$

$$(D - \rho + 2\epsilon)\phi_2 - (\delta^* + 2\pi)\phi_1 = 2\pi J_{m*}\,, \tag{3.3}$$

$$(\delta - \tau + 2\beta)\phi_2 - (\Delta + 2\mu)\phi_1 = 2\pi J_n\,, \tag{3.4}$$

where the ϕ's are the first-order test fields and $J_l = J_\mu l^\mu$, etc., with J_μ the 4-current density.

Operate on equation (3.1) with $(\delta - \beta - \alpha^* - 2\tau + \pi^*)$ and on equation (3.2) with $(D - \epsilon + \epsilon^* - 2\rho - \rho^*)$, and subtract one equation from the other. The identity (2.11) with $p = 0$ and $q = -2$ shows that the terms in ϕ_1 disappear, leaving a decoupled equation for ϕ_0:

$$\begin{aligned}[(D - \epsilon + \epsilon^* - 2\rho - \rho^*)(\Delta + \mu - 2\gamma)\\ - (\delta - \beta - \alpha^* - 2\tau + \pi^*)(\delta^* + \pi - 2\alpha)]\phi_0 = 2\pi J_0\,,\end{aligned} \tag{3.5}$$

$$J_0 = (\delta - \beta - \alpha^* - 2\tau + \pi^*)J_l - (D - \epsilon + \epsilon^* - 2\rho - \rho^*)J_m\,. \tag{3.6}$$

By interchanging $\boldsymbol{l}$ and $\boldsymbol{n}$, and $\boldsymbol{m}$ and $\boldsymbol{m}^*$, we obtain the equation for ϕ_2 (which is also derivable directly from eqs. [3.3] and [3.4]):

$$\begin{aligned}[(\Delta + \gamma - \gamma^* + 2\mu + \mu^*)(D - \rho + 2\epsilon)\\ - (\delta^* + \alpha + \beta^* + 2\pi - \tau^*)(\delta - \tau + 2\beta)]\phi_2 = 2\pi J_2\,,\end{aligned} \tag{3.7}$$

$$J_2 = (\Delta + \gamma - \gamma^* + 2\mu + \mu^*)J_{m*} - (\delta^* + \alpha + \beta^* + 2\pi - \tau^*)J_n\,. \tag{3.8}$$

Fackerell and Ipser (1972) derived an analogous decoupled equation for ϕ_1, but this equation does not appear to be separable in the Kerr case.

IV. SEPARATION OF THE EQUATIONS

The next step is to write out the equations in a particular coordinate system. In Boyer-Lindquist (1967) coordinates, and in units such that $c = G = 1$, the Kerr metric is

$$\begin{aligned}ds^2 = (1 - 2Mr/\Sigma)dt^2 + (4Mar\sin^2(\theta)/\Sigma)dtd\varphi - (\Sigma/\Delta)dr^2 - \Sigma d\theta^2\\ - \sin^2(\theta)(r^2 + a^2 + 2Ma^2r\sin^2(\theta)/\Sigma)d\varphi^2\,.\end{aligned} \tag{4.1}$$

Here M is the mass of the black hole, aM its angular momentum, $\Sigma = r^2 + a^2 \cos^2 \theta$, and[1] $\Delta = r^2 - 2Mr + a^2$. When $a = 0$, the metric reduces to the Schwarzschild metric, a nonrotating black hole.

Any NP tetrad must satisfy the following orthogonality relations:

$$\boldsymbol{l}\cdot\boldsymbol{n} = 1\,, \qquad \boldsymbol{m}\cdot\boldsymbol{m}^* = -1\,, \qquad \text{all other dot products zero}\,, \tag{4.2}$$

so the metric is

$$g^{\mu\nu} = l^\mu n^\nu + n^\mu l^\nu - m^\mu m^{*\nu} - m^{*\mu} m^\nu\,. \tag{4.3}$$

The relations (4.2) are preserved under the 6-parameter group of Lorentz transformations at each point of spacetime. A convenient decomposition of these six degrees of freedom is described in Appendix A. Choosing the directions of $\boldsymbol{l}$ and $\boldsymbol{n}$ so that equations (2.1) hold uses up four degrees of freedom (eqs. [A1] and [A2]). We choose to follow Kinnersley (1969) and use up the remaining freedom by making a "null rotation" (eq. [A3]) to set the spin coefficient $\epsilon = 0$. The resulting tetrad has $[t, r, \theta, \varphi]$ components:

$$l^\mu = [(r^2 + a^2)/\Delta, 1, 0, a/\Delta]\,, \qquad n^\mu = [r^2 + a^2, -\Delta, 0, a]/(2\Sigma)\,,$$
$$m^\mu = [ia \sin\theta, 0, 1, i/\sin\theta]/[2^{1/2}(r + ia\cos\theta)]\,. \tag{4.4}$$

The nonvanishing spin coefficients are

$$\rho = -1/(r - ia\cos\theta)\,, \qquad \beta = -\rho^* \cot\theta/(2\sqrt{2})\,, \qquad \pi = ia\rho^2 \sin\theta/\sqrt{2}\,,$$
$$\tau = -ia\rho\rho^* \sin\theta/\sqrt{2}\,,\ \mu = \rho^2\rho^*\Delta/2\,,\ \gamma = \mu + \rho\rho^*(r - M)/2\,,\ \alpha = \pi - \beta^*\,, \tag{4.5}$$

while

$$\psi_2 = M\rho^3\,. \tag{4.6}$$

We use these expressions, and the fact that[2] $D = l^\mu \partial/\partial x^\mu$, $\Delta = n^\mu \partial/\partial x^\mu$, and $\delta = m^\mu \partial/\partial x^\mu$, to write equations (2.12), (2.14), (3.5), and (3.7) as a single master equation—valid equally well for a test scalar field in the Kerr background ($s = 0$, not derived here), a test neutrino field ($s = \pm 1/2$, derived in Appendix B), a test electromagnetic field ($s = \pm 1$, derived in § III), or a gravitational perturbation ($s = \pm 2$, derived in § II):

$$\left[\frac{(r^2 + a^2)^2}{\Delta} - a^2 \sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\varphi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial\varphi^2}$$
$$- \Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - 2s\left[\frac{a(r - M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\varphi}$$
$$- 2s\left[\frac{M(r^2 - a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\psi}{\partial t} + (s^2\cot^2\theta - s)\psi = 4\pi\Sigma T\,. \tag{4.7}$$

Here s is a parameter called the "spin weight" of the field. Table 1 specifies the field quantities ψ which satisfy this equation, the corresponding values of s, and the source terms T.

[1] In §§ II and III, Δ denoted the NP operator $n^\mu\partial/\partial x^\mu$. In the remainder of the paper Δ is used in its other conventional sense, to denote the function $r^2 - 2Mr + a^2$ unless otherwise noted.

[2] See n. 1.

TABLE 1

FIELD QUANTITIES ψ, SPIN-WEIGHT s, AND SOURCE TERMS T FOR EQUATION (4.7)

ψ	s	T
Φ	0	$\Box\Phi = 4\pi T$
χ_0 $\rho^{-1}\chi_1$	$\frac{1}{2}$ $-\frac{1}{2}$	See references in Appendix B
ϕ_0 $\rho^{-2}\phi_2$	1 -1	J_0 (eq. [3.6]) $\rho^{-2}J_2$ (eq. [3.8])
$\psi_0{}^{\mathrm{B}}$ $\rho^{-4}\psi_4{}^{\mathrm{B}}$	2 -2	$2T_0$ (eq. [2.13]) $2\rho^{-4}T_4$ (eq. [2.15])

Consider first the vacuum case ($T = 0$). Then the master equation (4.7) can be separated by writing

$$\psi = e^{-i\omega t}e^{im\varphi}S(\theta)R(r)\,. \tag{4.8}$$

The equations for R and S are

$$\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{dR}{dr}\right) + \left(\frac{K^2 - 2is(r-M)K}{\Delta} + 4is\omega r - \lambda\right)R = 0\,, \tag{4.9}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right) + \left(a^2\omega^2\cos^2\theta - \frac{m^2}{\sin^2\theta} - 2a\omega s\cos\theta - \frac{2ms\cos\theta}{\sin^2\theta} - s^2\cot^2\theta + s + A\right)S = 0\,, \tag{4.10}$$

where $K \equiv (r^2 + a^2)\omega - am$ and $\lambda \equiv A + a^2\omega^2 - 2am\omega$. Equation (4.10), together with boundary conditions of regularity at $\theta = 0$ and π, constitutes a Sturm-Liouville eigenvalue problem for the separation constant $A = {}_sA^m{}_l(a\omega)$. For fixed s, m, and $a\omega$, we label the eigenvalues by l. The smallest eigenvalue has $l = \max(|m|, |s|)$. From Sturm-Liouville theory, the eigenfunctions ${}_sS^m{}_l$ are complete and orthogonal on $0 \le \theta \le \pi$ for each m, s, and $a\omega$. When $s = 0$, the eigenfunctions are the spheroidal wave functions $S^m{}_l(-a^2\omega^2, \cos\theta)$ (cf. Flammer 1957). When $a\omega = 0$, the eigenfunctions are the spin-weighted spherical harmonics ${}_sY^m{}_l = {}_sS^m{}_l(\theta)e^{im\varphi}$, and $A = (l - s)(l + s + 1)$ (cf. Goldberg *et al.* 1967). In the general case, we shall refer to the eigenfunctions as "spin-weighted spheroidal harmonics." The numerical calculation of these functions and the corresponding eigenvalues is described in Paper II of this series.

When sources are present ($T \neq 0$), we can use the eigenfunctions of equation (4.10) to separate equation (4.7) by expanding

$$4\pi\Sigma T = \int d\omega \sum_{l,m} G(r)_sS^m{}_l(\theta)e^{im\varphi}e^{-i\omega t}\,,$$

$$\psi = \int d\omega \sum_{l,m} R(r)_sS^m{}_l(\theta)e^{im\varphi}e^{-i\omega t}\,. \tag{4.11}$$

Then $R(r)$ satisfies equation (4.9) with $G(r)$ as source term on the right-hand side.

Equation (4.7) is also separable in Kerr coordinates (cf. eq. [5.7]), or any other coordinates related to Boyer-Lindquist by $\bar{t} = t + f_1(r) + f_2(\theta)$, $\bar{\varphi} = \varphi + g_1(r) + g_2(\theta)$, $\bar{r} = h(r)$, $\bar{\theta} = j(\theta)$.

The reason for the factors ρ^{-2} and ρ^{-4} in front of ϕ_2 and $\psi_4{}^B$ to achieve separable equations (cf. table 1) is related to the null rotation used to set $\epsilon = 0$. Had we made some other choice, there would in general be different factors in front of each of ϕ_0, ϕ_2, $\psi_0{}^B$, and $\psi_4{}^B$, but the master perturbation equation (4.7) would be left unchanged. (See Appendix A for the transformation properties of these quantities under null rotations.)

V. BOUNDARY CONDITIONS, ENERGY, AND POLARIZATION

To discuss the boundary conditions for the separated radial equation (4.9), it is useful to make the transformation

$$Y = \Delta^{s/2}(r^2 + a^2)^{1/2}R\,, \qquad dr^*/dr = (r^2 + a^2)/\Delta\,. \tag{5.1}$$

Then

$$Y_{,r*r*} + \{[K^2 - 2is(r - M)K + \Delta(4ir\omega s - \lambda)]/(r^2 + a^2)^2 - G^2 - G_{,r*}\}Y = 0\,, \tag{5.2}$$

where $G = s(r - M)/(r^2 + a^2) + r\Delta/(r^2 + a^2)^2$ and a comma denotes partial differentiation. As $r \to \infty$ ($r^* \to \infty$), equation (5.2) becomes

$$Y_{,r*r*} + (\omega^2 + 2i\omega s/r)Y \approx 0\,, \tag{5.3}$$

with asymptotic solutions $Y \sim r^{\pm s}e^{\mp i\omega r*}$, i.e., $R \sim e^{-i\omega r*}/r$ and $e^{i\omega r*}/r^{(2s+1)}$. This corresponds to

$$\Phi, \phi_2, \psi_4{}^B \sim e^{i\omega r*}/r\,, \quad \phi_0 \sim e^{i\omega r*}/r^3\,, \quad \psi_0{}^B \sim e^{i\omega r*}/r^5 \quad \text{(outgoing waves)};$$

$$\Phi, \phi_0, \psi_0{}^B \sim e^{-i\omega r*}/r\,, \quad \phi_2 \sim e^{-i\omega r*}/r^3\,, \quad \psi_4{}^B \sim e^{-i\omega r*}/r^5 \quad \text{(ingoing waves)}\,. \tag{5.4}$$

The different power-law fall-offs are dictated by the "peeling theorem" (cf. Newman and Penrose 1962). They necessitate special care in numerical integration of the equations to avoid losing the small solution in the roundoff error of the large solution. Such an integration is described in Paper II.

The event horizon is at $r = r_+$ ($r^* \to -\infty$), the larger root of $\Delta = 0$. Near the event horizon the transformed radial equation (5.2) becomes

$$Y_{,r*r*} + [k^2 - 2is(r_+ - M)k/(2Mr_+) - s^2(r_+ - M)^2/(2Mr_+)^2]Y \approx 0\,, \tag{5.5}$$

where $k = \omega - m\omega_+$, $\omega_+ = a/(2Mr_+)$. The asymptotic solutions are

$$Y \sim e^{\pm i[k - is(r_+ - M)/(2Mr_+)]r*} \sim \Delta^{\pm s/2}e^{\pm ikr*}\,,$$

$$\text{i.e., } R \sim e^{ikr*} \quad \text{or} \quad R \sim \Delta^{-s}e^{-ikr*}\,. \tag{5.6}$$

The correct boundary condition at the horizon[3] can be formulated in a number of equivalent ways. For example, way (i): require that a physically well-behaved observer at the horizon see nonspecial fields. (Nonspecial means neither singular nor identically zero.) Equivalently, way (ii): demand that the radial group velocity of a wave packet,

[3] We discuss here only the future horizon; the past horizon need not even exist if the black hole was formed by collapse.

as measured by a physically well-behaved observer, be negative (i.e., signals can travel into the hole, but cannot come out).

Every physical observer with 4-velocity $\boldsymbol{u}$ has associated with him an orthonormal tetrad, his local rest-frame with basis vectors $\{\boldsymbol{e}_{\hat{t}} = \boldsymbol{u}, \boldsymbol{e}_{\hat{r}}, \boldsymbol{e}_{\hat{\theta}}, \boldsymbol{e}_{\hat{\varphi}}\}$. Corresponding to this is a null tetrad: $\boldsymbol{l} = \boldsymbol{e}_{\hat{t}} - \boldsymbol{e}_{\hat{r}}$, $\boldsymbol{n} = (\boldsymbol{e}_{\hat{t}} + \boldsymbol{e}_{\hat{r}})/2$, $\boldsymbol{m} = (\boldsymbol{e}_{\hat{\theta}} + i\boldsymbol{e}_{\hat{\varphi}})/2^{1/2}$. Conversely, given a nonsingular null tetrad, there is a corresponding physical observer. Thus condition (i) can be reformulated as: NP field quantities on the horizon should be nonspecial for nonsingular null tetrads.

To examine the tetrad (4.4) on the horizon, we cannot use Boyer-Lindquist coordinates since they themselves are singular on the horizon. Hence, we transform to Kerr "ingoing" coordinates (cf. Misner, Thorne, and Wheeler 1973):

$$dv = dt + dr^* \,,$$

$$d\tilde{\varphi} = d\varphi + a(r^2 + a^2)^{-1} dr^* \,. \tag{5.7}$$

The tetrad (4.4) is still singular at $\Delta = 0$ when expressed in these well-behaved coordinates, but if we perform a null rotation with $\Lambda = \Delta/2(r^2 + a^2)$ (cf. Appendix A), the resulting tetrad has $[v, r, \theta, \tilde{\varphi}]$ components

$$l^\mu = [1, \tfrac{1}{2}\Delta/(r^2 + a^2), 0, a/(r^2 + a^2)] \,, \qquad n^\mu = [0, -(r^2 + a^2)/\Sigma, 0, 0] \,,$$

$$m^\mu = [ia \sin\theta, 0, 1, i/\sin\theta]/[2^{1/2}(r + ia\cos\theta)] \,, \tag{5.8}$$

which show that it is well behaved at $\Delta = 0$. Under this null rotation, the NP quantities of interest transform as follows (cf. Appendix A):

$$\psi \to \psi^{\mathrm{New}} = [\tfrac{1}{2}\Delta/(r^2 + a^2)]^s \psi \,. \tag{5.9}$$

On the horizon, the asymptotic solutions (5.6) have the forms $\psi^{\mathrm{New}} \sim e^{-i\omega t} e^{im\varphi} e^{-ikr^*}$ and $e^{-i\omega t} e^{im\varphi} e^{ikr^*} \Delta^s$. Clearly the first solution is the nonspecial one, as can be seen by writing it in the form $e^{-i\omega v} e^{im\tilde{\varphi}}$. The correct boundary condition is therefore

$$R \sim \Delta^{-s} e^{-ikr^*} \,. \tag{5.10}$$

The group and phase velocities of this solution are

$$v_{\mathrm{group}} = -dk/d\omega = -1 \,, \qquad v_{\mathrm{phase}} = -k/\omega = -1 + m\omega_+/\omega \,. \tag{5.11}$$

The group velocity agrees with condition (ii) above. Note that if $m\omega_+/\omega > 1$, then v_{phase} is positive. It turns out that the energy flow down the hole, while always inward as seen locally, is determined by v_{phase} for an observer at infinity. If $m\omega_+/\omega > 1$, energy flows out of the hole and the corresponding scattering wave mode is amplified, or "superradiantly scattered" (cf. Press and Teukolsky 1972, Misner 1972, and Zel'dovich 1972). A detailed discussion of electromagnetic and gravitational superradiance, including numerical values, will be given in a later paper in this series.

Turn now to the problem of extracting information from solutions of the perturbation equations. For scalar and electromagnetic fields, there is a well-defined energy-momentum tensor at every point of spacetime:

$$4\pi T_{\mu\nu}{}^{(\mathrm{scalar})} = \Phi_{;\mu}\Phi_{;\nu} - \tfrac{1}{2} g_{\mu\nu}\Phi_{;\alpha}\Phi^{;\alpha} \,,$$

$$4\pi T_{\mu\nu}{}^{(\mathrm{em})} = \{\phi_0\phi_0{}^* n_\mu n_\nu + 2\phi_1\phi_1{}^*[l_{(\mu}n_{\nu)} + m_{(\mu}m^*{}_{\nu)}] + \phi_2\phi_2{}^* l_\mu l_\nu$$
$$- 4\phi_0{}^*\phi_1 n_{(\mu}m_{\nu)} - 4\phi_1{}^*\phi_2 l_{(\mu}m_{\nu)} + 2\phi_2\phi_0{}^* m_\mu m_\nu\} + \text{c.c.} \,, \tag{5.12}$$

where parentheses on subscripts denote symmetrization. Note that when one has solved equation (4.7) for ϕ_2, say, ϕ_1 and ϕ_0 can be found from equations (3.1)–(3.4) which are then integrable Pfaffian equations in r and θ (cf. Fackerell and Ipser 1972). The only arbitrariness in the solution is the freedom to add $Q\rho^2$ to ϕ_1, which corresponds to adding a constant charge Q to the hole.

Often one is interested only in the energy carried off by outgoing waves at infinity. Using equations (5.4) and (5.12), we find that the total energy flux per unit solid angle can be found from ϕ_2 alone:

$$\frac{d^2E}{dtd\Omega} = \lim_{r\to\infty} r^2 T^r{}_t = \lim_{r\to\infty} \frac{r^2}{2\pi} |\phi_2|^2 \,. \tag{5.13}$$

For outgoing waves at infinity, the components of the electric and magnetic fields satisfy $E_{\hat\theta} = B_{\hat\phi}$, $E_{\hat\phi} = -B_{\hat\theta}$, so from equation (1.1) we find $\phi_2 \propto E_{\hat\theta} - iE_{\hat\phi}$. Thus the squares of the real and imaginary parts of ϕ_2 are proportional to the amounts of energy in the two linear polarization states along the directions $\boldsymbol{e}_{\hat\theta}$ and $\boldsymbol{e}_{\hat\phi}$, respectively.

For gravitational waves, one could in principle proceed as follows: Having solved equation (4.7) for $\psi_4{}^B$, say, solve the complete set of (nonseparable) NP equations for the perturbations in the metric. Then use the Isaacson (1968) stress-energy tensor to determine the energy-momentum flux at any point in spacetime. Unfortunately the equations are so complicated that this is an impractical task. One can, however, find the energy flux in the two most important cases: at infinity and on the horizon.

At infinity, one can use the standard equations of linearized theory (cf. Misner *et al.* 1973) to find the energy flux. For outgoing waves with frequency ω,

$$\psi_4{}^B = -(R^B{}_{\hat t\hat\theta\hat t\hat\theta} - iR^B{}_{\hat t\hat\theta\hat t\hat\phi}) = -\omega^2(h^B{}_{\hat\theta\hat\theta} - ih^B{}_{\hat\theta\hat\phi})/2 \,.$$

Therefore,

$$\frac{d^2E^{(\mathrm{out})}}{dtd\Omega} = \lim_{r\to\infty} \frac{r^2\omega^2}{16\pi} [(h^B{}_{\hat\theta\hat\theta})^2 + (h^B{}_{\hat\theta\hat\phi})^2] = \lim_{r\to\infty} \frac{r^2}{4\pi\omega^2} |\psi_4{}^B|^2 \,. \tag{5.14}$$

The squares of the real and imaginary parts of $\psi_4{}^B$ are proportional to the amounts of energy in the linear polarization states along $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_{\hat\phi}$, and $\boldsymbol{e}_\theta \pm \boldsymbol{e}_{\hat\phi}$, respectively. Similar results hold for $\psi_0{}^B$ and ingoing waves:

$$\frac{d^2E^{(\mathrm{in})}}{dtd\Omega} = \lim_{r\to\infty} \frac{r^2}{64\pi\omega^2} |\psi_0{}^B|^2 \,. \tag{5.15}$$

The extra factor of 1/16 comes from the 1/2 in the definition of $\boldsymbol{n}$ as opposed to $\boldsymbol{l}$.

Some problems require one to be able to find the ingoing energy at infinity from $\psi_4{}^B$ (or the outgoing energy from $\psi_0{}^B$). The method for doing this will be given in Paper II.

To calculate the gravitational wave energy flux on the horizon, one can use the results of Hartle and Hawking (1972). From $\psi_0{}^B$ on the horizon one can find the shear σ^B of the horizon. The shear gives the rate of change of the area of the horizon, dA/dt. The quantity dA/dt contains two terms: dM/dt and da/dt. (See Hartle and Hawking 1972 for details.) In our case, $d(aM)/dt = (m/\omega)dM/dt$, thus enabling us to find both dM/dt and da/dt from $\psi_0{}^B$ on the horizon.

For a stationary, nonaxisymmetric perturbation ($\omega = 0$, $dM/dt = 0$, $m \neq 0$, $da/dt \neq 0$), the radial wave equation (4.9) can be solved in terms of hypergeometric functions. This enables one to calculate the spin-down (loss of angular momentum) of a rotating black hole caused by such a perturbation. [See analyses by Press 1972 (scalar perturbation) and Hartle 1973 (gravitational perturbation with $a \ll M$).] The calculation for arbitrary a will be published in a later paper in this series.

VI. DISCUSSION

The important result presented in this paper is that there exists a tractable method of treating perturbations of a rotating black hole. One has to solve a relatively simple ordinary differential equation, the radial wave equation (4.9), subject to boundary conditions described in § V. The solution lends itself to direct physical interpretation, and can be related at infinity to the energy flux of gravitational or electromagnetic waves. A subsequent paper in this series will discuss and apply to this work the stronger result (due to Fackerell and Ipser 1972 for the electromagnetic case, and due to Wald 1973 for the gravitational case) that the solution of equation (4.9) in fact determines *all* nontrivial details of the full perturbation, at all radii outside the horizon.

Later papers in this series will deal primarily with applications of the equations in astrophysical contexts, including the dynamical stability of the Kerr metric (Paper II), the superradiant scattering of electromagnetic and gravitational waves by an astrophysical black hole, the spin-down of an arbitrarily rotating hole which is perturbed non-axisymmetrically by a distant massive object, and calculations of the gravitational waves emitted by accretion processes.

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APPENDIX A

The 6-parameter group of homogeneous Lorentz transformations, which preserves the tetrad orthogonality relations (4.2), can be decomposed into three Abelian subgroups (Janis and Newman 1965):

i)
$$\begin{aligned} &\boldsymbol{l} \to \boldsymbol{l}\,, \\ &\boldsymbol{m} \to \boldsymbol{m} + d\boldsymbol{l}\,, \\ &\boldsymbol{n} \to \boldsymbol{n} + d\boldsymbol{m}^* + d^*\boldsymbol{m} + dd^*\boldsymbol{l}\,; \end{aligned} \tag{A1}$$

ii)
$$\begin{aligned} &\boldsymbol{n} \to \boldsymbol{n}\,, \\ &\boldsymbol{m} \to \boldsymbol{m} + e\boldsymbol{n}\,, \\ &\boldsymbol{l} \to \boldsymbol{l} + e\boldsymbol{m}^* + e^*\boldsymbol{m} + ee^*\boldsymbol{n}\,; \end{aligned} \tag{A2}$$

iii)
$$\begin{aligned} &\boldsymbol{l} \to \Lambda\boldsymbol{l}\,, \\ &\boldsymbol{n} \to \Lambda^{-1}\boldsymbol{n}\,, \\ &\boldsymbol{m} \to \exp(i\theta)\boldsymbol{m}\,; \end{aligned} \tag{A3}$$

where d and e are complex numbers and Λ and θ are real. Under transformations of type (i),

$$\begin{aligned} &\psi_0 \to \psi_0\,, \qquad \psi_4 \to \psi_4 + 4d^*\psi_3 + 6d^{*2}\psi_2 + 4d^{*3}\psi_1 + d^{*4}\psi_0\,, \\ &\phi_0 \to \phi_0\,, \qquad \phi_2 \to \phi_2 + 2d^*\phi_1 + d^{*2}\phi_0\,. \end{aligned} \tag{A4}$$

For type (ii),

$$\begin{aligned} &\psi_0 \to \psi_0 + 4e\psi_1 + 6e^2\psi_2 + 4e^3\psi_3 + e^4\psi_4\,, \qquad \psi_4 \to \psi_4\,, \\ &\phi_0 \to \phi_0 + 2e\phi_1 + e^2\phi_2\,, \qquad \phi_2 \to \phi_2\,. \end{aligned} \tag{A5}$$

For type (iii),

$$\psi_0 \to \Lambda^2 \exp(2i\theta)\psi_0\,, \qquad \psi_4 \to \Lambda^{-2} \exp(-2i\theta)\psi_4\,,$$
$$\phi_0 \to \Lambda \exp(i\theta)\phi_0\,, \qquad \phi_2 \to \Lambda^{-1} \exp(-i\theta)\phi_2\,. \tag{A6}$$

The above relations can be used to prove that $\psi_0{}^B$ and $\psi_4{}^B$ are invariant under infinitesimal tetrad transformations; for, suppose d, e, $\Lambda - 1$, and θ are infinitesimal. Then

$$\psi_0{}^B \to \psi_0{}^B\,, \qquad \psi_4{}^B \to \psi_4{}^B + 4d^*\psi_3{}^A \qquad [\text{type (i)}]\,,$$
$$\psi_0{}^B \to \psi_0{}^B + 4e\psi_1{}^A\,, \qquad \psi_4{}^B \to \psi_4{}^B \qquad [\text{type (ii)}]\,,$$
$$\psi_0{}^B \to \psi_0{}^B + 2[(\Lambda - 1) + i\theta]\psi_0{}^A\,, \qquad \psi_4{}^B \to \psi_4{}^B - 2[(\Lambda - 1) + i\theta]\psi_4{}^A \qquad [\text{type (iii)}]\,. \tag{A7}$$

Since $\psi_0{}^A = \psi_1{}^A = \psi_3{}^A = \psi_4{}^A = 0$, $\psi_0{}^B$ and $\psi_4{}^B$ are invariant.

The quantities $\psi_0{}^B$ and $\psi_4{}^B$ are also invariant under gauge transformations (i.e., infinitesimal changes of coordinates which leave the tetrad unchanged at each point of spacetime). Locally, these transformations are the inhomogeneous part of the Lorentz group:

$$x^\mu \to x^\mu + \xi^\mu \tag{A8}$$

where ξ^μ is infinitesimal. Since the ψ's are scalars, they change as a function of coordinate location by

$$\psi \to \psi - \psi_{,\mu}\xi^\mu\,. \tag{A9}$$

Therefore,

$$\psi^B \to \psi^B - \psi^A{}_{,\mu}\xi^\mu = \psi^B\,, \tag{A10}$$

since $\psi_4{}^A = \psi_0{}^A = 0$.

APPENDIX B

In this Appendix we shall show that the neutrino equation, in two-component form, also leads to a separable wave equation. We shall not give any discussion of the source terms here, nor of the physical interpretation of the solutions. For these, the interested reader may refer to Hartle (1970), and Wainwright (1971) and references therein.

The sourceless neutrino equation (no coupling to electrons or muons) is

$$\nabla^{AA'}\Phi_A = 0\,, \tag{B1}$$

where Φ_A is a two-component spinor. (Our notation follows Pirani 1964.) This equation can be written in NP form by letting χ_0 and χ_1 denote the components of Φ_A along the dyad legs o and ι, respectively. Then

$$(\delta^* - \alpha + \pi)\chi_0 = (D - \rho + \epsilon)\chi_1\,, \tag{B2}$$

$$(\Delta + \mu - \gamma)\chi_0 = (\delta + \beta - \tau)\chi_1\,. \tag{B3}$$

Now consider Φ_A as a test field on the Kerr background. Operate on equation (B3) with $(D + \epsilon^* - \rho - \rho^*)$ and on equation (B2) with $(\delta - \alpha^* - \tau + \pi^*)$, and subtract one equation from the other. The identity (2.11) with $p = -1$ and $q = -1$ shows that the terms in χ_1 disappear, leaving

$$[(D + \epsilon^* - \rho - \rho^*)(\Delta - \gamma + \mu) - (\delta - \alpha^* - \tau + \pi^*)(\delta^* - \alpha + \pi)]\chi_0 = 0\,. \quad \text{(B4)}$$

The interchange $\boldsymbol{l} \leftrightarrow \boldsymbol{n}$, $\boldsymbol{m} \leftrightarrow \boldsymbol{m}^*$ gives

$$[(\Delta - \gamma^* + \mu + \mu^*)(D + \epsilon - \rho) - (\delta^* + \beta^* + \pi - \tau^*)(\delta + \beta - \tau)]\chi_1 = 0\,. \quad \text{(B5)}$$

When written out in Boyer-Lindquist coordinates, these equations are of the same form as the master equation (4.7), with $\psi = \chi_0$ ($s = \frac{1}{2}$) and $\psi = \rho^{-1}\chi_1$ ($s = -\frac{1}{2}$).

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