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Analytic Solutions of the Teukolsky Equation and Their Low Frequency Expansions

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Analytic solutions of the Teukolsky equation for arbitrary spin weight in Kerr geometry are presented in the form of series of hypergeometric functions and Coulomb wave functions. Relations between these solutions are established. The solutions provide a very powerful method not only for examining the general properties of solutions and physical quantities both analytically and numerically. The solutions can be regarded as series expansions in terms of a small parameter $\epsilon \equiv 2M\omega$, M being the mass of black hole, which corresponds to the Post-Minkowski expansion by G and to post-Newtonian expansion when they are applied to the gravitational radiation from a particle in circular orbit around a black hole. It is expected that these solutions will become a powerful weapon to construct accurate theoretical templates for LIGO and VIRGO projects.

§ 1. Introduction

There are growing interests in analytic solutions of the Teukolsky equation[1)] in the Schwarzshild and Kerr geometries in the connection with gravitational wave astrophysics. Since Teukolsky proposed the master equation for massless fields in the Kerr spacetime, many efforts have been made to obtain the analytic solutions. The analytic expressions valid for low frequencies were found by Page,[2)] Starobinsky and Churilov[3)] by matching the approximate solutions valid near horizon and far from it. Leaver[4)] made a systematic study to obtain the analytic solutions of the Teukolsky equation in the form of series of various functions. He found the solution in the form of series of Coulomb wave functions which is valid in the region far from the horizon and established the relation between that solution and the one in the form of the Jaffe type series which is valid near the horizon.

Recently, Tagoshi and Nakamura[5)] determined numerically the coefficients of the post-Newtonian expansion of the gravitational radiation by a particle traveling a circular orbit around a Schwarzshild black hole. Sasaki[6)] proposed a method of post-Newtonian expansion to solve the homogeneous Regge-Wheeler equation by using Bessel functions. Subsequently, the extensive study on this line was made by Tagoshi and Sasaki[7)] and the result was compared with the one by Tagoshi and Nakamura. The application of this method to the Kerr geometries was made by Shibata, Sasaki, Tagoshi and Tanaka.[8)] Various other applications were discussed by Poisson and Sasaki.[9)] Now the problem to obtain the analytic solutions and the examination of their behaviors in low frequencies became an important and urgent topic.

In this paper, we report that we obtained the analytic solutions of the Teukolsky equation in Kerr geometry in the form of series of hypergeometric functions and Coulomb wave functions. The series solution of hypergeometric type is shown to be convergent in the region except infinity, while that of Coulomb type is convergent in the region $|x|>1$, where $x=(r_+-r)/2M\sqrt{1-(a/M)^2}$ with r_+, M and a being the position of the outer horizon, the mass and the angular momentum of Kerr black hole, respectively. We establish the relation between the two solutions with different regions of convergences. The solutions are interesting not only for the investigation of general properties of solutions as mathematical physics, but also are for various applications to the gravitational wave astrophysics. The solutions are essentially given in the $\epsilon=2M\omega$ expansion where M and ω being the black hole mass and the angular frequency, which corresponds to the Post-Minkowskian G expansion and also corresponds to the post-Newtonian expansion when they are applied to the problem of the gravitational radiation from a particle in circular orbit around a black hole so that our solutions are quite powerful to examine the ϵ behavior of various physical quantities. Our solutions are expected to become a powerful machine for numerical computation also because the convergences of series are fast. Thus the solutions will become powerful weapons for the construction of theoretical templates used for gravitational wave observations by LIGO and VIRGO.

Our work was motivated by Sasaki's work.[6] We tried to improve his method for solving the Regge-Wheeler equation because his method has several disadvantages: (1) it is difficult to obtain the higher order terms of $\epsilon\equiv 2M\omega$, (2) the expansion is not really the Bessel expansion because coefficients are also variable dependent and (3) the convergence of the series was unknown. In order to improve these difficulties, we considered the solution in the form of series of hypergeometric functions for the solutions of the Regge-Wheeler equation and also for the Teukolsky equation in Schwarzshild spacetime and showed that the coefficients of series can be determined systematically in the expansion of ϵ due to the recurrence relations among hypergeometric functions which we found.[10] This solution is valid near the horizon and not at infinity so that away from the horizon we have to consider the solution in the form of series of Coulomb wave function which was found by Leaver.[4] By matching these two solutions in the intermediate region, we obtained a good solution in the entire region. After finishing our work, we happened to see the paper by Otchik[11] who discussed the analytic solutions of the Teukolsky equation in the form of series of hypergeometric functions and Coulomb wave functions. We found that our method is essentially identical to Otchik's method, but our solutions disagreed with his ones. We compared our solutions with his ones and found that although various formulas which he presented were incorrect, his story itself turns out to be true. Since our results are all different from these by Otchik[11] and the results themselves are quite important for the application, we present all results in this paper.

We start from the Teukolsky equation which is separated by writing

$$\psi=e^{-i\omega t}e^{im\phi}S_l^m(\theta)R_{\omega lm}(r)\,. \tag{1·1}$$

The equation for R is

$$\Delta R''+2(r-M)(s+1)R'+\left[\frac{K^2-2is(r-M)K}{\Delta}+4is\omega r-\lambda\right]R=0\,, \tag{1·2}$$

where M is the mass of the black hole, aM its angular momentum, $\Delta=r^2-2Mr+a^2$

$=(r-r_+)(r-r_-)$ with $r_\pm=M\pm\sqrt{M^2-a^2}$ where r_+ and r_- are positions of outer and inner horizons, respectively, $K=(r^2+a^2)\omega-am$, $\lambda=E-s(s+1)-2ma\omega+a^2\omega^2$. The function S_l^m is the spin weighted spheroidal harmonics which will be discussed in Appendix A with the eigenvalue E[12]

$$E=l(l+1)-\frac{2s^2m\xi}{l(l+1)}+[H(l+1)-H(l)-1]\xi^2+O(\xi^3)\,, \tag{1·3}$$

$$H(l)=\frac{2(l^2-m^2)(l^2-s^2)^2}{(2l-1)l^3(2l+1)}\,, \tag{1·4}$$

where $\xi=a\omega$ and l is the angular momentum which takes an integer or half-integer number which satisfies $l\geq\max(|m|,|s|)$.

In § 2, we give the discussion about how we arrive at the analytic solutions in terms of hypergeometric functions and discuss their properties. In § 3, the analytic solutios in terms of Coulomb wave functions are given following the work of Leaver. The relation between the two solutions in two different convergence regions is established in § 4. The low frequency expansion of these solutions is discussed in § 5. In § 6, a summary and remarks are given.

§ 2. Analytic solution in the form of series of hypergeometric functions

The radial Teukolsky equation has two regular singularities at $r=r_\pm$ and an irregular singularity at $r=\infty$. In order to obtain the solution in the form of series of hypergeometric functions, we have to deal with these regular singularities. Following the discussion in Appendix B, we take the form of R which satisfies the incoming boundary condition on the outer horizon. In particular, we choose the form given by (α_-,β_+) in the notation in Appendix B with the variable $x=\omega(r_+-r)/\epsilon\kappa$ as

$$R_{\rm in}^\nu=e^{i\epsilon\kappa x}(-x)^{-s-(i/2)(\epsilon+\tau)}(1-x)^{(i/2)(\epsilon-\tau)}p_{\rm in}^\nu(x)\,, \tag{2·1}$$

where $\epsilon=2M\omega$, $q=a/M$, $\kappa=\sqrt{1-q^2}$ and $\tau=(\epsilon-mq)/\kappa$. Then, the radial Teukolsky equation becomes

$$\begin{aligned}x(1-x)p_{\rm in}^{\nu\prime\prime}&+[1-s-i\epsilon-i\tau-(2-2i\tau)x]p_{\rm in}^{\nu\prime}+(\nu+i\tau)(\nu+1-i\tau)p_{\rm in}^\nu\\&=2i\epsilon\kappa[-x(1-x)p_{\rm in}^{\nu\prime}+(1-s+i\epsilon-i\tau)xp_{\rm in}^\nu]\\&\quad+[-\lambda-s(s+1)+\nu(\nu+1)+\epsilon^2-i\epsilon\kappa(1-2s)]p_{\rm in}^\nu\,.\end{aligned} \tag{2·2}$$

Here we introduced the parameter ν as the renormalized angular momentum which satisfies $\nu=l+O(\epsilon)$. Then, the right-hand side of Eq. (2·2) is of order ϵ so that this form of equation is suitable to obtain the solution in the expansion of ϵ. The zeroth order solution of Eq. (2·2) is the hypergeometric function.

From the structure of the above equation, the solution may be written in the form of series of hypergeometric functions as

$$p_{\rm in}^\nu(x)=\sum_{n=-\infty}^{\infty}a_n^\nu p_{n+\nu}(x)\,, \tag{2·3}$$

where

$$p_{n+\nu}(x)=F(n+\nu+1-i\tau,\ -n-\nu-i\tau;\ 1-s-i\epsilon-i\tau;\ x) \tag{2·4}$$

with the use of the renormalized angular momentum ν rather than l. We expect that the series will coincide with the ϵ expansion. In order for the coefficients of series (2·3) to be solved, it is essential that the coefficients a_n^ν satisfy the three term recurrence relation. For this, terms such as $x(1-x)p'_{n+\nu}$ and $xp_{n+\nu}$ must be expressed as linear combinations of $p_{n+\nu+1}$, $p_{n+\nu}$ and $p_{n+\nu-1}$. Amazingly enough, we found the following recurrennce relations,

$$xp_{n+\nu}=-\frac{(n+\nu+1-s-i\epsilon)(n+\nu+1-i\tau)}{2(n+\nu+1)(2n+2\nu+1)}p_{n+\nu+1}+\frac{1}{2}\left[1+\frac{i\tau(s+i\epsilon)}{(n+\nu)(n+\nu+1)}\right]p_{n+\nu}$$
$$-\frac{(n+\nu+s+i\epsilon)(n+\nu+i\tau)}{2(n+\nu)(2n+2\nu+1)}p_{n+\nu-1}\,, \tag{2·5}$$

$$x(1-x)p'_{n+\nu}=\frac{(n+\nu+i\tau)(n+\nu+1-i\tau)(n+\nu+1-s-i\epsilon)}{2(n+\nu+1)(2n+2\nu+1)}p_{n+\nu+1}$$
$$+\frac{1}{2}(s+i\epsilon)\left[1+\frac{i\tau(1-i\tau)}{(n+\nu)(n+\nu+1)}\right]p_{n+\nu}$$
$$-\frac{(n+\nu+1-i\tau)(n+\nu+i\tau)(n+\nu+s+i\epsilon)}{2(n+\nu)(2n+2\nu+1)}p_{n+\nu-1}\,, \tag{2·6}$$

which enable us to obtain the three term recurrence relation among a_n^ν. The above recurrence relations among hypergeometric functions can be proved by using the power series expansions. By substituting the form in Eq. (2·4) into the radial Teukolsky equation (2·2), we find that $p_{\rm in}^\nu$ becomes a solution if the following recurrence relation is satisfied:

$$\alpha_n^\nu a_{n+1}^\nu+\beta_n^\nu a_n^\nu+\gamma_n^\nu a_{n-1}^\nu=0\,, \tag{2·7}$$

where

$$\alpha_n^\nu=\frac{i\epsilon\kappa(n+\nu+1+s+i\epsilon)(n+\nu+1+s-i\epsilon)(n+\nu+1+i\tau)}{(n+\nu+1)(2n+2\nu+3)}\,, \tag{2·8}$$

$$\beta_n^\nu=-\lambda-s(s+1)+(n+\nu)(n+\nu+1)+\epsilon^2+\epsilon(\epsilon-mq)$$
$$+\frac{\epsilon(\epsilon-mq)(s^2+\epsilon^2)}{(n+\nu)(n+\nu+1)}\,, \tag{2·9}$$

$$\gamma_n^\nu=-\frac{i\epsilon\kappa(n+\nu-s+i\epsilon)(n+\nu-s-i\epsilon)(n+\nu-i\tau)}{(n+\nu)(2n+2\nu-1)}\,. \tag{2·10}$$

By introducing the continued fractions

$$R_n(\nu)=\frac{a_n^\nu}{a_{n-1}^\nu}\,,\quad L_n(\nu)=\frac{a_n^\nu}{a_{n+1}^\nu}\,, \tag{2·11}$$

we find

$$R_n(\nu)=-\frac{\gamma_n^\nu}{\beta_n^\nu+\alpha_n^\nu R_{n+1}(\nu)}, \quad L_n(\nu)=-\frac{\alpha_n^\nu}{\beta_n^\nu+\gamma_n^\nu L_{n-1}(\nu)}. \tag{2·12}$$

From these equations, we can evaluate the coefficients by taking the initial condition $a_0^\nu=1$. The renormalized angular momentum ν is determined by requiring that the coefficients obtained by using $R_n(\nu)$ agree with those by using $L_n(\nu)$, that is, by solving the transcendental equation for ν

$$R_n(\nu)L_{n-1}(\nu)=1. \tag{2·13}$$

If Eq. (2·13) is satisfied, we find

$$\lim_{n\to\infty} n\frac{a_n^\nu}{a_{n-1}^\nu}=-\lim_{n\to-\infty} n\frac{a_n^\nu}{a_{n+1}^\nu}=\frac{i\epsilon\kappa}{2}. \tag{2·14}$$

From the large n behavior of hypergeometric functions, we find by using the recurrence formula of hypergeometric functions (2·5) as[13]

$$\lim_{n\to\infty}\frac{p_{n+\nu}(x)}{p_{n+\nu-1}(x)}=\lim_{n\to-\infty}\frac{p_{n+\nu}(x)}{p_{n+\nu+1}(x)}=1-2x+((1-2x)^2-1)^{1/2}. \tag{2·15}$$

From Eqs. (2·14) and (2·15), we find

$$\lim_{n\to\infty}\frac{na_n^\nu p_{n+\nu}(x)}{a_{n-1}^\nu p_{n+\nu-1}(x)}=-\lim_{n\to-\infty}\frac{na_n^\nu p_{n+\nu}(x)}{a_{n+1}^\nu p_{n+\nu+1}(x)}=\frac{i\epsilon\kappa}{2}[1-2x+((1-2x)^2-1)^{1/2}]. \tag{2·16}$$

Thus the series converges in all over the complex plane of x except for $x=\infty$.

As for the recurrence relation (2·8), we find $\alpha_{-n}^{-\nu-1}=\gamma_n^\nu$ and $\gamma_{-n}^{-\nu-1}=\alpha_n^\nu$ so that $a_{-n}^{-\nu-1}$ satisfies the same recursion relation as a_n^ν does. Thus if we choose $a_0^\nu=a_0^{-\nu-1}=1$, we have

$$a_n^\nu=a_{-n}^{-\nu-1}. \tag{2·17}$$

Also, we find

$$R_n(-\nu-1)L_{n-1}(-\nu-1)=R_{-n+1}(\nu)L_{-n}(\nu)=1, \tag{2·18}$$

which means that if ν is the solution of Eq. (2·13), then $-\nu-1$ is also the solution.

It is easily seen that the solution R_{in}^ν is symmetric under the exchange of ν with $-\nu-1$ as follows. By using the formula

$$\begin{aligned}p_{n+\nu}(x)=&\frac{\Gamma(1-s-i\epsilon-i\tau)\Gamma(2n+2\nu+1)}{\Gamma(n+\nu+1-i\tau)\Gamma(n+\nu+1-s-i\epsilon)}(-x)^{n+\nu+i\tau}\\&\times F\left(-n-\nu-i\tau,-n-\nu+s+i\epsilon;-2n-2\nu;\frac{1}{x}\right)\\&+\frac{\Gamma(1-s-i\epsilon-i\tau)\Gamma(-2n-2\nu-1)}{\Gamma(-n-\nu-i\tau)\Gamma(-n-\nu-s-i\epsilon)}(-x)^{-n-\nu+i\tau}\\&\times F\left(n+\nu+1-i\tau,n+\nu+1+s+i\epsilon;2n+2\nu+2;\frac{1}{x}\right),\end{aligned} \tag{2·19}$$

we can show

$$R_{\text{in}}^{\nu}=R_0{}^{\nu}+R_0{}^{-\nu-1}\,, \tag{2·20}$$

where

$$R_0{}^{\nu}=e^{i\epsilon\kappa x}(-x)^{\nu-s-(i/2)(\epsilon-\tau)}(1-x)^{(i/2)(\epsilon-\tau)}$$
$$\times\sum_{n=-\infty}^{\infty}a_n{}^{\nu}\frac{\Gamma(1-s-i\epsilon-i\tau)\Gamma(2n+2\nu+1)}{\Gamma(n+\nu+1-i\tau)\Gamma(n+\nu+1-s-i\epsilon)}$$
$$\times(-x)^nF\Big(-n-\nu-i\tau,\,-n-\nu+s+i\epsilon;\,-2n-2\nu;\frac{1}{x}\Big)\,. \tag{2·21}$$

The behavior of R_{in}^{ν} on the outer horizon ($x=0$) is

$$R_{\text{in}}^{\nu}\to(-x)^{s-(i/2)(\epsilon+\tau)}\sum_{n=-\infty}^{\infty}a_n{}^{\nu}\,, \tag{2·22}$$

which gives the normalization of our solution.

We can also show that $R_0{}^{\nu}$ and $R_0{}^{-\nu-1}$ are solutions which are independent of each other. To see this explicitly, we consider the solution which satisfies the outgoing boundary condition on the outer horizon. The outgoing solution which corresponding to (α_+, β_-) in the notation defined in Appendix B can be written as

$$R_{\text{out}}^{\nu}=e^{i\epsilon\kappa x}(-x)^{(i/2)(\epsilon+\tau)}(1-x)^{-s-(i/2)(\epsilon-\tau)}p_{\text{out}}^{\nu}\,. \tag{2·23}$$

Now we expand p_{out}^{ν} as

$$p_{\text{out}}^{\nu}(x)=\sum_{n=-\infty}^{\infty}\tilde{a}_n{}^{\nu}\tilde{p}_{n+\nu}(x)\,, \tag{2·24}$$

where

$$\tilde{p}_{n+\nu}(x)=F(n+\nu+1+i\tau,\,-n-\nu+i\tau;\,1+s+i\epsilon+i\tau;\,x)\,. \tag{2·25}$$

Similar to the solution satisfying the incoming boundary condition, we find that the above series becomes a solution if the following recurrence relation is satisfied:

$$\tilde{\alpha}_n{}^{\nu}\tilde{a}_{n+1}^{\nu}+\beta_n{}^{\nu}\tilde{a}_n{}^{\nu}+\tilde{\gamma}_n{}^{\nu}\tilde{a}_{n-1}^{\nu}=0\,, \tag{2·26}$$

$$\tilde{\alpha}_n{}^{\nu}=\frac{(n+\nu+1-s-i\epsilon)(n+\nu+1-i\tau)}{(n+\nu+1+s+i\epsilon)(n+\nu+1+i\tau)}\alpha_n{}^{\nu}\,, \tag{2·27}$$

$$\tilde{\gamma}_n{}^{\nu}=\frac{(n+\nu+s+i\epsilon)(n+\nu+i\tau)}{(n+\nu-s-i\epsilon)(n+\nu-i\tau)}\gamma_n{}^{\nu}\,, \tag{2·28}$$

where $\alpha_n{}^{\nu}$, $\beta_n{}^{\nu}$ and $\gamma_n{}^{\nu}$ are defined in Eqs. (2·8)~(2·10). By inspection, we see that this recurrence relation is reduced to the one in Eq. (2·7) by redefining systematically the coefficients as

$$\tilde{a}_n{}^{\nu}=\frac{\Gamma(\nu+1-s-i\epsilon)\Gamma(\nu+1-i\tau)\Gamma(n+\nu+1+s+i\epsilon)\Gamma(n+\nu+1+i\tau)}{\Gamma(\nu+1+s+i\epsilon)\Gamma(\nu+1+i\tau)\Gamma(n+\nu+1-s-i\epsilon)\Gamma(n+\nu+1-i\tau)}a_n{}^{\nu}\,, \tag{2·29}$$

where we chose $\tilde{a}_0{}^{\nu}=1$. Now we take $\tilde{a}_0{}^{-\nu-1}=\tilde{a}_0{}^{\nu}=1$, then after some computation we find

$$R_{\rm out}^{\nu}=A_{\nu}R_0{}^{\nu}+A_{-\nu-1}R_0{}^{-\nu-1}\,, \tag{2·30}$$

where

$$A_{\nu}=\frac{\Gamma(1+s+i\epsilon+i\tau)\Gamma(\nu+1-s-i\epsilon)\Gamma(\nu+1-i\tau)}{\Gamma(1-s-i\epsilon-i\tau)\Gamma(\nu+1+s+i\epsilon)\Gamma(\nu+1+i\tau)}\,. \tag{2·31}$$

This relation explicitly shows that $R_0{}^{\nu}$ and $R_0{}^{-\nu-1}$ are independent solutions of Eq. (2·1).

§ 3. Analytic solutions in the form of series of Coulomb wave functions

Analytic solution in the form of series of Coulomb wave functions are given by Leaver.[4] Here, we follow the discussion in Appendix B and start the parameterization to remove the singularity at $r=r_-$. By using a variable $z=\omega(r-r_+)=-\epsilon\kappa x$, we take the following form:

$$R_{\rm C}{}^{\nu}=z^{-1-s}\left(1+\frac{\epsilon\kappa}{z}\right)^{(i/2)(\epsilon-\tau)}f_{\nu}(z)\,. \tag{3·1}$$

Then, we find

$$\begin{aligned}
&z^2f_{\nu}''+[z^2+2(\epsilon+is)z-\nu(\nu+1)]f_{\nu}\\
&\quad=-\epsilon\kappa z(f_{\nu}''+f_{\nu})+\epsilon\kappa(1+s+i\epsilon-i\tau)f_{\nu}'-\frac{\epsilon\kappa(1+s+i\epsilon)(1-i\tau)}{z}f_{\nu}\\
&\qquad+[\lambda+s(s+1)-\nu(\nu+1)-2\epsilon^2+\epsilon mq-\epsilon\kappa(\epsilon+is)]f_{\nu}\,.
\end{aligned} \tag{3·2}$$

If we consider ν to be $\nu=l+O(\epsilon)$, the right-hand side of Eq. (3·2) is a quantity of order ϵ so that this equation is a suitable one to obtain the solution in the expansion of ϵ.

Here we aim to obtain the exact solution by expanding $f_{\nu}(z)$ in terms of Coulomb functions with the renormalized angular momentum ν,

$$f_{\nu}=\sum_{n=-\infty}^{\infty}b_n{}^{\nu}F_{n+\nu}(z)\,, \tag{3·3}$$

where $F_{n+\nu}$ is the unnormalized Coulomb wave function,

$$F_{n+\nu}=e^{-iz}(2z)^{n+\nu}z\frac{\Gamma(n+\nu+1-s+i\epsilon)}{\Gamma(2n+2\nu+2)}\Phi(n+\nu+1-s+i\epsilon,2n+2\nu+2;2iz)\,, \tag{3·4}$$

where Φ is the regular confluent hypergeometric function.[13] It is essential for the solution of Coulomb wave function to be related to the one of hypergeometric functions, in order that the renormalized angular momentum ν takes the same value for both cases.

By substituting Eq. (3·3) into Eq. (3·2) and using the recurrence relations satisfied by the Coulomb wave functions,

$$\frac{1}{z}F_{n+\nu}=\frac{(n+\nu+1+s-i\epsilon)}{(n+\nu+1)(2n+2\nu+1)}F_{n+\nu+1}+\frac{is+\epsilon}{(n+\nu)(n+\nu+1)}F_{n+\nu}$$
$$+\frac{(n+\nu-s+i\epsilon)}{(n+\nu)(2n+2\nu+1)}F_{n+\nu-1}, \tag{3·5}$$

$$F'_{n+\nu}=-\frac{(n+\nu)(n+\nu+1+s-i\epsilon)}{(n+\nu+1)(2n+2\nu+1)}F_{n+\nu+1}+\frac{is+\epsilon}{(n+\nu)(n+\nu+1)}F_{n+\nu}$$
$$+\frac{(n+\nu+1)(n+\nu-s+i\epsilon)}{(n+\nu)(2n+2\nu+1)}F_{n+\nu-1}, \tag{3·6}$$

we obtain the three term recursion relation of coefficients $b_n{}^\nu$,

$$\alpha_n'^\nu b_{n+1}^\nu+\beta_n{}^\nu b_n{}^\nu+\gamma_n'^\nu b_{n-1}^\nu=0, \tag{3·7}$$

$$\alpha_n'^\nu=-i\frac{(n+\nu+1-s+i\epsilon)(n+\nu+1-s-i\epsilon)}{(n+\nu+1+s+i\epsilon)(n+\nu+1+s-i\epsilon)}\alpha_n{}^\nu, \tag{3·8}$$

$$\gamma_n'^\nu=i\frac{(\nu+n+s+i\epsilon)(n+\nu+s-i\epsilon)}{(n+\nu-s+i\epsilon)(n+\nu-s-i\epsilon)}\gamma_n{}^\nu, \tag{3·9}$$

where $\alpha_n{}^\nu$, $\beta_n{}^\nu$ and $\gamma_n{}^\nu$ are defined in Eqs. (2·8)~(2·10). By inspection, we see that this recurrence relation is deformed to the one in Eq. (2·7) if we systematically redefine the coefficients as

$$b_n{}^\nu=i^n\frac{\Gamma(\nu+1-s+i\epsilon)\Gamma(\nu+1-s-i\epsilon)\Gamma(n+\nu+1+s+i\epsilon)\Gamma(n+\nu+1+s-i\epsilon)}{\Gamma(\nu+1+s+i\epsilon)\Gamma(\nu+1+s-i\epsilon)\Gamma(n+\nu+1-s+i\epsilon)\Gamma(n+\nu+1-s-i\epsilon)}a_n{}^\nu, \tag{3·10}$$

where we chose the initial condition $b_0{}^\nu=1$. Since the recurrence relation obtained for the Coulomb expansion case is identical to the one for the hypergeometric case, the renormalized angular momenta ν derived from both solutions are the same which allows us to relate these two solutions.

As for the convergence of series in Eq. (3·3), we find

$$\lim_{n\to\infty} n\frac{b_n{}^\nu}{b_{n-1}^\nu}=\lim_{n\to-\infty} n\frac{b_n{}^\nu}{b_{n+1}^\nu}=-\frac{\epsilon\kappa}{2}, \tag{3·11}$$

and from the recurrence relation (3·5)

$$\lim_{n\to\infty}\frac{F_{n+\nu}(z)}{nF_{n+\nu-1}(z)}=\lim_{n\to-\infty}\frac{F_{n+\nu}(z)}{nF_{n+\nu+1}(z)}=\frac{2}{z}, \tag{3·12}$$

so that

$$\lim_{n\to\infty}\frac{b_n{}^\nu F_{n+\nu}(z)}{b_{n-1}^\nu F_{n+\nu-1}(z)}=\lim_{n\to-\infty}\frac{b_n{}^\nu F_{n+\nu}(z)}{b_{n+1}^\nu F_{n+\nu+1}(z)}=-\frac{\epsilon\kappa}{z}. \tag{3·13}$$

Thus we find that the series converges for $z>\epsilon\kappa$ or $|x|>1$.

In order to derive the asymptotic behavior of the Coulomb solution $R_C{}^\nu$, it is useful to rewrite as

$$R_C{}^\nu=R_{C\,\text{in}}^\nu+R_{C\,\text{out}}^\nu \tag{3·14}$$

where

$$R^{\nu}_{C\,\mathrm{in}}=e^{-iz}z^{\nu-s}\left(1+\frac{\epsilon\kappa}{z}\right)^{(i/2)(\epsilon-\tau)}2^{\nu}e^{i\pi(\nu+1-s+i\epsilon)}$$

$$\times\sum_{n=-\infty}^{\infty}b_n{}^{\nu}(-2z)^n\frac{\Gamma(n+\nu+1-s+i\epsilon)}{\Gamma(n+\nu+1+s-i\epsilon)}\Psi(n+\nu+1-s+i\epsilon,2n+2\nu+2;2iz)\,, \tag{3·15}$$

$$R^{\nu}_{C\,\mathrm{out}}=e^{iz}z^{\nu-s}\left(1+\frac{\epsilon\kappa}{z}\right)^{(i/2)(\epsilon-\tau)}2^{\nu}e^{-i\pi(\nu+1+s-i\epsilon)}$$

$$\times\sum_{n=-\infty}^{\infty}b_n{}^{\nu}(-2z)^n\Psi(n+\nu+1+s-i\epsilon,2n+2\nu+2;-2iz)\,, \tag{3·16}$$

where Ψ is the irregular confluent hypergeometric function.[13)]

Another independent solution is obtained by replacing ν with $-\nu-1$ because $-\nu-1$ is the renormalized angular momentum if ν is the solution of Eq. (2·13). Thus, we have another independent solution by $R_C{}^{-\nu-1}$. The coefficients $b_{-n}^{-\nu-1}$ are obtained from $b_n{}^{\nu}$ by the relation

$$b_{-n}^{-\nu-1}=(-1)^n b_n{}^{\nu} \tag{3·17}$$

by choosing $b_0{}^{\nu}=b_0{}^{-\nu-1}=1$ in conformity with Eq. (3·10). With the use of Eq. (3·14), we find by using the identity $\Psi(-L\pm s\mp i\epsilon,-2L;x)=x^{2L+1}\Psi(L+1\pm s\mp i\epsilon,2L+2;x)$,

$$R^{-\nu-1}_{C\,\mathrm{in}}=-ie^{-i\pi\nu}\frac{\sin\pi(\nu-s+i\epsilon)}{\sin\pi(\nu+s-i\epsilon)}R^{\nu}_{C\,\mathrm{in}}\,, \tag{3·18}$$

$$R^{-\nu-1}_{C\,\mathrm{out}}=ie^{i\pi\nu}R^{\nu}_{C\,\mathrm{out}}\,. \tag{3·19}$$

Thus the solution $R_C{}^{-\nu-1}$ is expressed by

$$R_C{}^{-\nu-1}=-ie^{-i\pi\nu}\frac{\sin\pi(\nu-s+i\epsilon)}{\sin\pi(\nu+s-i\epsilon)}R^{\nu}_{C\,\mathrm{in}}+ie^{i\pi\nu}R^{\nu}_{C\,\mathrm{out}}\,. \tag{3·20}$$

§ 4. The relation between two solutions

First we note that $R_0{}^{\nu}$ and $R_C{}^{\nu}$ are solutions of the Teukolsky equation. Second we see that if we expand these solutions in Laurent series of $x=-z/\epsilon\kappa$, both solutions give the series with the same characteristic exponent at $x\to\infty$. Thus, $R_0{}^{\nu}$ must be proportional to $R_C{}^{\nu}$,

$$R_0{}^{\nu}=K_{\nu}R_C{}^{\nu}\,. \tag{4·1}$$

The constant factor K_{ν} is determined by comparing like terms of these series. We find

$$K_\nu=\frac{(\epsilon\kappa)^{-\nu-r+s}2^{-\nu-r}(-i)^r\Gamma(1-s-i\epsilon-i\tau)}{\Gamma(1+r+\nu+i\tau)\Gamma(1+r+\nu-s-i\epsilon)\Gamma(1+r+\nu-s+i\epsilon)}$$
$$\times\sum_{n=r}^{\infty}\frac{\Gamma(n+\nu+1+i\tau)\Gamma(n+r+2\nu+1)}{(n-r)!\Gamma(n+\nu+1-i\tau)}a_n{}^\nu$$
$$\times\left[\sum_{n=-\infty}^{r}(-i)^n\frac{b_n{}^\nu}{(r-n)!\Gamma(n+r+2\nu+2)}\right]^{-1}, \tag{4·2}$$

where r is an arbitrary integer.

By using these relations, $R_{\rm in}^\nu$ can be written by using the Coulomb expansion solutions as

$$R_{\rm in}^\nu=(K_\nu R_{\rm C\ in}^\nu+K_{-\nu-1}R_{\rm C\ in}^{-\nu-1})+(K_\nu R_{\rm C\ out}^\nu+K_{-\nu-1}R_{\rm C\ out}^{-\nu-1})$$
$$=\left(K_\nu-ie^{-i\pi\nu}\frac{\sin\pi(\nu-s+i\epsilon)}{\sin\pi(\nu+s-i\epsilon)}K_{-\nu-1}\right)R_{\rm C\ in}^\nu+(K_\nu+ie^{i\pi\nu}K_{-\nu-1})R_{\rm C\ out}^\nu\,. \tag{4·3}$$

The asymptotic behavior at $z\to\infty$ is

$$R_{\rm in}^\nu=A_{\rm out}^{s\nu}e^{iz}z^{-2s-1+i\epsilon}+A_{\rm in}^{s\nu}e^{-iz}z^{-1-i\epsilon}\,, \tag{4·4}$$

where $A_{\rm out}^{s\nu}$ and $A_{\rm in}^{s\nu}$ are amplitudes of the outgoing and incoming waves at infinity of the solution which satisfies the incoming boundary condition at the outer horizon. They are given by

$$A_{\rm out}^{s\nu}=e^{(i/2)\pi(\nu+1+s-i\epsilon)}2^{-1-s+i\epsilon}(K_\nu+ie^{i\pi\nu}K_{-\nu-1})\sum_{n=-\infty}^{\infty}b_n{}^\nu(-i)^n \tag{4·5}$$

and

$$A_{\rm in}^{s\nu}=e^{-(i/2)\pi(-\nu-1+s-i\epsilon)}2^{-1+s-i\epsilon}\left(K_\nu-ie^{-i\pi\nu}\frac{\sin\pi(\nu-s+i\epsilon)}{\sin\pi(\nu+s-i\epsilon)}K_{-\nu-1}\right)$$
$$\times\sum_{n=-\infty}^{\infty}b_n{}^\nu i^n\frac{\Gamma(n+\nu+1-s+i\epsilon)}{\Gamma(n+\nu+1+s-i\epsilon)}\,. \tag{4·6}$$

One application of these amplitudes is to derive the absorption coefficients. By using the method given in Ref. 14), the absorption coefficient Γ can be expressed in terms of $A_s{}^{\rm in}$ and $A_s{}^{\rm out}$ as follows:

$$\Gamma^{s\nu}=1-\left|\frac{A_{\rm out}^{-s\nu}A_{\rm out}^{s\nu}}{A_{\rm in}^{-s\nu}A_{\rm in}^{s\nu}}\right|. \tag{4·7}$$

In the end of this section, we show how the upgoing solution which satisfies the outgoing boundary condition at infinity is expressed in terms of $R_0{}^\nu$ and $R_0{}^{-\nu-1}$ defined in Eq. (2·21). From Eq. (2·30), we find

$$R_{\rm out}^\nu=\left(A_\nu K_\nu-ie^{-i\pi\nu}\frac{\sin\pi(\nu-s+i\epsilon)}{\sin\pi(\nu+s-i\epsilon)}A_{-\nu-1}K_{-\nu-1}\right)R_{\rm C\ in}^\nu$$
$$+(A_\nu K_\nu+ie^{i\pi\nu}A_{-\nu-1}K_{-\nu-1})R_{\rm C\ out}^\nu\,. \tag{4·8}$$

By using Eqs. (4·3) and (4·8), we obtain

$$R_{\mathrm{up}}^{\nu}=R_{\mathrm{C\ out}}^{\nu}$$
$$=\left[\frac{\sin\pi(\nu-s+i\epsilon)}{\sin\pi(\nu+s-i\epsilon)}(K_{\nu})^{-1}R_0{}^{\nu}-ie^{i\pi\nu}(K_{-\nu-1})^{-1}R_0{}^{-\nu-1}\right]$$
$$\times\left[e^{2i\pi\nu}+\frac{\sin\pi(\nu-s+i\epsilon)}{\sin\pi(\nu+s-i\epsilon)}\right]^{-1}. \tag{4·9}$$

§ 5. Low frequency expansions of solutions

In this section, we discuss how to derive the solution in the expansion of the small parameter $\epsilon=2M\omega$. In order to find the solution in Eqs. (2·1) and (4·3) up to some power of ϵ, we have to calculate ν and $a_n{}^{\nu}$ to that order by using Eq. (2·12) and (2·13) with the condition (2·14) and $a_0{}^{\nu}=a_0{}^{-\nu-1}=1$. Other coefficients $b_n{}^{\nu}$ can be calculated from $a_n{}^{\nu}$ by using the formula (3·10).

For $a_n{}^{\nu}$ with $n\geq 1$, the equation for $R_n(\nu)$ is useful. Since $\alpha_n{}^{\nu}$, $\gamma_n{}^{\nu}\sim O(\epsilon)$ and $\beta_n{}^{\nu}\simeq n(n+2l+1)\sim O(1)$, we find $R_n{}^{\nu}\sim O(\epsilon)$ for all positive integer n. As a result with $a_0{}^{\nu}=1$, we find

$$a_n{}^{\nu}\sim O(\epsilon^n)\quad\text{for } n\geq 1\,. \tag{5·1}$$

Before discussing the coefficients for $n<0$, we derive the renormalized angular momentum ν up to $O(\epsilon^2)$. For this, it is convenient to use the constraint for $n=1$, $R_1(\nu)L_0(\nu)=1$. We note that $R_1(\nu)\sim O(\epsilon)$ so that $L_0(\nu)$ must behave as $O(1/\epsilon)$, which requires that $\beta_0{}^{\nu}+\gamma_0{}^{\nu}L_{-1}(\nu)\sim O(\epsilon^2)$ because $\alpha_0{}^{\nu}\sim O(\epsilon)$. In order to obtain ν up to $O(\epsilon)$, we need to know the information of $\beta_0{}^{\nu}$ up to $O(\epsilon^2)$ where the second order term of ν is involved. Thus, we need the information about $R_1(\nu)$, $L_{-1}(\nu)$, $\alpha_0{}^{\nu}$ and $\gamma_0{}^{\nu}$ up to $O(\epsilon)$. Here we assume that $L_{-2}(\nu)\sim O(\epsilon)$ whose validity will be discussed later. In this situation, $R_1(\nu)$, $L_{-1}(\nu)$, $\alpha_0{}^{\nu}$ and $\gamma_0{}^{\nu}$ can be calculated immediately. By substituting these to the constraint equation $R_1(\nu)L_0(\nu)=1$, we find

$$\nu=l+\frac{1}{2l+1}\left[-2-\frac{s^2}{l(l+1)}+\frac{[(l+1)^2-s^2]^2}{(2l+1)(2l+2)(2l+3)}-\frac{(l^2-s^2)^2}{(2l-1)2l(2l+1)}\right]\epsilon^2$$
$$+O(\epsilon^3)\,. \tag{5·2}$$

The fact that the correction term of ν starts from the second order term of ϵ simplifies the calculation of the coefficients up to $O(\epsilon^2)$.

Now we discuss the coefficients for negative integer n for $s\neq 0$ which are derived by using the equation for $L_n(\nu)$. For large negative value of $|n|$, $L_n(\nu)\simeq -i\epsilon\kappa/2n$. Most of the negative integer value of n, $L_n(\nu)\sim O(\epsilon)$. There arise some exceptions for certain values of n because the denominator of $\alpha_n{}^{\nu}$ vanishes at $n=-l-1$ or $-l-3/2$ and also $\beta_n{}^{\nu}$ vanishes at $n=-2l-1$ in the zeroth order of ϵ. Because of this, we find for integers l,

$$L_{-l-1}(\nu)\sim O(1)\,,$$

$$L_{-2l-1}(\nu)\sim O(1/\epsilon)\,,$$

$$L_n(\nu)\sim O(\epsilon) \quad \text{for all others}. \tag{5·3}$$

We also find for half-integers n

$$L_{-(l+1/2)-1}(\nu)\sim O(1/\epsilon),$$
$$L_{-2(l+1/2)}(\nu)\sim O(1/\epsilon),$$
$$L_n(\nu)\sim O(\epsilon) \quad \text{for all others}. \tag{5·4}$$

From the above estimates, we find for a integer l,

$$a_n^{\nu}\sim O(\epsilon^{|n|}) \quad \text{for} \quad -1\geq n\geq -l,$$
$$a_{-l-1}^{\nu}\sim O(\epsilon^{l}),$$
$$a_n^{\nu}\sim O(\epsilon^{|n|-1}) \quad \text{for} \quad -l-2\geq n\geq -2l,$$
$$a_{-2l-1}^{\nu}\sim O(\epsilon^{2l-2}),$$
$$a_n^{\nu}\sim O(\epsilon^{|n|-3}) \quad \text{for} \quad -2l-2\geq n, \tag{5·5}$$

and for a half-integer l,

$$a_n^{\nu}\sim O(\epsilon^{|n|}) \quad \text{for} \quad n\geq -\left(l+\frac{1}{2}\right),$$
$$a_{-(l+1/2)-1}^{\nu}\sim O(\epsilon^{(l+1/2)-1})$$
$$a_n^{\nu}\sim O(\epsilon^{|n|-2}) \quad \text{for} \quad -\left(l+\frac{1}{2}\right)-2\geq n\geq -2\left(l+\frac{1}{2}\right)-1$$
$$a_{-2(l+1/2)}^{\nu}\sim O(\epsilon^{2(l+1/2)-4}),$$
$$a_n^{\nu}\sim O(\epsilon^{|n|-4}) \quad \text{for} \quad -2\left(l+\frac{1}{2}\right)-1\geq n. \tag{5·6}$$

With the above order estimates, we see that how many terms should be needed to calculate the coefficients with the specified accuracy of ϵ.

Coming back to ν, we assumed that $L_{-2}(\nu)\sim O(\epsilon)$ which is valid if we consider $l\geq 3/2$. However, this speciality is due to the fact that we solved the constraint equation for $n=1$ in Eq. (2·13). Since ν is independent of what n we used for solving the constraint equation, the result in Eq. (5·2) should be valid for all angular momentum case. In fact the result is nonsingular for all integer and half-integer values of l.

The coefficients a_n^{ν} and also b_n^{ν} up to $O(\epsilon^2)$ (which are valid for $l\geq 3/2$) are obtained explicitly by

$$a_1^{\nu}=i\frac{(l+1-s)^2[(l+1)\kappa+imq]}{2(l+1)^2(2l+1)}\epsilon + \frac{(l+1-s)^2}{2(l+1)^2(2l+1)}\left[1-i\frac{(l+1)\kappa+imq}{l(l+1)^2(l+2)}mqs^2\right]\epsilon^2+O(\epsilon^3), \tag{5·7}$$

$$a_2{}^{\nu}=-\frac{(l+1-s)^2(l+2-s)^2[(l+1)\kappa+imq][(l+2)\kappa+imq]}{4(l+1)^2(l+2)(2l+1)(2l+3)^2}\epsilon^2+O(\epsilon^3) \tag{5·8}$$

$$a_{-1}^{\nu}=i\frac{(l+s)^2[l\kappa-imq]}{2l^2(2l+1)}\epsilon$$
$$-\frac{(l+s)^2}{2l^2(2l+1)}\left[1+i\frac{l\kappa-imq}{(l-1)l^2(l+1)}mqs^2\right]\epsilon^2+O(\epsilon^3)\,, \tag{5·9}$$

$$a_{-2}^{\nu}=-\frac{(l-1+s)^2(l+s)^2[(l-1)\kappa-imq][l\kappa-imq]}{4(l-1)l^2(2l-1)^2(2l+1)}\epsilon^2+O(\epsilon^3)\,. \tag{5·10}$$

The coefficients $b_n{}^{\nu}$ are given from Eq. (3·10) by

$$b_1{}^{\nu}=i\frac{(l+1+s)^2}{(l+1-s)^2}a_1{}^{\nu}+O(\epsilon^3)\,, \tag{5·11}$$

$$b_2{}^{\nu}=-\frac{(l+1+s)^2(l+2+s)^2}{(l+1-s)^2(l+2-s)^2}a_2{}^{\nu}+O(\epsilon^3)\,, \tag{5·12}$$

$$b_{-1}^{\nu}=-i\frac{(l-s)^2}{(l+s)^2}a_{-1}^{\nu}+O(\epsilon^3)\,, \tag{5·13}$$

$$b_{-2}^{\nu}=-\frac{(l-1-s)^2(l-s)^2}{(l-1+s)^2(l+s)^2}a_{-2}^{\nu}+O(\epsilon^3)\,. \tag{5·14}$$

By using these coefficients, we can evaluate the ingoing and the outgoing amplitudes at infinity. From Eq. (4·2), we find by taking $r=0$ that $K_\nu\sim O(\epsilon^{-l+s})$. On the other hand, the estimate of $K_{-\nu-1}$ needs some care. By taking into account of the singular behaviors of gamma functions and the fact that the deviation of ν from l starts from the second order of ϵ, we find that $K_{-\nu-1}\sim O(\epsilon^{l-1+s}\sin i\pi\tau)$. Thus we obtain

$$\frac{K_{-\nu-1}}{K_\nu}\sim O(\epsilon^{2l-1}\sin i\pi\tau)\,, \tag{5·15}$$

where $\tau=(\epsilon-mq)/\kappa$. In the approximation up to $O(\epsilon^2)$, we can neglect $K_{-\nu-1}$ term when we restrict $l\geq 3/2$. We note that for the Schwarzshild case, $\tau=\epsilon$ so that the ratio in Eq. (5·15) is of order ϵ^{2l}.

Thus for $l\geq 3/2$, we get the simple expressions for the outgoing and the incoming amplitudes as follows:

$$A_{\rm out}^{s\nu}=-ie^{-(i/2)\pi(\nu+s-i\epsilon)}2^{-1-s+i\epsilon}K_\nu\sum_{n=-2}^{2}b_n{}^{\nu}(-i)^n \tag{5·16}$$

and

$$A_{\rm in}^{s\nu}=ie^{-(i/2)\pi(-\nu+s-i\epsilon)}2^{-1+s-i\epsilon}K_\nu\sum_{n=-2}^{2}b_n{}^{\nu}i^n\frac{\Gamma(n+\nu+1-s+i\epsilon)}{\Gamma(n+\nu+1+s-i\epsilon)}\,. \tag{5·17}$$

By substituting the coefficients, we can easily calculate the amplitudes up to the order ϵ^2. Since the explicit expressions are complicated, we present the amplitudes up to $O(\epsilon^2)$ explicitly. We find

$$A_{\rm out}^{s\nu}=-ie^{-(i/2)\pi(\nu+s-i\epsilon)}2^{-1-s+i\epsilon}K_\nu \exp\left\{i\left[\frac{\kappa}{2}\left(1+\frac{s^2}{l(l+1)}\right)\epsilon+\phi_2\epsilon^2\right]\right.$$

$$\left.+s\left[-\frac{mq}{l(l+1)}\epsilon+\psi_2\epsilon^2\right]\right\}\left[1+\frac{mqs^2}{2l^2(l+1)^2}\epsilon+d_2\epsilon^2\right] \tag{5·18}$$

and

$$A_{\rm in}^{s\nu}=ie^{-(i/2)\pi(-\nu+s-i\epsilon)}2^{-1+s-i\epsilon}K_\nu \exp i\left\{-\frac{\kappa}{2}\left(1-\frac{s^2}{l(l+1)}\right)\epsilon+\phi_2'\epsilon^2\right\}$$

$$\times\left[1+\frac{mqs^2}{2l^2(l+1)^2}\epsilon+d_2\epsilon^2\right], \tag{5·19}$$

where

$$\phi_2=\frac{\kappa mq}{2(2l+1)}\left[(l-s)^2\left\{\frac{(l-1-s)^2}{2(l-1)l^2(2l-1)}-\frac{s^2}{(l-1)l^3(l+1)}\right\}\right.$$

$$\left.+\frac{1}{4(2l+1)}\left\{\frac{(l-s)^2}{l}+\frac{(l+1+s)^2}{l+1}\right\}\left\{\frac{(l-s)^2}{l^2}-\frac{(l+1+s)^2}{(l+1)^2}\right\}\right]$$

$$+(l\to -l-1), \tag{5·20}$$

$$\psi_2=\frac{1}{2l+1}\left[\frac{1}{l}+(mqs)^2\left\{\frac{1}{(l-1)l^3(l+1)}+\frac{2l+1}{4l^3(l+1)^3}\right\}\right.$$

$$\left.+\frac{[\kappa(l-1)l-(mq)^2][(l-1)l+s^2]}{2(l-1)l^2(2l-1)}\right]+(l\to -l-1), \tag{5·21}$$

$$\phi_2'=\frac{mq}{2l+1}\left[\frac{1}{l}+\frac{\kappa s^2(l^2-s^2)}{2(l-1)l^3(l+1)}+\frac{\kappa[(l-1)^2-s^2][l^2-s^2]}{4(l-1)l^2(2l-1)}\right.$$

$$\left.+\frac{\kappa s^2}{8l^2(l+1)^2}\left\{1-\frac{s^2}{l(l+1)}\right\}\right]+(l\to -l-1), \tag{5·22}$$

$$d_2=\left[\frac{(mqs)^2}{4l^2(l+1)^2}-\frac{l^2+s^2}{2l^2(2l+1)}\left\{1+\frac{(mqs)^2}{(l-1)l^2(l+1)}\right\}\right.$$

$$-\frac{[\kappa^2(l-1)l-(mq)^2][((l-1)^2+s^2)(l^2+s^2)+4(l-1)ls^2]}{4(l-1)l^2(2l-1)^2(2l+1)}$$

$$\left.+\frac{\kappa^2}{16}\left\{1+\frac{s^2}{l(l+1)}\right\}^2\right]+(l\to -l-1). \tag{5·23}$$

The above result shows that the absorption coefficient Γ in Eq. (4·7) is zero up to the order ϵ^2 for Kerr black hole.

§ 6. Summary and remarks

Analytic solutions of the Teukolsky equation are obtained in the form of series of hypergeometric functions and Coulomb wave functions. The convergence of these solutions is examined. The series solution of hypergeometric type is convergent in the region except infinity, while the one of Coulomb type is convergent when $|x|>1$.

The renormalized angular momentum ν turns out to be identical for these two solutions. This fact enabled us to relate these two solutions analytically.

We examined the ϵ dependence of a_n^{ν} and found that the series corresponds essentially to ϵ except for some negative integer n where some anomalous behaviors occurred for which we need to take care to evaluate the coefficients. We explicitly calculated ν, the coefficients, $A^{s\nu}_{\rm out}$ and $A^{s\nu}_{\rm in}$ up to the order ϵ^2.

The solutions are useful not only to discuss the low frequency behavior of various physical quantities in applications, but also to know the general properties of solutions. For example, we consider the ϵ dependence of the renormalized angular momentum ν in the Schwarzschild geometry. ν is determined by solving the transcendental equation (2·13) which is composed of β_k^{ν} and $\alpha_k^{\nu}\gamma_{k+1}^{\nu}$. These quantities are even functions of ϵ in the Schwarzschild geometry because $\tau=\epsilon$. Therefore, we conclude that ν is an even function of ϵ, i.e., $\nu(-\epsilon)=\nu(\epsilon)$. This property is the special one and not valid for the Kerr geometry. The fact that the solutions are given by the ϵ expansion is important because the ϵ expansion corresponds to the Post-Minkowskian G expansion and also to the post-Newtonian expansion when they are applied to the gravitational radiation from a particle in circular orbit around a black hole. The solutions can be used for the analysis of the gravitational radiation from coalescing compact binary systems. Since the analytical properties and the convergences are known, the solution will give a powerful method for numerical computation and will contribute to construction of accurate theoretical templates for the gravitational wave observation by LIGO and VIRGO.

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Appendix A
—— *Spheroidal Teukolsky Equation* ——

To expand the radial Teukolsky equation, we have to derive the eigenvalue of the spheroidal Teukolsky equation. Fortunately, our method is also available in expansion of the spheroidal Teukolsky equation. Fackerell expands the spheroidal Teukolsky equation in terms of Jacobi functions.[12] In our expansion method, we can derive the eigenvalue of the spheroidal field equation which appears in the radial Teukolsky equation. The separated spheroidal Teukolsky equation is

$$\left[(1-x^2)\frac{d^2}{dx^2}-2x\frac{d}{dx}+\xi^2x^2-\frac{m^2+s^2-2msx}{1-x^2}-2s\xi x+E\right]S(x)=0\,, \qquad (\mathrm{A}\cdot1)$$

where $\xi=a\omega$, $x=\cos\theta$. We make transformation as

$$S(x)=e^{\xi x}\left(\frac{1-x}{2}\right)^{\alpha}\left(\frac{1-x}{2}\right)^{\beta}u(x)\,, \qquad (\mathrm{A}\cdot2)$$

where $\alpha=|m+s|$, $\beta=|m-s|$, then we recast the equation:

$$(1-x^2)u''+[\beta-\alpha-(2+\alpha+\beta)x]u'+\left[E-\frac{\alpha+\beta}{2}\left(\frac{\alpha+\beta}{2}+1\right)\right]u$$
$$=\xi[-2(1-x^2)u'+(\alpha+\beta+2s+2)xu-(\xi+\beta-\alpha)u]\,. \tag{A·3}$$

As a solution of the first order equation, we use a Jacobi function which is the solution of the equation

$$(1-x^2)U_n{}^{(\alpha,\beta)''}+[\beta-\alpha-(\alpha+\beta+2)x]U_n{}^{(\alpha,\beta)'}+n(n+\alpha+\beta+1)U_n{}^{(\alpha,\beta)}=0\,. \tag{A·4}$$

By using recursion relations of Jacobi functions, we can analytically expand u in terms of $U_n{}^{(\alpha,\beta)}$,[12]

$$u_n=\sum_{j=-\infty}^{\infty}c_jU_{n+j}^{(\alpha,\beta)}\,, \tag{A·5}$$

where $l=n+(\alpha+\beta)/2$. We can expand c_j, E in ξ:

$$c_j=\sum_{k=0}^{\infty}c_j{}^{(k)}\xi^k\,,\quad E=\sum_{k=0}^{\infty}E^{(k)}\xi^k\,, \tag{A·6}$$

where we set $c_0=1$, $c_{n\neq0}^{(0)}=0$ and $E^{(0)}=l(l+1)$.

$$c_1=\frac{(2l+2)^2-(\alpha+\beta)^2}{(2l+1)(2l+2)^2}(l+s+1)\xi\,, \tag{A·7}$$

$$c_{-1}=\frac{(2l)^2-(\alpha-\beta)^2}{(2l)^2(2l+1)}(l-s)\xi\,, \tag{A·8}$$

$$c_2=\frac{[(2l+4)^2-(\alpha+\beta)^2][(2l+2)^2-(\alpha+\beta)^2]}{4(2l+1)(2l+2)^2(2l+3)^2(2l+4)}(l+s+1)(l+s+2)\xi^2\,, \tag{A·9}$$

$$c_{-2}=\frac{[(2l-2)^2-(\alpha-\beta)^2][(2l)^2-(\alpha-\beta)^2]}{4(2l-2)^2(2l-1)(2l)(2l+1)^2}(l-s)(l-s-1)\xi^2 \tag{A·10}$$

and E in Eq. (1·3) which is identical to that of Fackerell[12] who also has shown that the convergency of the expansion in terms of Jacobi functions.

Appendix B
—— *Derivations of Equations (2·2) and (3·2)* ——

The radial Teukolsky equation is written by using the variable $y=\omega r$ with $y_+=\omega r_+$ and $y_-=\omega r_-$ as

$$\frac{d^2R}{dy^2}+(s+1)\left(\frac{1}{y-y_+}+\frac{1}{y-y_-}\right)\frac{dR}{dy}$$
$$+\left[1+\frac{1}{y-y_+}\left(\epsilon+is+\frac{\epsilon+2is}{\kappa}\right)+\frac{1}{y-y_-}\left(\epsilon+is-\frac{\epsilon+2is}{\kappa}\right)\right.$$
$$+\frac{1}{(y-y_+)^2}\frac{(\epsilon-is+\tau)^2+s^2}{4}+\frac{1}{(y-y_-)^2}\frac{(\epsilon-is-\tau)^2+s^2}{4}$$
$$\left.+\frac{1}{(y-y_+)(y-y_-)}\left(-\lambda+\frac{\epsilon^2}{2}-\frac{\tau^2}{2}-i\epsilon s-\epsilon mq\right)\right]R=0\,. \tag{B·1}$$

(a) The expansion in terms of hypergeometric functions

We define a new variable x by

$$y-y_{+}=\epsilon\kappa(-x)\,,\quad y-y_{-}=\epsilon\kappa(1-x)\,. \tag{B·2}$$

We rewrite R as

$$R=(-x)^{\alpha}(1-x)^{\beta}\tilde{R} \tag{B·3}$$

in order to eliminate the terms proportional to $1/x^2$ and $1/(1-x)^2$. Then, α and β are determined to be one of the following values,

$$\alpha_{\pm}=\frac{1}{2}(-s\pm i(\epsilon-is+\tau))\,,\quad \beta_{\pm}=\frac{1}{2}(-s\pm i(\epsilon-is-\tau))\,, \tag{B·4}$$

respectively. With these choices of α and β and the change of $\tilde{R}$ in the following form:

$$\tilde{R}=e^{i\epsilon\kappa x}p\,, \tag{B·5}$$

the equation for p is expressed as

$$\begin{aligned}&x(1-x)p''+[(2\alpha+s+1)-2(\alpha+\beta+s+1)x]p'-abp\\&\quad=-2i\epsilon\kappa x(1-x)p'+2i\epsilon\kappa(\alpha+\beta+1+i\epsilon)xp+[-\lambda+2\alpha\beta+(s+1)(\alpha+\beta)\\&\qquad-i\epsilon\kappa(2\alpha+s+1)+\epsilon\kappa(\epsilon+is)+\frac{3}{2}\epsilon^2-\frac{1}{2}\tau^2+i\epsilon s-\epsilon mq-ab]p\,,\end{aligned} \tag{B·6}$$

where a and b are chosen such that the equality

$$a+b=2(\alpha+\beta)+2s+1 \tag{B·7}$$

is satisfied and also they take some simple forms.

If we take one of choices (α_{-}, β_{+}) and (α_{-}, β_{-}), the form of R defined in Eq. (B·3) becomes a suitable form for the solution which satisfies the incoming boundary condition on the outer horizon. On the other hand, the choice of (α_{+}, β_{-}) or (α_{+}, β_{+}) is suitable to obtain the solution which satisfies the outgoing boundary condition. In the text, we took (α_{-}, β_{+}) for the solution satisfying the incoming boundary condition in which case the above equation (B·6) reduces to the one in Eq. (2·2), by taking $a=\nu+1-i\tau$, $b=-\nu-i\tau$. For the solution satisfying the outgoing boundary condition, we took (α_{+}, β_{-}) in which case the equation takes a similar form.

(b) The expansion in terms of Coulomb wave functions

We take the parameterization

$$R=[(y-y_{+})(y-y_{-})]^{-(1+s)/2}\left(\frac{y-y_{+}}{y-y_{-}}\right)^{\gamma}f \tag{B·8}$$

and determine γ to eliminate the singularity proportional to $1/(y-y_{-})^2$. Then we find γ should take one of the following values:

$$\gamma_{\pm}=\frac{1}{2}[-1\pm i(\epsilon-is-\tau)]\,. \tag{B·9}$$

If we take one of these values of γ, the equation for f becomes with $z=y-y_+=-\epsilon\kappa x$

$$z^2 f''+[z^2+2(\epsilon+is)z]f=-\epsilon\kappa z(f''+f)-2\gamma\epsilon\kappa f'-\frac{\epsilon\kappa[2\gamma-\tau(\epsilon-is)]}{z}f$$
$$+[\lambda+s(s+1)-2\epsilon^2+\epsilon mq-\epsilon\kappa(\epsilon+is)]f\,. \qquad (\mathrm{B}\cdot 10)$$

Equation (3·2) in the text is obtained by taking $\gamma=\gamma_-$. The choice $\gamma=\gamma_-$ gives a Coulomb type solution which matches with the hypergeometric type solution with (α_-, β_+) as we saw in Eq. (4·1). We can also obtain the solution by choosing γ_+ which matches with the hypergeometric one with (α_-, β_-).

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