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An analytic representation for the quasi-normal modes of Kerr black holes

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The gravitational quasi-normal frequencies of both stationary and rotating black holes are calculated by constructing exact eigensolutions to the radiative boundary-value problem of Chandrasekhar and Detweiler. The method is that employed by Jaffé in his determination of the electronic spectra of the hydrogen molecule ion in 1934, and analytic representations of the quasi-normal mode wavefunctions are presented here for the first time. Numerical solution of Jaffé's characteristic equation indicates that for each l-pole there is an infinite number of damped Schwarzschild quasi-normal modes. The real parts of the corresponding frequencies are bounded, but the imaginary parts are not. Figures are presented that illustrate the trajectories the five least-damped of these frequencies trace in the complex frequency plane as the angular momentum of the black hole increases from zero to near the Kerr limit of maximum angular momentum per unit mass, $a = M$, where there is a coalescence of the more highly damped frequencies to the purely real value of the critical frequency for superradiant scattering.

1. INTRODUCTION

Complex resonant frequencies characteristic of the Schwarzschild geometry were first discovered in calculations of the scattering of gravitational waves by black holes (Vishveshwara 1970). Recent speculation as to the role that black holes might play in a variety of astrophysical processes has created considerable interest in methods of computing these resonant (or quasi-normal) frequencies. In this paper the problem of determining the gravitational quasi-normal frequencies is cast, after the manner of Zerilli (1970), Chandrasekhar & Detweiler (1975, 1976), and Detweiler (1977, 1980), in the form of a linearized boundary-value problem on a stationary black hole background. Specifically, Teukolsky's equations describing small perturbations to the Kerr geometry are shown to be generalized spheroidal wave equations of the type solved by George Jaffé (1934) in his classic determination of the electronic spectra of the hydrogen molecule ion. A solution of Jaffé's form, as generalized by Baber & Hassé (1935), can be applied to the Chandrasekhar–Detweiler problem, and yields the complete scalar, electromagnetic, and gravitational quasi-normal frequency spectra of Kerr black holes. Jaffé's representation defines the eigensolutions (quasi-normal mode functions) of

Teukolsky's equations analytically, and eliminates the tedious (and frequently inaccurate) numerical integrations that have characterized previous methods. Instead, the quasi-normal frequencies and angular separation constants are defined as the simultaneous roots of two characteristic continued fraction equations, and these may be solved numerically with high precision.

Present results may be summarized as follows: (i) for each l-pole moment a Schwarzschild black hole possesses an infinity of distinct complex quasi-normal frequencies $\{\omega_n : n = 1, 2 \ldots\}$; (ii) for fixed l and large n these frequencies become evenly spaced along an asymptote parallel the imaginary ω axis; (iii) this relationship among the higher-order (large n) modes changes markedly as the angular momentum of the black hole increases to the extreme Kerr limit. For non-zero azimuthal separation constant m, all but a possibly finite number of the high-order modes coalesce to one undamped mode. The frequency of this undamped coalescence mode is simply $\omega_c = mc^3/2GM$, the critical frequency for superradiant scattering.

2. Schwarzschild quasi-normal modes

In this section I review the Chandrasekhar–Detweiler radiative boundary-value problem, and produce the Schwarzschild quasi-normal modes as its eigenfunctions. The problem has been solved previously by Chandrasekhar & Detweiler (1975), who employed a numerical integration scheme to solve the separated partial differential equation with sufficient accuracy to allow the determination of the under-damped (i.e. $|\mathrm{Re}(\omega)| > |\mathrm{Im}(\omega)|$) quasi-normal frequencies, and by Ferrari & Mashhoon (1984), who obtained approximate values for the fundamental (least damped) quasi-normal frequency via a potential inversion method that was amenable to W.K.B. analysis at large values of the multipole moment l. Other important results were obtained by Press (1971) and Cunningham *et al.* (1978), who estimated quasi-normal frequencies after numerical integration of the time-dependent wave equation. Difficulties inherent to numerical integration methods are discussed by Detweiler (1979). The method presented here is similar to the original one of Chandrasekhar & Detweiler, but uses analytic solutions to the differential equation. It will be seen that this approach allows an essentially complete characterization of the quasi-normal frequencies both for static and for rotating black holes.

Choose Schwarzschild coordinates and let $\psi(t, r, \theta, \phi)$ denote a component of a perturbation to a massless spin s field. Understanding of wave equations obeyed by ψ has come from studies by Wheeler (1955), Regge & Wheeler (1957), Zerilli (1970), Bardeen & Press (1973), Chandrasekhar (1975), and Chandrasekhar & Detweiler (1975). If $\psi(t, r, \theta, \phi)$ is Fourier analysed and expanded in spherical harmonics as

$$\psi(t, r, \theta, \phi) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega t} \left(\sum_l \frac{1}{r} \psi_l(r, \omega)\, Y_{lm}(\theta, \phi) \right) \mathrm{d}\omega, \tag{1}$$

then it suffices to write the ordinary differential equation satisfied by $\psi_l(r, \omega)$ in the form, where t and r are scaled such that $c = G = 2M = 1$,

$$r(r-1)\,\psi_{l,\,rr} + \psi_{l,\,r} + \left[\frac{\omega^3 r^3}{r-1} - l(l+1) + \frac{\epsilon}{r} \right] \psi_l = 0. \tag{2}$$

The index ϵ is one less than the square of the field's spin weight, and takes the values $-1, 0$, or $+3$ depending on whether ψ represents, respectively, a component of a scalar, electromagnetic, or gravitational field.

Equation (2) is a second-order ordinary differential equation with two regular singular points and one confluently irregular singular point. It belongs to a class of differential equations known as *generalized spheroidal wave equations* (Wilson 1928). The regular singular points are at $r = 0$ and at $r = 1$ (the event horizon). The irregular singularity is at $r = \infty$. The singular point at $r = 0$ has indices of $1 \pm (\epsilon+1)^{\frac{1}{2}}$, and the singular point at $r = 1$ has indices $\pm \mathrm{i}\omega$. The asymptotic solutions to (2) are $\psi_l \rightarrow \exp[\pm \mathrm{i}\omega(r+\ln r)]$ as $r \rightarrow \infty$. The boundary conditions for the exterior eigenvalue problem (the quasi-normal mode problem) are that $\psi_l \rightarrow (r-1)^{-\mathrm{i}\omega}$ as $r \rightarrow 1$, and that $\psi_l \rightarrow \exp[\mathrm{i}\omega(r+\ln r)]$ as $r \rightarrow \infty$. These boundary conditions ensure that the field radiate only inward at the horizon and only outward at spatial infinity.

It is notationally convenient to introduce a new frequency variable ρ by $\rho = -\mathrm{i}\omega$. Then the boundary value problem may be expressed as the differential equation

$$r(r-1)\,\psi_{l,\,rr} + \psi_{l,\,r} - \left[\frac{\rho^2 r^3}{r-1} + l(l+1) - \frac{\epsilon}{r}\right]\psi_l = 0 \tag{3}$$

subject to boundary conditions

$$\psi_l \xrightarrow{r\rightarrow 1} (r-1)^{\rho} \quad \text{and} \quad \psi_l \xrightarrow{r\rightarrow\infty} r^{-\rho}\,\mathrm{e}^{-\rho r}. \tag{4}$$

A solution to equation (3) that has the desired behaviour at the event horizon ($r = 1$) can be written in the form

$$\psi_l = (r-1)^{\rho}\, r^{-2\rho}\, \mathrm{e}^{-\rho(r-1)} \sum_{n=0}^{\infty} a_n \left(\frac{r-1}{r}\right)^n \tag{5}$$

(Baber & Hassé 1935, p. 568). The sequence of expansion coefficients $\{a_n : n = 1, 2 \ldots\}$ is determined by a three-term recurrence relation starting with $a_0 = 1$:

$$\alpha_0\, a_1 + \beta_0\, a_0 = 0, \tag{6}$$

$$\alpha_n\, a_{n+1} + \beta_n\, a_n + \gamma_n\, a_{n-1} = 0, \quad n = 1, 2 \ldots. \tag{7}$$

The recurrence coefficients α_n, β_n, and γ_n are simple functions of n and the parameters of the differential equation:

$$\left.\begin{aligned} \alpha_n &= n^2 + (2\rho+2)\,n + 2\rho + 1, \\ \beta_n &= -(2n^2 + (8\rho+2)\,n + 8\rho^2 + 4\rho + l(l+1) - \epsilon), \\ \gamma_n &= n^2 + 4\rho n + 4\rho^2 - \epsilon - 1. \end{aligned}\right\} \tag{8}$$

The boundary condition at spatial infinity will be satisfied for those values of $\omega = \omega_n$ (the quasi-normal frequencies) for which the series in (5) is absolutely convergent, i.e. for which $\sum a_n$ exists and is finite.

The theory of three-term recurrence relations (Gautschi 1967) may by invoked to determine the conditions under which this sum of coefficients converges. Baber

& Hassé's analysis of the large n behaviour of the expansion coefficients a_n indicates

$$\frac{a_{n+1}}{a_n} \overset{n\to\infty}{\to} 1 \pm \frac{(2\rho)^{\frac{1}{2}}}{n^{\frac{1}{2}}} + \frac{2\rho - \frac{3}{4}}{n} + \dots. \tag{9}$$

The series in (5) will converge uniformly only if the minus sign is obtained in (9), which will happen only for eigenvalues ρ corresponding to quasi-normal frequencies. The a_n are then said to form a 'solution sequence to the recurrence relation (7) that is minimal as $n \to \infty$' (Gautschi 1967), and the ratio of successive a_n will be given by the infinite continued fraction

$$\frac{a_{n+1}}{a_n} = \cfrac{-\gamma_{n+1}}{\beta_{n+1} - \cfrac{\alpha_{n+1}\gamma_{n+2}}{\beta_{n+2} - \cfrac{\alpha_{n+2}\gamma_{n+3}}{\beta_{n+3} - \dots}}}.$$

The usual notation for such a continued fraction is

$$\frac{a_{n+1}}{a_n} = \frac{-\gamma_{n+1}}{\beta_{n+1}-}\,\frac{\alpha_{n+1}\gamma_{n+2}}{\beta_{n+2}-}\,\frac{\alpha_{n+2}\gamma_{n+3}}{\beta_{n+3}-}\dots. \tag{10}$$

Equation (10) may be thought of as an '$n = \infty$ boundary condition' on the sequence a_n. We obtain a characteristic equation for the quasi-normal frequencies by evaluating (10) at $n = 0$, and using equation (6) as an '$n = 0$ boundary condition' on the ratio a_1/a_0. Specifically, we have two expressions that must be satisfied:

$$\frac{a_1}{a_0} = -\frac{\beta_0}{\alpha_0} \tag{11}$$

and

$$\frac{a_1}{a_0} = \frac{-\gamma_1}{\beta_1-}\,\frac{\alpha_1\gamma_2}{\beta_2-}\,\frac{\alpha_2\gamma_3}{\beta_3-}\dots. \tag{12}$$

We equate the right-hand sides of (11) and (12) to obtain the desired (implicit) characteristic equation for the quasi-normal frequencies:

$$0 = \beta_0 - \frac{\alpha_0\gamma_1}{\beta_1-}\,\frac{\alpha_1\gamma_2}{\beta_2-}\,\frac{\alpha_2\gamma_3}{\beta_3-}\dots. \tag{13}$$

The α_n, β_n, and γ_n are explicit functions of the frequency $\rho = -\mathrm{i}\omega$, and are given by (8).

Equation (13) may be inverted an arbitrary number of times, n, to yield an equality between two continued fractions, one of infinite length, as in (13), and the other finite:

$$\left[\beta_n - \frac{\alpha_{n-1}\gamma_n}{\beta_{n-1}-}\,\frac{\alpha_{n-2}\gamma_{n-1}}{\beta_{n-2}-}\dots - \frac{\alpha_0\gamma_1}{\beta_0}\right] = \left[\frac{\alpha_n\gamma_{n+1}}{\beta_{n+1}-}\,\frac{\alpha_{n+1}\gamma_{n+2}}{\beta_{n+2}-}\,\frac{\alpha_{n+2}\gamma_{n+3}}{\beta_{n+3}-}\dots\right] \quad (n = 1, 2 \dots). \tag{14}$$

For every $n > 0$, (13) and (14) are completely equivalent in that every solution to (13) is also a solution to (14), and vice versa. Either one may be taken as the defining equation for the Schwarzschild quasi-normal frequencies ω_n, and the determination of those frequencies is now reduced to the numerical problem of finding the roots of this equation. That equation (13) involves an infinite continued fraction whose elements are each a different function of the frequency, similar in form to the elements of the continued fraction that determines the spectra of the hydrogen molecule-like ions, leads to the suspicion that the equation should have an infinite number of roots. Although I have no formal proof of this infinity, I have calculated sixty roots for $l = 2$ and $l = 3$ gravitational fields. These are plotted in figure 1, and tend to support the idea of an infinity of quasi-normal frequencies that asymptotically approach the values $(\pm 0.15, -\frac{1}{2}n+0.20)$ for $l = 2$, and $(\pm 0.16, -\frac{1}{2}(n-1)+0.13)$ for $l=3$. (More accurate computations will be necessary before better representations of these asymptotes can be deduced.)

Although each inversion of (14) has the same solutions as (13), the topology of the function on the right-hand side of the equation changes markedly with the number of inversions n. The nth quasi-normal mode is usually found to be numerically the most stable root of the nth inversion.

The first ten lowest-order modes were computed for $l = 4$ to $l = 12$, and the frequencies plotted in figure 2. The values of the fundamental frequencies for the larger values of l approach the $2(\pm l+\frac{1}{2}, -n-\frac{1}{2})/(27)^{\frac{1}{2}}$ asymptote obtained by Ferrari & Mashhoon (1984) through a W.K.B. argument. Some details of the root-search algorithm are presented in §4. Further discussion of the Jaffé representation of the quasi-normal mode wavefunctions and the three-term recurrence relation (7) that generates it will be found in Leaver (1985*a*, *b*).

3. Kerr quasi-normal modes

The analysis of the preceding section may readily be generalized to the case of rotating black holes. The relevant partial differential equation is given by Teukolsky (1972). Coordinates are the Boyer–Lindquist t, r, θ, and ϕ. We again scale t and r such that $c = G = 2M = 1$. Teukolsky denotes the field quantities by ψ, and separates the wave equation by writing

$$\psi(t,r,\theta,\phi) = \frac{1}{2\pi}\int \mathrm{e}^{-\mathrm{i}\omega t} \sum_{l=|s|}^{\infty} \sum_{m=-l}^{l} \mathrm{e}^{\mathrm{i}m\phi} S_{lm}(u)\, R_{lm}(r)\, \mathrm{d}\omega. \tag{15}$$

The separated differential equations for R_{lm} and S_{lm} are

$$[(1-u^2)\, S_{lm,\,u}]_{,u} + \left[a^2\omega^2u^2 - 2a\omega su + s + A_{lm} - \frac{(m+su)^2}{1-u^2}\right] S_{lm} = 0, \tag{16}$$

which is Teukolsky's equation 8, but with $u = \cos\theta$, and

$$\Delta R_{lm,\,rr} + (s+1)\,(2r-1)\, R_{lm,\,r} + V(r)\, R_{lm} = 0, \tag{17}$$

where

$$V(r) = \{[(r^2+a^2)^2\,\omega^2 - 2am\omega r + a^2m^2 + \mathrm{i}s(am(2r-1) - \omega(r^2-a^2))]\,\Delta^{-1} + [2\mathrm{i}s\omega r - a^2\omega^2 - A_{lm}]\},$$

which is Teukolsky's equation 7. The rotation parameter a is the angular momentum per unit mass $(0 \leqslant a \leqslant \frac{1}{2})$, and $\Delta = r^2 - r + a^2$. The field spin-weight parameter s takes the values 0, -1, -2, respectively, for outgoing scalar, electromagnetic, and gravitational fields. A_{lm} is the angular separation constant for (16), and reduces to $l(l+1)-s(s+1)$ at the Schwarzschild limit (see below).

Boundary conditions for (16) are that S_{lm} be finite at the regular singular points $u=+1$ and $u=-1$, where the indices are $\pm\frac{1}{2}(m+s)$ and $\pm\frac{1}{2}(m-s)$, respectively. A solution to (16) may be expressed as

$$S_{lm}(u) = e^{a\omega u}(1+u)^{\frac{1}{2}|m-s|}(1-u)^{\frac{1}{2}|m+s|}\sum_{n=0}^{\infty} a_n(1+u)^n \tag{18}$$

(Baber & Hassé 1935, equation 34). The expansion coefficients are related by a three-term recurrence relation, and the boundary condition at $u=+1$ will be satisfied only by its minimal solution sequence. The recurrence relation is

$$\left.\begin{aligned} &\alpha_0^\theta a_1 + \beta_0^\theta a_0 = 0, \\ &\alpha_n^\theta a_{n+1} + \beta_n^\theta a_n + \gamma_n^\theta a_{n-1} = 0, \quad n = 1, 2 \ldots \end{aligned}\right\} \tag{19}$$

where the superscript θ is used to denote association with the 'angular' equation, and the recurrence coefficients are, with $k_1 = \frac{1}{2}|m-s|$ and $k_2 = \frac{1}{2}|m+s|$,

$$\left.\begin{aligned} \alpha_n^\theta &= -2(n+1)(n+2k_1+1), \\ \beta_n^\theta &= n(n-1)+2n(k_1+k_2+1-2a\omega) \\ &\quad -[2a\omega(2k_1+s+1)-(k_1+k_2)(k_1+k_2+1)]-[a^2\omega^2+s(s+1)+A_{lm}], \\ \gamma_n^\theta &= 2a\omega(n+k_1+k_2+s). \end{aligned}\right\} \tag{20}$$

For a given a, m, ω, and s the minimal solution sequence will satisfy (19) if the separation constant A_{lm} is a root of the continued fraction equation

$$0 = \beta_0^\theta - \frac{\alpha_0^\theta \gamma_1^\theta}{\beta_1^\theta -}\,\frac{\alpha_1^\theta \gamma_2^\theta}{\beta_2^\theta -}\,\frac{\alpha_2^\theta \gamma_3^\theta}{\beta_3^\theta -}\cdots \tag{21}$$

or any of its inversions (cf. equation (14)). Note that at the Schwarzschild limit ($a=0$) the γ_n are zero for all n, and the recursion will stop whenever A_{lm} is such that β_n is zero for some n. This will happen when $A_{lm} = n(n+1)-s(s+1)$, that is, when $n = l$ (Teukolsky 1972).

A solution $R_{lm}(r)$ to (17) may be found in a manner similar to our solution to (2) since both are generalized spheroidal wave equations with similar boundary conditions. Teukolsky defines the regular singular points r_+ and r_- as the roots of Δ, so that $\Delta = r^2 - r + a^2 \equiv (r-r_-)(r-r_+)$. It is useful to define an auxiliary rotation parameter $b = (1-4a^2)^{\frac{1}{2}}$, so that b ranges from 1 to 0 as a ranges from 0 to $\frac{1}{2}$ (Kerr limit). Then $r_\pm = \frac{1}{2}(1 \pm b)$. The event horizon is at the larger of these values, $r = r_+$. The indices at $r = r_+$ are $i\sigma_+$ and $-s-i\sigma_+$, where $\sigma_+ = (\omega r_+ - am)/b$. It is the second index that corresponds to in-going radiation.

Asymptotic solutions to (17) are

$$\lim_{r\to\infty} R_{lm}(r) \sim r^{-1-i\omega}e^{-i\omega r} \quad \text{and} \quad \lim_{r\to\infty} R_{lm}(r) \sim r^{-1-2s+i\omega}e^{+i\omega r}$$

(Teukolsky 1972), the latter being outgoing according to the sign convention of

(15). The radial equation boundary conditions for the quasi-normal mode problem are then

$$R_{lm}(r) \overset{r\to r_+}{\to} (r-r_+)^{-s-\mathrm{i}\sigma_+} \quad \text{and} \quad R_{lm}(r) \overset{r\to\infty}{\to} r^{-1-2s+\mathrm{i}\omega}\,\mathrm{e}^{\mathrm{i}\omega r}. \tag{22}$$

Our solution may be expressed as

$$R_{lm} = \mathrm{e}^{\mathrm{i}\omega r}(r-r_-)^{-1-s+\mathrm{i}\omega+\mathrm{i}\sigma_+}(r-r_+)^{-s-\mathrm{i}\sigma_+} \sum_{n=0}^{\infty} d_n \left(\frac{r-r_+}{r-r_-}\right)^n, \tag{23}$$

where the expansion coefficients are again defined by a three-term recursion relation:

$$\left.\begin{aligned} \alpha_0^r d_1 + \beta_0^r d_0 &= 0, \\ \alpha_n^r d_{n+1} + \beta_n^r d_n + \gamma_n^r d_{n-1} &= 0, \quad n = 1, 2 \ldots \end{aligned}\right\} \tag{24}$$

The recursion coefficients are

$$\left.\begin{aligned} \alpha_n^r &= n^2 + (c_0+1)\,n + c_0, \\ \beta_n^r &= -2n^2 + (c_1+2)\,n + c_3, \\ \gamma_n^r &= n^2 + (c_2-3)\,n + c_4 - c_2 + 2, \end{aligned}\right\} \tag{25}$$

and the intermediate constants c_n are defined by

$$\left.\begin{aligned} c_0 &= 1 - s - \mathrm{i}\omega - \frac{2\mathrm{i}}{b}\left(\frac{\omega}{2} - am\right), \\ c_1 &= -4 + 2\mathrm{i}\omega(2+b) + \frac{4\mathrm{i}}{b}\left(\frac{\omega}{2} - am\right), \\ c_2 &= s + 3 - 3\mathrm{i}\omega - \frac{2\mathrm{i}}{b}\left(\frac{\omega}{2} - am\right), \\ c_3 &= \omega^2(4+2b-a^2) - 2am\omega - s - 1 + (2+b)\mathrm{i}\omega - A_{lm} + \frac{4\omega+2\mathrm{i}}{b}\left(\frac{\omega}{2} - am\right), \\ c_4 &= s + 1 - 2\omega^2 - (2s+3)\,\mathrm{i}\omega - \frac{4\omega+2\mathrm{i}}{b}\left(\frac{\omega}{2} - am\right). \end{aligned}\right\} \tag{26}$$

The series in (23) converges and the $r = \infty$ boundary condition (22) is satisfied if, for a given a, m, A_{lm}, and s, the frequency ω is a root of the continued fraction equation

$$0 = \beta_0^r - \frac{\alpha_0^r \gamma_1^r}{\beta_1^r -} \frac{\alpha_1^r \gamma_2^r}{\beta_2^r -} \frac{\alpha_2^r \gamma_3^r}{\beta_3^r -} \ldots, \tag{27}$$

or any of its inversions.

It can be shown that in the limit as $a \to 0$, the β_n^r of (25) equal the β_n of (8), and that the product $\alpha_n^r \gamma_{n+1}^r$ of (25) equals the product $\alpha_n \gamma_{n+1}$ of (8). Since $A_{lm} \to l(l+1) - s(s+1)$ as $a \to 0$, we have the necessary result that (27) reduces to (13) at the Schwarzschild limit.

Equations (21) and (27) are two equations for the unknowns A_{lm} and ω. They may be solved simultaneously by standard nonlinear root-search algorithms. I used Argonne Laboratory's MINPACK subroutine HYBRD. The continued fractions were evaluated by Steed's algorithm (Barnett *et al.* 1974), with the Numerical Algorithm Group's sequence accelerator subroutine C06BAF used to speed convergence of the approximants.

It is interesting to note that the apparently singular nature of the recursion coefficients (25) at the Kerr limit can be avoided if $\lim_{b\to 0}\omega = m$, which in the normalized units used here corresponds to the critical frequency ω_c for the superradiant scattering of an incident wave of spheroidal multipole m (Detweiler 1977; Chandrasekhar & Detweiler 1976). We should not be surprised to find that as $b \to 0$ the value $\omega_c = m$ is indeed a root of (21) and (27), at least for $m \geqslant 1$ (see figure 3). In fact, the numerical results suggest that in the Kerr limit the number of damped low-order modes may become finite (although some quite imaginative extrapolation would be necessary to infer the exact number from the present data), and the frequencies of the highest order modes coalesce to the single undamped frequency ω_c. Detweiler (1980) has given an analytic proof that ω_c is an accumulation point for quasi-normal frequencies at the Kerr limit. The present study suggests the likelihood that each of the infinity of frequencies clustered near ω_c can be mapped to one of the infinity of Schwarzschild quasi-normal frequencies as the rotation parameter decreases from $a = \frac{1}{2}$ to $a = 0$.

Jaffé's method can express the quasi-normal mode wavefunctions for all values of the rotation parameter a less than the Kerr limit $a = \frac{1}{2}$, but fails when $a \equiv \frac{1}{2}$ because there $r_+ = r_-$ and the sum $\Sigma\, d_n[(r-r_+)/(r-r_-)]^n$ becomes useless as a solution to differential equation (17). As $a \to \frac{1}{2}$ the regular singular points $r = r_+$ and $r = r_-$ of (17) coalesce to form an irregular (confluent) singular point. Analytic solutions to the differential equation in this case do exist, and are described in Leaver (1985*a*,*b*).

4. Complex conjugate symmetry and discussion of results

Consideration of quasi-normal frequencies as the poles of the Green function that propagates the perturbations requires the quasi-normal modes to appear as complex conjugate pairs of the frequency variable $\rho = -\mathrm{i}\omega$, for the only way a real perturbation can excite a complex mode characterized by a complex frequency to give purely real radiation is if that real perturbation simultaneously excites a symmetry mode that is complex conjugate to the first. Ferrari and Mashhoon (1984) attain this requirement by treating the quasi-normal frequencies as the poles of the reflection amplitude of radiation scattered by the Regge–Wheeler potential. For the Schwarzschild geometry this symmetry is explicit in (14) since the continued fractions are real when ρ is real. The complex conjugate symmetry of the roots is then assured by the Schwartz reflection principle.

The symmetry of the Schwarzschild quasi-normal frequencies about the imaginary ω-axis is shown explicitly in figure 1 for $l = 2$ and $l = 3$. I include both branches in this figure to illustrate the crossings the branches make of the imaginary ω-axis (for example, at $(0, -3.998)$ for $l = 2$). The values of some of these frequencies are listed in table 1. Only the right-hand frequency branches (for l values 4 to 12) are plotted in figure 2.

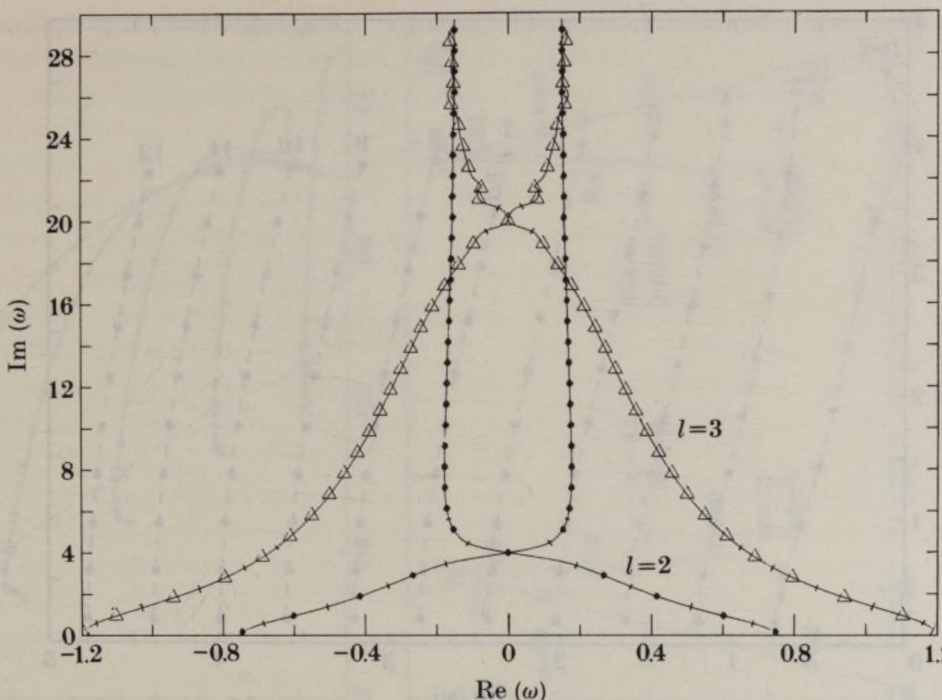

FIGURE 1. First 60 Schwarzschild quasi-normal frequencies for $l = 2$ and $l = 3$. The odd-order frequencies are prominently marked; a few-even order frequencies are indicated as short bars perpendicular to the curves connecting the points.

TABLE 1. REPRESENTATIVE SCHWARZSCHILD GRAVITATIONAL QUASI-NORMAL FREQUENCIES FOR $l = 2$ AND $l = 3$

(Note the near-coincidence of the ninth $l = 2$ and the forty-first $l = 3$ frequencies with the 'algebraically special' values $\frac{1}{6}(l-1)\ l(l+1)(l+2)$ discussed by Chandrasekhar (1984).)

	$l = 2$	$l = 3$
n	ω_n	ω_n
1	(0.747343, −0.177925)	(1.198887, −0.185406)
2	(0.693422, −0.547830)	(1.165288, −0.562596)
3	(0.602107, −0.956554)	(1.103370, −0.958186)
4	(0.503010, −1.410296)	(1.023924, −1.380674)
5	(0.415029, −1.893690)	(0.940348, −1.831299)
6	(0.338599, −2.391216)	(0.862773, −2.304303)
7	(0.266505, −2.895822)	(0.795319, −2.791824)
8	(0.185617, −3.407676)	(0.737985, −3.287689)
9	(0.000000, −3.998000)	(0.689237, −3.788066)
10	(0.126527, −4.605289)	(0.647366, −4.290798)
11	(0.153107, −5.121653)	(0.610922, −4.794709)
12	(0.165196, −5.630885)	(0.578768, −5.299159)
20	(0.175608, −9.660879)	(0.404157, −9.333121)
30	(0.165814, −14.677118)	(0.257431, −14.363580)
40	(0.156368, −19.684873)	(0.075298, −19.415545)
41	(0.154912, −20.188298)	(−0.000259, −20.015653)
42	(0.156392, −20.685530)	(0.017662, −20.566075)
50	(0.151216, −24.693716)	(0.134153, −24.119329)
60	(0.148484, −29.696417)	(0.163614, −29.135345)

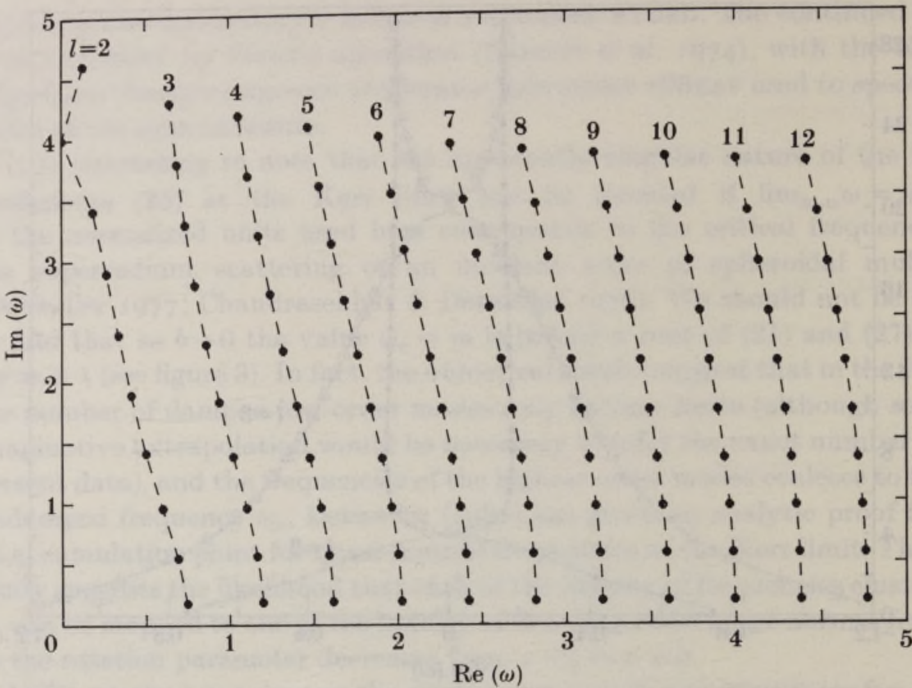

FIGURE 2. First 10 Schwarzschild gravitational quasi-normal frequencies for $l = 2$ to $l = 12$.

The equations for the Kerr geometry ((16) and (17)) are more complicated than the equation (2) for Schwarzschild's geometry, but still contain, in a slightly less direct form, the desired symmetry for the quasi-normal frequencies. If $\rho_{n,m} = -\mathrm{i}\omega_{n,m}$ and $A_{l,m}$ are a quasi-normal frequency and corresponding angular separation constant for azimuthal index m, then $\rho_{n,-m} = \rho^*_{n,m}$ and $A_{l,-m} = A^*_{l,m}$ are a quasi-normal frequency and angular separation constant for azimuthal index $-m$. This satisfies the requirement of complex conjugate pairing since the sum in expression (15) is over both positive and negative values of m.

The functional dependence of A_{lm} on the rotation parameter a is shown in tables 2 and 3 for $l = 2$, $m = 0$, and $l = 2$, $m = 1$. Figure 3 plots the trajectories the five lowest gravitational quasi-normal frequencies trace as the rotation of the black hole increases from the Schwarzschild limit to near the Kerr limit, and illustrates the degree to which the $(2l+1)$-fold azimuthal degeneracy is lifted by the rotation for $l = 2$. To save space both the right (Re $(\omega) > 0$) branch and the left (Re $(\omega) < 0$) branches were plotted on the same graph, the values of the real parts of the left branches being replaced by their absolute magnitudes. Thus the right branch appears to the right of the Schwarzschild limit (indicated by the broken line), and the values for the left branches typically appear to the left. The exception here is the case of $m = 0$, where, as in the Schwarzschild limit, the quasi-normal frequencies are symmetric about the imaginary ω-axis and the left branch superimposes the right: what appear in each of the figures are the images of the left branches as they reflect through the imaginary axis. The values for all the Kerr frequencies as plotted are again reflected through the imaginary axis when m is replaced by $-m$. Comparison of the tabulated values for the $l = 2$, $m = 0$ and the

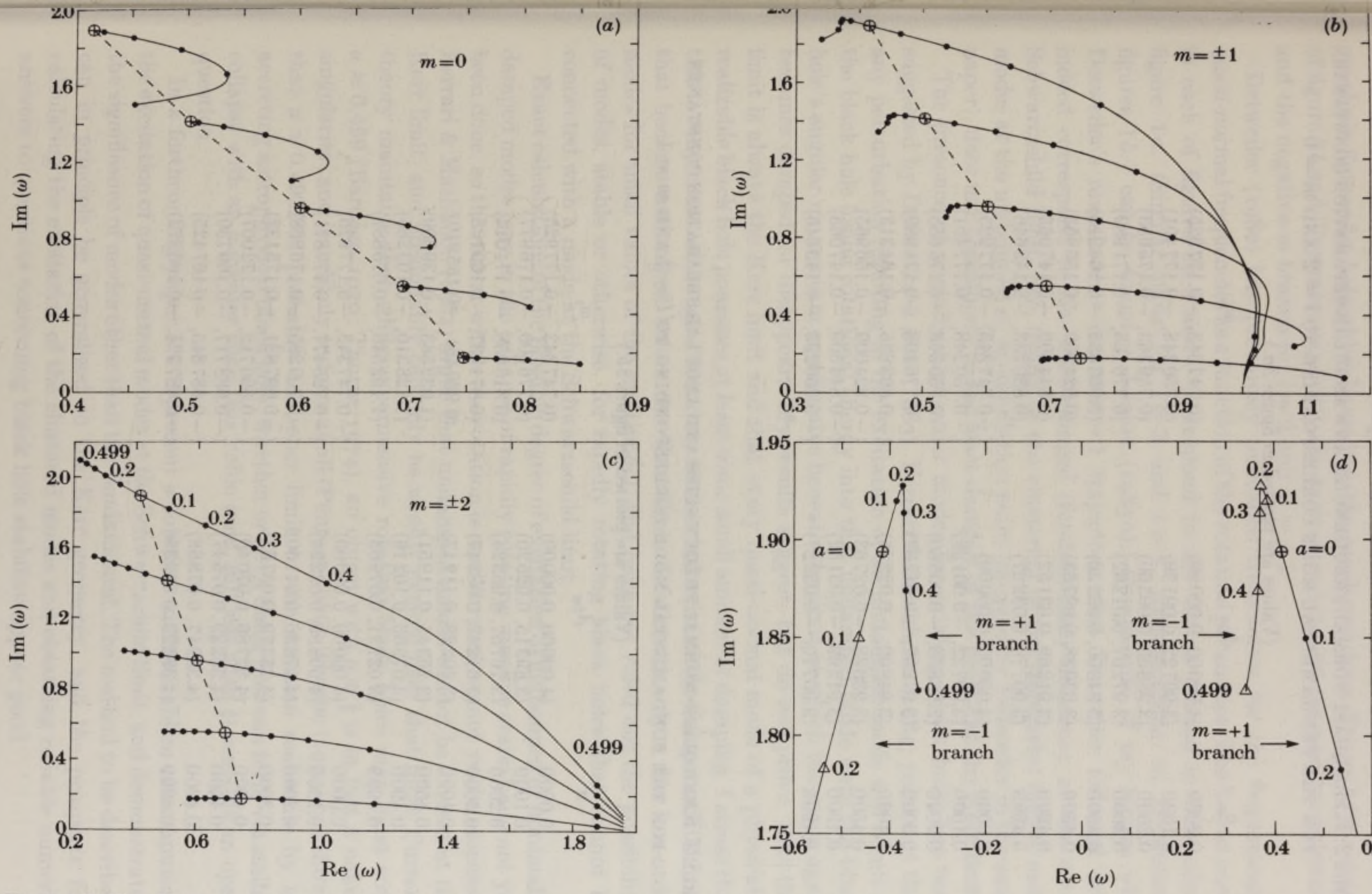

FIGURE 3. Five Kerr quasi-normal frequencies for $l = 2$ and $m = 0$, ± 1, and ± 2. The curves are parametrized by the value of the rotation parameter a. The broken line indicates the Schwarzschild limit of $a = 0$, and the ends of each curve are near the Kerr limit with $a = 0.4999$. The solid dots occur at $a = 0.1$, 0.2, 0.3, 0.4 and 0.4990. The lowest curve in each of these plots corresponds to the curves in figure 1 a of Detweiler (1980). Tables 2 and 3 list some of the frequencies plotted in (a) and (b). Part (d) is a detail of the top curve in (b) and illustrates the complex conjugate symmetry of the Kerr quasi-normal frequencies.

TABLE 2. KERR QUASI-NORMAL FREQUENCIES AND ANGULAR SEPARATION CONSTANTS FOR THE FUNDAMENTAL MODE CORRESPONDING TO $l = 2$ AND $m = 0$

(Values are plotted in figure 3*a*.)

a	A_{lm}	ω_1
0.0000	(4.00000, 0.00000)	(0.747343, −0.177925)
0.1000	(3.99722, 0.00139)	(0.750248, −0.177401)
0.2000	(3.98856, 0.00560)	(0.759363, −0.175653)
0.3000	(3.97297, 0.01262)	(0.776108, −0.171989)
0.4000	(3.94800, 0.02226)	(0.803835, −0.164313)
0.4500	(3.93038, 0.02763)	(0.824009, −0.156965)
0.4900	(3.91269, 0.03152)	(0.844509, −0.147065)
0.4999	(3.90770, 0.03227)	(0.850233, −0.143646)
0.0000	(4.00000, 0.00000)	(−0.747343, −0.177925)
0.1000	(3.99722, −0.00139)	(−0.750248, −0.177401)
0.2000	(3.98856, −0.00560)	(−0.759363, −0.175653)
0.3000	(3.97297, −0.01262)	(−0.776108, −0.171989)
0.4000	(3.94800, −0.02226)	(−0.803835, −0.164313)
0.4500	(3.93038, −0.02763)	(−0.824009, −0.156965)
0.4900	(3.91269, −0.03152)	(−0.844509, −0.147065)
0.4999	(3.90770, −0.03227)	(−0.850233, −0.143646)

TABLE 3. KERR QUASI-NORMAL FREQUENCIES AND ANGULAR SEPARATION CONSTANTS FOR THE FUNDAMENTAL MODE CORRESPONDING TO $l = 2$ AND $m = 1$

(Values are plotted in figure 3*b*.)

a	A_{lm}	ω_1
0.0000	(4.00000, 0.00000)	(0.747343, −0.177925)
0.1000	(3.89315, 0.02520)	(0.776500, −0.176977)
0.2000	(3.76757, 0.05324)	(0.815958, −0.174514)
0.3000	(3.61247, 0.08347)	(0.871937, −0.169128)
0.4000	(3.40228, 0.11217)	(0.960461, −0.155910)
0.4500	(3.25345, 0.11951)	(1.032583, −0.139609)
0.4900	(3.07966, 0.10216)	(1.128310, −0.103285)
0.4999	(3.02131, 0.07903)	(1.162546, −0.076881)
0.0000	(4.00000, 0.00000)	(−0.747343, −0.177925)
0.1000	(4.09389, 0.02224)	(−0.725477, −0.177871)
0.2000	(4.17836, 0.04150)	(−0.709265, −0.176968)
0.3000	(4.25579, 0.05767)	(−0.697821, −0.175132)
0.4000	(4.32786, 0.07049)	(−0.690712, −0.172007)
0.4500	(4.36229, 0.07547)	(−0.688717, −0.169730)
0.4900	(4.38917, 0.07868)	(−0.687845, −0.167425)
0.4999	(4.39573, 0.07935)	(−0.687724, −0.166772)

$l = 2$, $m = 1$ Kerr quasi-normal frequencies (tables 2 and 3) with their respective graphs in figures $3a, b$ should indicate how the graph was done. Figure $3d$ is a detail of figure $3b$ and illustrates explicitly the symmetry between the positive m branch and the negative m branch for $l = 2$ and $m = \pm 1$.

Detweiler (1980) has previously published trajectories for the fundamental quasi-normal frequencies as a function of the rotation parameter. The lowest curve in each of figures $3a, b$ and c correspond to the curves plotted in Detweiler's figure $1a$. (Similar plots for $l = 3$ and $l = 4$, corresponding to Detweiler's figures $1b, c$ can be found in Leaver (1985a).) Comparison of my figures with Detweiler's reveals that the $m = -1$ trajectory that Detweiler followed does indeed correspond to the least-damped (fundamental) frequency at the $a = 0$ Schwarzschild limit, but, owing to the clustering of the (formerly) higher order modes at the undamped ω_c accumulation point (derived by Detweiler in the same paper), does not correspond to the least-damped of the modes at the Kerr limit.

The presence of unstable high-order modes for rotating black holes has been suggested by Detweiler & Ove (1983). Ferrari & Mashhoon (1984) point out that any perturbation resulting in the excitation of an undamped mode will result in the black hole losing rotational energy into that undamped mode until the black hole's angular momentum equilibrates beneath the limit at which the mode again becomes damped, or disappears. My results suggest, but do not prove, that this limit is always the Kerr limit and that every quasi-normal mode of a physically realizable black hole possesses at least some small amount of damping. I stress that this is conjecture: although I found no modes that exist at the Schwarzschild limit that become undamped before the Kerr limit, I followed only a few low-order modes for small values of the multipole l, and have not ruled out the possibility of modes, stable or otherwise, for rapidly rotating black holes that cannot be connected with a mode at the Schwarzschild limit.

Exact calculations of the possible degree of excitation of undamped or minimally damped modes (say in the collapse of rapidly rotating massive stars) have not yet been done, so the rate at which the rotation is equilibrated cannot yet be assessed. Ferrari & Mashhoon (1984) argue that undamped modes cannot be excited at the Kerr limit, and presumably can only be weakly excited near that limit. Current theory maintains that while supermassive rotating black holes may exist with $a \approx 0.499$ (Bardeen 1970; Thorne 1974), an object with $a \equiv \frac{1}{2}$ is in fact a naked singularity, and is unlikely to form at all (Penrose 1969). Thorne (1974) has shown that $a \approx 0.4992$ is probably an upper limit to the rotation realizable by an accreting astrophysical black hole; whether or not a black hole can form via stellar collapse with an a greater than this value (and still less than $\frac{1}{2}$) remains an open question.

In a forthcoming paper I will present a solution to the problem of computing the excitation of quasi-normal modes at the Schwarzschild limit, and demonstrates the significance of modes other than the fundamental. The method to be described can in principle be generalized to the Kerr geometry, and the prospects for calculating the excitation of the clustered modes and obtaining reliable numeric answers to questions concerning black hole stability appear good.

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