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STABILITY OF STATIONARY, SPHERICAL ACCRETION ONTO A SCHWARZSCHILD BLACK HOLE[1]

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ABSTRACT

We formulate the general problem of perturbing a (non–self-gravitating) perfect fluid potential flow in an arbitrary background gravitational field. We then specialize to the case of perturbing stationary, spherical accretion onto a Schwarzschild black hole and derive the following stability results: (i) no unstable normal modes exist which extend outside the sound horizon of the background flow; and (ii) there are no unstable modes which represent a standing shock at the sound horizon.

We also derive the high-frequency (JWKB) approximation to traveling wave perturbations and show that these approximate solutions are regular across the black hole's event horizon.

Subject headings: black holes — hydrodynamics — stars: accretion

I. INTRODUCTION

In this paper we consider the adiabatic accretion of a perfect fluid onto a nonrotating black hole. We study especially the stability of those stationary, spherically symmetric solutions for which the density is nonzero and the matter is at rest at spatial infinity. Each of these solutions has a spherical sound horizon, outside the black hole's event horizon, across which the flow becomes supersonic.

We first show that if the entropy and vorticity perturbations are of bounded extent on some initial hypersurface, then they merely advect into the hole (with entropy driving the vorticity), leaving a pure potential flow perturbation in their wake. We then show that the potential perturbations (sound waves) have the following stability properties:

i) A suitable energy norm of the perturbation outside the sound horizon remains bounded by its initial value; in particular there are no unstable normal modes extending outside the sound horizon.

ii) There is no unstable normal mode corresponding to a standing shock at the sound horizon; thus unstable modes are excluded from the supersonic region as well.

We also give the high-frequency (JWKB) approximation to traveling wave perturbations and verify the regularity of these approximate solutions down to the black hole's event horizon.

We also discuss, in a qualitative way, the corresponding stability problem for potential flow onto a rotating black hole. In this case the sound horizon is surrounded by a region in which the superradiance of sound waves becomes possible. This region is somewhat analogous to the ergoregion of a Kerr black hole and indeed coincides with it in the special case of a stiff fluid ($p = \rho$) accretion.

Our motivation for studying the stability of spherical accretion flows lies in the possibility of their relevance to the rapid time variations of certain compact X-ray sources. The same methods used for the black hole problem can be adapted to the study of accretion onto a star or to the problem of a stellar wind. For these problems (except perhaps for neutron stars) a nonrelativistic treatment is usually sufficient. We therefore sketch the Newtonian analog of our relativistic method for completeness. The stellar accretion and wind problems have, of course, different boundary conditions and require independent stability analyses from that for the black hole. We shall briefly discuss some of the expected results for these cases.

The Newtonian stability problem for spherical accretion has been previously studied by a number of workers (see Garlick 1978; Petterson, Silk, and Ostriker 1978; Balazs 1972; Parker 1966; Bondi 1952). In addition there have been numerical analyses (see Stellingwerf and Buff 1977; Cowie, Stark, and Ostriker 1978). Most of this earlier work considers only the spherically symmetrical perturbations. Our work treats the general case and includes the relativistic effects needed for any discussion of black holes.

II. RELATIVISTIC POTENTIAL FLOW

a) *Background Solutions*

Consider the flow of a perfect fluid in an arbitrary gravitational field (with metric $g_{\mu\nu}$). The equations of motion are

$$T^{\mu\nu}{}_{;\nu} = 0\,, \qquad (nu^{\mu})_{;\mu} = 0\,, \tag{II-1}$$

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where

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}\,, \qquad u^\mu u_\mu = -1\,, \qquad dp = ndh - nTds\,, \qquad h = (p + \rho)/n\,. \tag{II-2}$$

Here n, p, ρ, h, T, s have their usual meanings, a semicolon signifies covariant differentiation with respect to $g_{\mu\nu}$, and the above equations imply

$$u^\mu s_{;\mu} = 0\,, \tag{II-3}$$

the absence of heat exchange between different elements of the fluid. We shall also neglect radiative cooling, magnetic effects, and the self-gravity of the fluid.

A particular solution to the above equations will be said to represent a (local) potential flow if it satisfies

$$s = s_0 = \text{const}\,, \qquad \tilde{\omega}_{\alpha\beta} = P_\alpha{}^\mu P_\beta{}^\nu \omega_{\mu\nu} = 0\,, \tag{II-4}$$

where

$$\omega_{\mu\nu} = (hu_\mu)_{;\nu} - (hu_\nu)_{;\mu} \tag{II-5}$$

and $P_\alpha{}^\beta$ is the projection tensor

$$P_\alpha{}^\beta = \delta_\alpha{}^\beta + u_\alpha u^\beta\,. \tag{II-6}$$

Equations (II-1) imply that

$$\tilde{\omega}_{\alpha\beta} = \omega_{\alpha\beta} + Tu_\beta s_{;\alpha} - Tu_\alpha s_{;\beta} \tag{II-7}$$

so that $s =$ const. and $\tilde{\omega}_{\alpha\beta} = 0$ imply that

$$\omega_{\alpha\beta} = (hu_\alpha)_{;\beta} - (hu_\beta)_{;\alpha} = 0\,. \tag{II-8}$$

Therefore the vector field hu_μ is (locally at least) expressible as a gradient

$$hu_\mu = \psi_{,\mu} \tag{II-9}$$

(here and throughout, a subscript comma mu $(,\mu) \equiv \partial/\partial x^\mu$). If the region of spacetime occupied by the fluid is simply connected, then hu_μ is globally expressible as a gradient. For the accretion problems we consider, this will be so. A family of fluid disk solutions for which hu_μ is locally but not globally a gradient has been constructed by Fishbone (1977).

For potential flow the Euler equations and equation of continuity simplify to the single scalar equation

$$\left(\frac{n}{h}\psi^{;\mu}\right)_{;\mu} = 0\,, \tag{II-10}$$

where n is expressed as a function of h through the chosen equation of state,

$$n = \left(\frac{\partial p}{\partial h}\right)\bigg|_{s=s_0}\,, \tag{II-11}$$

and where h is reexpressed as

$$h = +(-g^{\alpha\beta}\psi_{,\alpha}\psi_{,\beta})^{1/2}\,, \tag{II-12}$$

which follows from equation (II-9) and the normalization condition, $u^\alpha u^\beta g_{\alpha\beta} = -1$, on the fluid velocity. Equation (II-10) is typically a nonlinear equation for ψ, though it linearizes in the special case of a stiff ($p = \rho$) equation of state. In that case $n \propto h$ and equation (II-10) reduces to the scalar wave equation. Another simple limiting case is that of dust ($p = 0$). The problem of dust accretion onto a Kerr black hole has been treated extensively by Shapiro (1974).

These cases require special treatment, so we shall assume that $0 < v_s^2 < c^2$ for the fluids considered here.

b) Perturbations of a Potential Flow

Now consider small (Eulerian) perturbations of a background potential flow solution. Since by assumption the background is isentropic, we see that the entropy perturbation obeys

$$u^\mu \delta s_{,\mu} = 0\,. \tag{II-13}$$

For a general flow (not necessarily potential) the tensor $\omega_{\mu\nu}$ obeys

$$(\mathscr{L}_v \omega)_{\alpha\beta} = (hT)_{;\beta} s_{;\alpha} - (hT)_{;\alpha} s_{;\beta}\,, \tag{II-14}$$

where

$$\tilde{v} = h\tilde{u}\,, \qquad (\tilde{u})^\alpha = u^\alpha\,, \tag{II-15}$$

and $(\mathscr{L}_{\tilde{v}}\omega)_{\alpha\beta}$ is the Lie derivative of $\omega_{\alpha\beta}$ with respect to $\tilde{v}$:

$$(\mathscr{L}_{\tilde{v}}\omega)_{\mu\nu} = v^{\alpha}\omega_{\mu\nu;\alpha} + v^{\alpha}{}_{;\mu}\omega_{\alpha\nu} + v^{\alpha}{}_{;\nu}\omega_{\mu\alpha} \, . \tag{II-16}$$

Perturbing equation (II-14) about a background potential flow gives

$$(\mathscr{L}_{\tilde{v}}\delta\omega)_{\alpha\beta} = (hT)_{;\beta}\delta s_{;\alpha} - (hT)_{;\alpha}\delta s_{;\beta} \, , \tag{II-17}$$

where

$$\delta\omega_{\alpha\beta} = [\delta(hu_{\alpha})]_{;\beta} - [\delta(hu_{\beta})]_{;\alpha} \tag{II-18}$$

and h, T, $\tilde{v} = h\tilde{u}$ are background quantities.

In the following we shall suppose that the perturbations δs and $\delta\omega_{\alpha\beta}$ are of bounded extent on some initial hypersurface (i.e., vanish outside a sufficiently large sphere). We shall also suppose that the background flow is totally accreting, (i.e., that every fluid particle eventually falls into the black hole). In this case equation (II-13) shows that δs is dragged into the black hole by the background flow. After δs has disappeared into the hole, any vorticity perturbation $\delta\omega_{\alpha\beta}$ propagates freely according to (the homogeneous form of) equation (II-17). This equation shows that $\delta\omega_{\alpha\beta}$ is Lie dragged into the hole along the background flow. Thus both δs and $\delta\omega_{\alpha\beta}$ are swept away by advection. To the future of some spacelike hypersurface intersecting the black hole's event horizon, the perturbation in the flow becomes purely potential.

If $\delta\omega_{\alpha\beta}$ vanishes throughout some region, then

$$(\delta v_{\alpha})_{,\beta} - (\delta v_{\beta})_{,\alpha} = 0 \, , \tag{II-19}$$

where

$$\delta v_{\alpha} = \delta(hu_{\alpha}) = u_{\alpha}\delta h + h\delta u_{\alpha} \, , \tag{II-20}$$

and so δv_{α} is expressible as a gradient throughout that region,

$$\delta v_{\alpha} = \delta\psi_{,\alpha} \, . \tag{II-21}$$

The equation of motion for $\delta\psi$ may be derived by perturbing equation (II-10) or, equivalently, from Schutz's velocity potential formalism (Schutz 1972) specialized to the case of potential flow. This basic perturbation equation has the simple form

$$(-\det\mathfrak{G})^{-1/2}\frac{\partial}{\partial x^{\mu}}[(-\det\mathfrak{G})^{1/2}\mathfrak{G}^{\mu\nu}\delta\psi_{,\nu}] = 0 \tag{II-22}$$

where $\mathfrak{G}_{\mu\nu}$ is the Lorentzian metric given by

$$\mathfrak{G}_{\mu\nu} = \frac{n}{h}\left(\frac{c}{v_s}\right)\left[g_{\mu\nu} + \left(1 - \frac{v_s^2}{c^2}\right)u_{\mu}u_{\nu}\right] , \tag{II-23}$$

where v_s is the sound speed of the background flow:

$$\left(\frac{v_s}{c}\right)^2 = \left(\frac{\partial p}{\partial\rho}\right)_s \tag{II-24}$$

(we shall demand that $0 < v_s^2 < c^2$ holds throughout the background flow). Several useful relations involving $\mathfrak{G}_{\mu\nu}$ are

$$\mathfrak{G}^{\mu\nu} = \frac{h}{n}\left(\frac{v_s}{c}\right)\left[g^{\mu\nu} - \left(\frac{c^2}{v_s^2} - 1\right)u^{\mu}u^{\nu}\right] , \qquad (-\det\mathfrak{G})^{1/2} = \left(\frac{n}{h}\right)^2\frac{c}{v_s}(-\det g)^{1/2} \, , \tag{II-25}$$

where

$$\mathfrak{G}^{\mu\nu}\mathfrak{G}_{\nu\gamma} = \delta^{\mu}{}_{\gamma} \, , \qquad g^{\mu\nu}g_{\nu\gamma} = \delta^{\mu}{}_{\gamma} \, , \qquad u^{\mu} = g^{\mu\nu}u_{\nu} \, , \tag{II-26}$$

and $(\det\mathfrak{G})$ and $(\det g)$ are the determinants of $\mathfrak{G}_{\mu\nu}$ and $g_{\mu\nu}$, respectively. Equation (II-22) is obviously derivable from the variational integral

$$\mathscr{J} = -\tfrac{1}{2}\int d^4x[(-\det\mathfrak{G})^{1/2}\mathfrak{G}^{\mu\nu}\delta\psi_{,\mu}\delta\psi_{,\nu}] \, . \tag{II-27}$$

c) *Some Geometry of the Sound Metric*

The causal properties of sound propagation are determined by the metric $\mathfrak{G}_{\mu\nu}$. In particular the null hypersurfaces (relative to $\mathfrak{G}_{\mu\nu}$) are those across which discontinuities in the derivatives of $\delta\psi$ (i.e., shocks) may occur. They are the characteristic surfaces for sound wave propagation. In addition $\mathfrak{G}_{\mu\nu}$ will share those symmetries which are common to $g_{\mu\nu}$ and the background fluid flow. More precisely, suppose $\tilde{X}$ is a Killing field of $g_{\mu\nu}$,

$$(\mathscr{L}_X g)_{\mu\nu} = X_{\mu;\nu} + X_{\nu;\mu} = 0 \tag{II-28}$$

and that

$$\mathscr{L}_X h = \mathscr{L}_X \tilde{u} = 0 .$$

It follows that

$$(\mathscr{L}_X \mathfrak{G})_{\mu\nu} = 0 . \tag{II-29}$$

(Note that since s = const., all other thermodynamic functions are invariant provided h is.) This result will be useful in the discussion of conservation laws.

We have mentioned that $\mathfrak{G}_{\mu\nu}$ is always a Lorentzian metric (provided $0 < v_s^2 \leq c^2$). This follows from noting that any vector $\tilde{t}$ which is timelike relative to $\mathfrak{G}_{\mu\nu}$ is timelike relative to $g_{\mu\nu}$, i.e.,

$$t^\mu t^\nu \mathfrak{G}_{\mu\nu} < 0 \rightarrow t^\mu t^\nu g_{\mu\nu} < 0 , \tag{II-30}$$

and that a triad of vectors orthogonal to $\tilde{u}$ (relative to $g_{\mu\nu}$) is also spacelike with respect to $\mathfrak{G}_{\mu\nu}$. A corollary of this argument is that the sound cones of $\mathfrak{G}_{\mu\nu}$ lie inside or (if $v_s^2 = c^2$) on the light cones of $g_{\mu\nu}$. Since the event horizon of a black hole defines a boundary to the region which can send light signals to an arbitrarily distant observer, there must also exist a boundary to the region which can send sound signals to a distant observer (who is immersed in the fluid at some large distance where the flow is subsonic). In the stationary case this boundary surface (the sound horizon H_s) defines the transition from subsonic to supersonic flow.

Now consider the special case of a stationary background so that, in suitable coordinates $g_{\mu\nu}$, $\mathfrak{G}_{\mu\nu}$, h, etc., are all independent of the time coordinate t. In addition to the sound horizon H_s we may define a surface B_s (the stationary boundary relative to $\mathfrak{G}_{\mu\nu}$) on which the Killing field $\tilde{X} = \partial/\partial t$ becomes null, $\mathfrak{G}_{tt}|B_s = 0$. (A more refined definition would be needed if $\mathfrak{G}_{tt}$ vanished on more than one three-dimensional surface or throughout an open region. We shall not consider these "degenerate" cases.)

In the spherically symmetric case H_s and B_s will coincide. However, if the accreting fluid flow were not purely radial or if the black hole were rotating or both, then the two surfaces would not in general coincide and B_s would lie outside of H_s. As we shall see below, the energy density of the sound perturbations becomes indefinite inside this region. This occurrence signals, as it does in the Kerr spacetime, the possibility of superradiant scattering. Indeed, for the special (though artificial) case of a stiff ($p = \rho$) fluid, the sound metric $\mathfrak{G}_{\mu\nu}$ reduces to (a constant multiple of) the spacetime metric $g_{\mu\nu}$ so that the superradiant features of sound wave propagation are identical, in the Kerr metric, to those of a massless scalar field.

d) The Spherically Symmetric Case

We now specialize to the case of stationary, spherical accretion onto a Schwarzschild black hole. This solution may be expressed as

$$hu_t = -h_\infty = \text{const.} \neq 0 , \qquad hu_r = \frac{h\alpha}{nr^2(1 - 2GM/c^2r)} , \tag{II-31}$$

where

$$\alpha = -\frac{\dot{N}}{4\pi} = \text{const.} \neq 0 \tag{II-32}$$

with $\dot{N} > 0$ the particle accretion rate. The enthalpy h is determined (implicitly) from

$$\left(1 - \frac{2GM}{c^2r}\right)h^2 = h_\infty{}^2 - \frac{h^2\alpha^2}{n^2r^4} \tag{II-33}$$

where

$$n = \left(\frac{\partial p}{\partial h}\right)\bigg|_{s=s_0} .$$

In requiring that

$$\lim_{r\to\infty} h = h_\infty \neq 0 \tag{II-34}$$

we force

$$\lim_{r\to\infty} u^r = 0 , \tag{II-35}$$

so that the flow becomes subsonic at sufficiently large distances. For accretion onto a black hole we must demand regularity of the fields h, u, etc., across the black hole's event horizon. Finiteness of h as $r \to 2GM/c^2$ gives, from equation (II-33), the condition

$$\lim_{r\to 2MG/c^2}\left(\frac{h^2}{n^2r^4}\right) = \frac{h_\infty{}^2}{\alpha^2}\,. \tag{II-36}$$

This condition will generally freeze one of the adjustable constants (h_∞ and $\alpha = -\dot{N}/4\pi$) in terms of the other. Not every equation of state admits a solution satisfying both boundary conditions. For accretion onto a star, condition (II-36) is absent, but it must be replaced by an appropriate boundary condition at the star's surface.

III. CONSERVATION LAWS AND STABILITY RESULTS

From the perturbation equation (II-22) one can derive

$$\nabla_\nu \mathfrak{I}_\mu{}^\nu = 0\,, \tag{III-1}$$

where

$$\mathfrak{I}_\mu{}^\nu = \tfrac{1}{2}[\delta\psi_{,\mu}\delta\psi_{,\gamma}\mathfrak{G}^{\nu\gamma} - \tfrac{1}{2}\delta_\mu{}^\nu\mathfrak{G}^{\gamma\sigma}\delta\psi_{,\gamma}\delta\psi_{,\sigma}] \tag{III-2}$$

and in which ∇_μ signifies the covariant derivative with respect to $\mathfrak{G}_{\mu\nu}$. $\mathfrak{I}_\mu{}^\nu$ is the usual energy-momentum tensor associated with the scalar wave equation.

If $\tilde{X}$ is a Killing field of $\mathfrak{G}_{\mu\nu}$, we have

$$\frac{\partial}{\partial x^\nu}[(-\det\mathfrak{G})^{1/2}X^\mu\mathfrak{I}_\mu{}^\nu] = 0 \tag{III-3}$$

and thus obtain the associated conservation law. In particular if $\tilde{X} = \partial/\partial t$ is a timelike Killing field, then

$$E_\Omega = -2\int_\Omega d^3x(-\det\mathfrak{G})^{1/2}\mathfrak{I}_t{}^t \tag{III-4}$$

is the energy contained in the volume Ω of a t = constant hypersurface. The energy density is given explicitly by

$$\mathfrak{E} = -2(-\det\mathfrak{G})^{1/2}\mathfrak{I}_t{}^t = (-\det\mathfrak{G})^{1/2}\{-\tfrac{1}{2}\mathfrak{G}^{tt}(\delta\psi_{,t})^2 + \tfrac{1}{2}\mathfrak{G}^{ij}\,\delta\psi_{,i}\delta\psi_{,j}\}\,, \tag{III-5}$$

where i, j range over the spatial coordinate labels. In this formula $(-\mathfrak{G}^{tt}) > 0$ since the surfaces t = constant are spacelike by assumption (it is straightforward to show that surfaces spacelike with respect to $g_{\mu\nu}$ are also spacelike with respect to $\mathfrak{G}_{\mu\nu}$ provided $0 < v_s{}^2 \leq c^2$).

The symmetric form $\mathfrak{G}^{ij}$ becomes indefinite inside that surface (the sonic stationary boundary B_s) across which $\mathfrak{G}_{tt}$ changes sign. This follows from the identity

$$-N_iN_j{}^{(3)}\mathfrak{G}^{ij}\mathfrak{G}_{tt} = N^2\mathfrak{G}^{ij}N_iN_j\,, \tag{III-6}$$

where

$$N = (-\mathfrak{G}^{tt})^{-1/2}\,, \qquad N_i = \mathfrak{G}_{ti}\,, \qquad {}^{(3)}\mathfrak{G}_{ij} = \mathfrak{G}_{ij} \tag{III-7}$$

and ${}^{(3)}\mathfrak{G}^{ij}$ is the inverse of the (strictly positive definite) metric ${}^{(3)}\mathfrak{G}_{ij}$. For accretion problems $N_i = \mathfrak{G}_{ti}$ is nonvanishing so that the sign of $\mathfrak{G}^{ij}N_iN_j$ must change whenever that of $\mathfrak{G}_{tt}$ does. Furthermore,

$$\mathfrak{G}^{ij}Y_iY_j = {}^{(3)}\mathfrak{G}^{ij}Y_iY_j > 0 \tag{III-8}$$

for any nonzero Y_i which satisfies

$$Y_iN_j{}^{(3)}\mathfrak{G}^{ij} = 0 \tag{III-9}$$

so that $\mathfrak{G}^{ij}$ has signature $(-1, +1, +1)$ in the interior region.

In the spherically symmetric case it is easy to show (using $\tilde{u}\cdot\tilde{u} = -1$) that (in Schwarzschild coordinates for $g_{\mu\nu}$)

$$\mathfrak{G}^{rr} = -\left(\frac{h}{n}\right)^2\mathfrak{G}_{tt} \tag{III-10}$$

so that the surface $r = r_s$ = const. at which $\mathfrak{G}_{tt} = 0$ is a null surface. It is in fact the sound horizon H_s of the background flow (we shall assume for simplicity the nondegenerate case in which $\mathfrak{G}_{tt} = 0$ defines a unique three-dimensional surface outside the event horizon of the black hole).

From equations (III-2) and (III-3) specialized to the spherical case we see that the energy $E_{(r_0,r_1)}$ contained between the spherical surfaces $r = r_0$ and $r = r_1 > r_0$ obeys

$$\frac{d}{dt}E_{(r_0,r_1)} = \int_{S^2} d\theta d\varphi[2(-\det \mathfrak{G})^{1/2}\mathfrak{J}_t{}^r]\Big|_{r_0}^{r_1}$$
$$= \int_{S^2} d\theta d\varphi[(-\det \mathfrak{G})^{1/2}\delta\psi_{,t}(\mathfrak{G}^{rt}\delta\psi_{,t} + \mathfrak{G}^{rr}\delta\psi_{,r})]\Big|_{r_0}^{r_1}$$
$$\equiv F_{r_1} - F_{r_0}\,. \tag{III-11}$$

As a special case, we take $r_0 = r_s$ and $r_1 \to \infty$. From the boundary conditions discussed in § II*d* and the requirement of finiteness of the energy, which implies that

$$\delta\psi_{,t} \sim \frac{a}{r^{3/2+\epsilon}}\,, \qquad \delta\psi_{,r} \sim \frac{a'}{r^{3/2+\epsilon}} \tag{III-12}$$

as $r \to \infty$ (where a, a', and ϵ are constants and $\epsilon > 0$), we find that

$$F_{r_1} \xrightarrow[r_1 \to \infty]{} 0\,. \tag{III-13}$$

Using the results derived above for the sound horizon we get

$$(-\det \mathfrak{G})^{1/2}\mathfrak{G}^{rr} \xrightarrow[r \to r_s]{} 0 \tag{III-14}$$

and

$$(-\det \mathfrak{G})^{1/2}\mathfrak{G}^{rt} \xrightarrow[r \to r_s]{} \left[r^2 \sin\theta\left(\frac{n}{h}\right)\left(1 - \frac{c^2}{v_s{}^2}\right)u^r u^t\right]\Big|_{r_s} = \sin\theta\left[\frac{n}{h}\left(1 - \frac{c^2}{v_s{}^2}\right)\frac{h_\infty \alpha}{nh(1 - 2GM/c^2r)}\right]_{r_s} \geq 0\,, \tag{III-15}$$

where the inequality follows from $\alpha = -\dot{N}/4\pi < 0$ (for accretion) and $c^2 > v_s{}^2$. Thus

$$\frac{dE}{dt}(r_s, \infty) = -\frac{\dot{N}}{4\pi}\int_{S^2} d\theta d\varphi \sin\theta\left[\frac{h_\infty}{h^2}\frac{(c^2/v_s{}^2 - 1)}{(1 - 2GM/c^2r)}(\delta\psi_{,t})^2\right]\Big|_{r_s} \leq 0\,, \tag{III-16}$$

so that energy outside the sound horizon can only remain constant or decrease (as it flows inward across the horizon).

Since $dE(r_s, \infty)/dt \leq 0$ for all finite energy solutions, it follows that the L_2 norm of $(\delta\psi)_{,\mu}$ defined by the energy functional (eqs. [III-4], [III-5]) remains bounded by its initial value $E_0 = E(t_0)$ for all $t > t_0$. Since

$$\delta\psi_{,\mu} = \delta v_\mu = (u_\mu \delta h + h\delta u_\mu)\,, \tag{III-17}$$

we thus get the bound (holding for $t \geq t_0$)

$$\|\delta\bar{v}\|^2 = \frac{1}{2}\int_\Omega d^3x(-\det \mathfrak{G})^{1/2}\left[+\frac{1}{N^2}(\delta h u_t + h\delta u_t)^2 + \mathfrak{G}^{ij}(\delta h u_i + h\delta u_i)(\delta h u_j + h\delta u_j)\right] \leq E_0 \tag{III-18}$$

(recall that $\mathfrak{G}^{ij}$ is positive definite outside the sound horizon).

In particular no unstable normal mode solution [i.e., solution with time dependence $\exp(\lambda t)$, $\mathrm{Re}(\lambda) > 0$] with finite energy can extend beyond (i.e., have δv_μ nonvanishing outside of) the sound horizon since it would then contradict the above result (by having exponentially growing energy).

Since the sound horizon H_s is a characteristic surface, it might seem possible to have an unstable normal mode solution which vanishes outside H_s but which has a discontinuity across this surface. Such a solution would represent an unstable standing shock at the horizon. We shall now show, by a purely local argument, that such solutions do not exist.

First we separate the angle variables by seeking an elementary solution of equation (II-22) of the form

$$\delta\psi = \delta\psi(r)e^{\lambda t}Y_{LM}(\theta, \varphi)\,, \tag{III-19}$$

where $Y_{LM}(\theta, \varphi)$ is the usual spherical harmonic. The eigenvalue equation for $\delta\psi(r)$ is:

$$0 = \lambda^2 r^2 \frac{n}{h}\left[-\frac{1}{(1 - 2GM/c^2r)} + \left(1 - \frac{c^2}{v_s{}^2}\right)u^t u^t\right]\delta\psi + \lambda r^2 \frac{n}{h}\left(1 - \frac{c^2}{v_s{}^2}\right)u^t u^r \delta\psi_{,r} - L(L+1)\frac{n}{h}\delta\psi$$
$$+ \lambda\frac{\partial}{\partial r}\left[r^2\frac{n}{h}\left(1 - \frac{c^2}{v_s{}^2}\right)u^t u^r \delta\psi\right] + \frac{\partial}{\partial r}\left[r^2\frac{n}{h}\left(\left(1 - \frac{2GM}{c^2 r}\right) + \left(1 - \frac{c^2}{v_s{}^2}\right)u^r u^r\right)\delta\psi_{,r}\right]\,. \tag{III-20}$$

Now consider a function $\delta\psi(r)$ which vanishes for all $r \geq r_s$, is continuous at $r = r_s$, but has discontinuous first derivative at $r = r_s$. Thus

$$\delta\psi \xrightarrow[r\to r_s^-]{} 0\,, \qquad \delta\psi_{,r} \xrightarrow[r\to r_s^-]{} \kappa \neq 0\,. \tag{III-21}$$

Taking the limit of equation (III-20) as $r \to r_s^-$ (i.e., from the interior to r_s), we get

$$0 = 2\lambda\left[r^2\frac{n}{h}\left(1 - \frac{c^2}{v_s^2}\right)u^t u^r\right]\Bigg|_{r_s}\kappa + \left[\left[\frac{\partial}{\partial r}\left\{r^2\frac{n}{h}\left[\left(1 - \frac{2GM}{c^2 r}\right) + \left(1 - \frac{c^2}{v_s^2}\right)u^r u^l\right]\right\}\right]\right]\Bigg|_{r_s}\kappa\,, \tag{III-22}$$

where no term in $\delta\psi_{,rr}(r_s)$ (which is assumed to be finite) occurs because of the vanishing of $\mathfrak{G}^{rr}(r_s)$. Thus since $\kappa \neq 0$ by assumption and since

$$\mathfrak{G}^{rr}(r_s) = 0\,, \qquad \mathfrak{G}^{rr}{}_{,r}(r_s) \geq 0\,, \qquad \mathfrak{G}^{tr}(r_s) > 0\,, \tag{III-23}$$

we obtain

$$2\lambda = -\left[\frac{1}{\mathfrak{G}^{tr}}\frac{\partial}{\partial r}(\mathfrak{G}^{rr})\right]\Bigg|_{r_s} \leq 0\,, \tag{III-24}$$

we get λ strictly less than zero in the (nondegenerate) case $\mathfrak{G}^{rr}{}_{,r}(r_s) > 0$.

Thus the only normal mode solutions which represent a standing shock at the sound horizon are necessarily either stable ($\lambda < 0$) or (in the degenerate case) stationary ($\lambda = 0$). Of course, such solutions might actually be excluded by further considerations (e.g., the inner boundary condition at the horizon of the black hole), but such considerations are not required to exclude the unstable case.

We can exclude the possibility of an unstable shock of "higher order" by a similar argument. Suppose that $\delta\psi(r)$ and all of its derivatives up to the nth are zero as $r \to r_s^-$ but that

$$\frac{\partial^n}{\partial r^n}\,\delta\psi(r) \xrightarrow[r\to r_s]{} \kappa \neq 0\,.$$

Then by differentiating equation (III-20) $n - 1$ times and taking the limit as above, we can derive equation (III-24) exactly as before.

We conclude this section with a sketch of the Newtonian analog of our approach. In this limit, potential flow means simply

$$\boldsymbol{v} = \nabla\psi\,, \qquad s = s_0 = \text{const}\,. \tag{III-25}$$

The (Eulerian) perturbations of entropy and vorticity can be handled as in the relativistic problem; they leave a purely potential perturbation in their wake. The wave equation for $\delta\psi$ may be derived by combining the perturbed Bernoulli equation and the perturbed continuity equation. A variational integral for this wave equation is given by

$$\mathscr{I} = \tfrac{1}{2}\int d^3x dt(\det g)^{1/2}\left\{\frac{\rho}{v_s^2}(\delta\psi_{,t} + v^j\delta\psi_{,j})^2 - \rho g^{ij}\delta\psi_{,i}\delta\psi_{,j}\right\}, \tag{III-26}$$

where g_{ij} is the metric of the Euclidean three-space (with $\det g$ its determinant) and where ρ and $v_s^2 = (\partial p/\partial\rho)_s$ and v^i are background fluid quantities. One can identify the Lorentzian sound metric $\mathfrak{G}_{\mu\nu}$ for the Newtonian case by writing

$$\mathscr{I} = -\tfrac{1}{2}\int d^4x(-\det\mathfrak{G})^{1/2}\mathfrak{G}^{\mu\nu}\delta\psi_{,\mu}\delta\psi_{,\nu} \tag{III-27}$$

and proceeding as in the relativistic problem. For the case of a stationary background one has the conserved energy (Hamiltonian) function

$$H = \frac{1}{2}\int_\Omega dx^3\left\{\frac{v_s^2}{\rho(\det g)^{1/2}}\left(p_\psi - \frac{\rho(\det g)^{1/2}}{v_s^2}v^j\delta\psi_{,j}\right)^2 + (\det g)^{1/2}\rho\left(g^{ij}\delta\psi_{,i}\delta\psi_{,j} - \frac{1}{v_s^2}(v^i\delta\psi_{,i})^2\right)\right\}, \tag{III-28}$$

where p_ψ (the conjugate momentum to $\delta\psi$) is given by

$$p_\psi = (\det g)^{1/2}\frac{\rho}{v_s^2}(\delta\psi_{,t} + v^j\delta\psi_{,j}) = -(\det g)^{1/2}\,\delta\rho\,. \tag{III-29}$$

The Hamiltonian obeys

$$\frac{dH}{dt} = \int_\Omega d^3x\frac{\partial}{\partial x_j}\left\{(\det g)^{1/2}\rho\delta\psi_{,t}\left[-\frac{v^j}{v_s^2}\delta\psi_{,t} + \left(g^{ij} - \frac{v^j v^i}{v_s^2}\right)\delta\psi_{,i}\right]\right\} \tag{III-30}$$

which, by Gauss's theorem, may be reexpressed as a surface integral over the boundary of the volume Ω.

If we specialize to the case of spherical symmetry (stationary purely radial flow) and put $v^r = u$, then the energy (between two spheres at $r = r_0$ and $r = r_1$) obeys

$$\frac{dE}{dt} = \int_{S^2} d\theta d\varphi \left\{ (\det g)^{1/2} \rho \left[-\frac{u}{v_s^2} (\delta\psi_{,t})^2 + \delta\psi_{,t}\delta\psi_{,r}\left(1 - \frac{u^2}{v_s^2}\right)\right] \Bigg|_{r_0}^{r_1} \right\}, \quad \text{(III-31)}$$

which is the Newtonian analog of equation (III-11). The stability argument can be carried out exactly as in the relativistic case by noting that (for accretion) $u < 0$ and $(1 - u^2/v_s^2)$ vanishes at the sound horizon.

IV. ASYMPTOTIC SOLUTIONS

We can construct approximate (JWKB) solutions to the eigenvalue equation (III-20) in the limit of large (real) frequency. Such elementary waves reduce to plane waves at large radius and so have infinite total energy. One could construct finite energy solutions from them in the usual way by taking Fourier integrals. The corresponding solutions for the Newtonian problem have been given by others for adiabatic accretion (see Petterson, Silk, and Ostriker 1978) and for the case of a stellar wind (see Parker 1966). Our computation is essentially the same except that we include the effects of spacetime curvature and verify the regularity of our approximate solutions at the event horizon of the accreting black hole. Our approximate solutions are not regular near the sonic point $r = r_s$ (which is a singular point of the differential eq. [III-20]) but could be matched across this singularity with some additional effort. We put $\lambda = i\omega$ (with ω real) in equation (III-20) and look for a solution of the (JWKB) form

$$\delta\psi(r) = \exp\left\{ i\omega\left[k_0(r) + \frac{k_1(r)}{i\omega} + \frac{k_2(r)}{(i\omega)^2} + \cdots \right]\right\}. \quad \text{(IV-1)}$$

Substituting this into equation (III-20) and collecting terms in various powers of ω gives equations which successively determine the functions $k_i(r)$. The first two terms give

$$k_0^{\pm} = \int dr \left\{ \frac{(c^2/v_s^2 - 1)u^t u^r \pm c/v_s}{[(1 - 2GM/c^2 r) - (c^2/v_s^2 - 1)u^r u^r]} \right\} + \text{const}. \quad \text{(IV-2)}$$

and

$$k_1 = \ln\left[\left(r^4 \left(\frac{n}{h}\right)^2 \frac{c^2}{v_s^2}\right)^{-1/4}\right] + \text{const.}, \quad \text{(IV-3)}$$

the latter being valid for both k_0^+ and k_0^-. Thus to a first approximation (for large ω) the elementary monochromatic waves have the form

$$\delta\psi = \frac{1}{r(n/h)^{1/2}(c/v_s)^{1/2}} \{\alpha_+ \exp[i\omega(k_0^+ + t)] + \alpha_- \exp[i\omega(k_0^- + t)]\} Y_{LM}(\theta, \varphi). \quad \text{(IV-4)}$$

At large radii, $r \gg r_s$, we have the limiting forms

$$k_0^{\pm} \sim \pm \left(\frac{c}{v_s}\right)_\infty r^* + (\text{lower order terms}), \quad \text{(IV-5)}$$

where r^* is the "tortoise coordinate" defined by

$$r^* = \int \frac{dr}{(1 - 2GM/c^2 r)}. \quad \text{(IV-6)}$$

Thus α_+ is the amplitude of the incoming wave while α_- is that of the outgoing wave.

For the region far inside the sound horizon the same form of the asymptotic solution applies, but this form becomes singular at the sonic radius itself. One could match the two asymptotic expansions to find the relation between the inner and outer values of (α_+, α_-), but we shall be content to verify the regularity of the inner solutions as $r \to 2GM/c^2$ (i.e., at $r^* \to -\infty$).

Consider a nonrotating orthonormal tetrad "carried" by observers riding with the background fluid flow. Such a tetrad may be defined by the vector fields

$$\tilde{h}_{(t)} = \tilde{u}, \quad \tilde{h}_{(r)} = u_t \frac{\partial}{\partial r} - u_r \frac{\partial}{\partial t}, \quad \tilde{h}_{(\theta)} = \frac{1}{r}\frac{\partial}{\partial \theta}, \quad \tilde{h}_{(\varphi)} = \frac{1}{r \sin\theta}\frac{\partial}{\partial\varphi}, \quad \text{(IV-7)}$$

we shall show that the functions

$$h_{(\alpha)}{}^{\mu}\delta\psi_{,\mu} = h_{(\alpha)}{}^{\mu}\delta v_\mu \quad \text{(IV-8)}$$

have finite limits at the black hole's event horizon.

First note that $h_{(\theta)}{}^{\mu}\delta\psi_{,\mu}$ and $h_{(\varphi)}{}^{\mu}\delta\psi_{,\mu}$ will have finite limits provided that $\delta\psi$ itself does. By expanding the integrand in equation (IV-2) near $r = 2GM/c^2$ we may derive the limiting form

$$k_0{}^{\pm} \sim r^* + c_{\pm}(r)\,, \tag{IV-9}$$

where $c_{\pm}(r)$ is a slowly varying function of r with a finite limit at the horizon (provided $v_s{}^2 < c^2$ outside and on the horizon). Thus, near $r = 2GM/c^2$,

$$\delta\psi \sim \left[\frac{1}{r(n/h)^{1/2}(c/v_s)^{1/2}}\right]\{\alpha_+ \exp\,[i\omega(t + r^* + c_+(r))] + \alpha_- \exp\,[i\omega(t + r^* + c_-(r))]\} \equiv \alpha_+\delta\psi^+ + \alpha_-\delta\psi^-\,; \tag{IV-10}$$

and since $v = t + r^*$ is finite at the future horizon, the two phase functions in $\delta\psi$ are finite on the horizon. Recalling the limit (II-36), we see that the overall amplitude function is also finite.

Similarly we find, as $r \to 2GM/c^2$,

$$h^{\mu}{}_{(t)}\delta\psi^{\pm}{}_{,\mu} \sim \left\{i\omega\left[\frac{1}{2|\alpha/nr^2|}\left(1 + \frac{(1 \mp c/v_s)^2}{(1 - c^2/v_s{}^2)} + \cdots\right)\right] - \frac{1}{4}u^r\frac{\partial}{\partial r}\ln\left(\frac{r^4n^2c^2}{h^2v_s{}^2}\right) + \cdots\right\}\delta\psi^{\pm}\,, \tag{IV-11}$$

which is finite by virtue of the finiteness of $\delta\psi$, the regularity of the background flow across the horizon, and the requirement that $v_s{}^2$ be bounded away from c^2 in our background solution. Finally we get

$$h_{(r)}{}^{\mu}\delta\psi_{,\mu}{}^{\pm} \sim -\frac{h_\infty}{h}\left\{i\omega\left[\frac{1}{(1 - 2GM/c^2r)}\left(1 + \frac{\alpha h}{nr^2h_\infty}\right) - \frac{n^2r^4}{2\alpha^2}\left[\left(1 \mp \frac{c}{v_s}\right)^2\Big/\left(1 - \frac{c^2}{v_s{}^2}\right)\right] + \cdots\right]\right.$$
$$\left. - \frac{1}{4}\left(\frac{nr^2}{\alpha}\right)u^r\frac{\partial}{\partial r}\ln\left(\frac{r^4n^2c^2}{h^2v_s{}^2}\right) + \cdots\right\}\delta\psi^{\pm}\,. \tag{IV-12}$$

The only potentially troublesome term is the first; but, recalling equation (II-33), we have

$$\left(1 + \frac{\alpha h}{nr^2h_\infty}\right)\Big/\left(1 - \frac{2GM}{c^2r}\right) = \left(\frac{h}{h_\infty}\right)^2\frac{1}{[1 - (h/h_\infty)\alpha/nr^2]}\xrightarrow[r\to 2GM/c^2]{}\frac{1}{2}\left(\frac{h_{2M}}{h_\infty}\right)^2, \tag{IV-13}$$

where h_{2M} is the value of h on the horizon.

It is worth noting that since $u^\mu\delta u_\mu = 0$,

$$h_{(t)}{}^{\mu}(\delta\psi)_{,\mu} = u^\mu(\delta h u_\mu + h\delta u_\mu) = -\delta h\,.$$

V. CONCLUDING REMARKS

One can study the stability of spherical accretion onto nonrotating stars by the methods used here for black holes. If the star's surface lies inside a sound horizon, the arguments given in § III apply just as in the black hole problem. We can conclude as before that no unstable modes exist which either extend into the subsonic region or exhibit standing shocks at the horizon. On the other hand, if the star's surface lies in the subsonic region, then a more detailed consideration of the inner boundary condition would be needed.

For the problem of a stellar wind the density approaches zero while the velocity remains nonzero as $r \to \infty$. In this case the energy outside the sound horizon (assuming such a surface exists) is fed by the interior region (i.e., the flux of energy at r_s has opposite sign from that in the accretion problem). No simple energy argument for stability applies unless one has, on physical grounds, a bound on the energy flux at the sound horizon.

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