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Spacetime approach to force-free magnetospheres

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ABSTRACT

Force-free electrodynamics (FFE) describes magnetically dominated relativistic plasma via non-linear equations for the electromagnetic field alone. Such plasma is thought to play a key role in the physics of pulsars and active black holes. Despite its simple covariant formulation, FFE has primarily been studied in 3+1 frameworks, where spacetime is split into space and time. In this paper, we systematically develop the theory of force-free magnetospheres taking a spacetime perspective. Using a suite of spacetime tools and techniques (notably exterior calculus), we cover (1) the basics of the theory, (2) exact solutions that demonstrate the extraction and transport of the rotational energy of a compact object (in the case of a black hole, the Blandford–Znajek mechanism), (3) the behaviour of current sheets, (4) the general theory of stationary, axisymmetric magnetospheres, and (5) general properties of pulsar and black hole magnetospheres. We thereby synthesize, clarify, and generalize known aspects of the physics of force-free magnetospheres, while also introducing several new results.

Key words: black hole physics – MHD – plasmas – methods: analytical – pulsars: general.

1 INTRODUCTION

Soon after the discovery of pulsars (Hewish et al. 1968) it became clear that they must be rapidly rotating, highly magnetized neutron stars (Gold 1968; Pacini 1968) whose magnetosphere is filled with plasma (Goldreich & Julian 1969). The plasma mass density is many orders of magnitude lower than the electromagnetic field energy density, so one may neglect the plasma four-momentum and set the Lorentz four-force density to zero. The resulting autonomous dynamics for the electromagnetic field, known as *force-free electrodynamics* (FFE), forms a foundation for studies of the pulsar magnetosphere.

Quasars were discovered several years before pulsars (Schmidt 1963), and while supermassive black holes were soon suspected as the energy source, more than a decade passed before the discovery of a viable mechanism for extracting the energy. The breakthrough was the seminal work of Blandford & Znajek (1977, BZ), who argued that black holes immersed in magnetic fields could have a force-free plasma. BZ showed that the presence of plasma enables a magnetic Penrose process in which even *stationary* fields can efficiently extract energy from a spinning black hole.

Despite this important progress, little further was done on the subject for several years. MacDonald & Thorne (1982) diagnosed the difficulty as a problem of language. In addition to significantly extending the theory, they recast the work of BZ in a $3 + 1$ decomposition designed to render the equations and concepts more familiar to astrophysicists. The efficacy of their cure is well supported by the significant progress on the problem that has been made since then, nearly all of it using the $3 + 1$ approach.

But even the best medicines can have side effects. From the relativist's point of view, the use of $3 + 1$ methods obscures intrinsic structures and creates unnecessary complications by introducing artificial ones. For a subject in which curved spacetime and highly relativistic phenomena play central roles, one might expect that the impressive arsenal of spacetime techniques developed over the last century could be profitably exploited. However, very few general relativity theorists have become involved, and little work of this nature has been pursued. It may be that the unfamiliar language and phenomena of plasma physics, together with their casting in $3 + 1$ language, have made the subject largely inaccessible to relativists.

The beginnings of the field were in fact rather relativistic in flavour, with Znajek's (1977) use of a null tetrad formalism and BZ's tensor component calculations. Since then however there has been little use of spacetime techniques on black hole and pulsar force-free magnetospheres, notable exceptions being the work of Carter (1979) and Uchida (1997a,b,c,d, 1998). Our own involvement began recently when we noticed that some apparently disparate exact solutions shared the property of having four-current along a geodesic, shear-free null congruence (Brennan, Gralla & Jacobson 2013). We made a null current ansatz and immediately found a large class of non-stationary, non-axisymmetric exact solutions in the Kerr spacetime, which can also be used in flat spacetime

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in modelling pulsar magnetospheres. This rapid progress suggested to us that translation of magnetospheric physics into spacetime language may be more than a matter of words, and that a geometrical perspective on FFE could lead to powerful insights and significant new results.

This paper has a number of distinct purposes. One is to present the theory of force-free magnetospheres with a spacetime perspective from the ground up. In this way we hope both to introduce relativists to the subject and to introduce plasma astrophysicists to potentially powerful new techniques. We focus on intrinsic properties, avoiding the introduction of arbitrary structures – such as a time function or a reference frame – that have no intrinsic relation to either the spacetime geometry or the particular electromagnetic field being discussed. The other purposes of our paper are to present new insights, techniques, and results, as well as the convenient methods of exterior calculus we have made use of.

The paper is organized into nine sections and five appendices:

1. Introduction
2. Astrophysical setting
3. Force-free electrodynamics
4. Poynting flux solutions
5. Monopole magnetospheres
6. Current sheets and split monopoles
7. Stationary, axisymmetric magnetospheres
8. Pulsar magnetosphere
9. Black hole magnetosphere

A. Differential forms
B. Poynting flux examples
C. Kerr metric
D. Euler potentials with symmetry
E. Conserved Noether current associated with a symmetry.

We now provide a detailed description of the contents of each section.

In Section 2, we sketch the relevant astrophysical settings and discuss the basic reasoning that accounts for the validity of the force-free approximation. We then present the basic features and mathematical structure of FFE and degenerate electromagnetic fields (Section 3). This section is primarily a review and synthesis of previous research, focusing on the spacetime approaches of Carter and especially Uchida, who formulated the theory in terms of two scalar Euler potentials. We have found that the use of differential forms (with wedge product and exterior derivative) together with Euler potentials provides an elegant and computationally efficient method to handle the mathematics, and we focus on this approach throughout the paper. Appendix A covers the properties of differential forms needed in the paper. We emphasize the geometrical role of certain timelike 2-surfaces that, for degenerate magnetically dominated fields, extend the notion of field line to a spacetime object. These are called 'flux surfaces' in the literature, but we adopt here the more suggestive name 'field sheets'. In particular, we observe that the induced metric on these sheets governs the dynamics for particles and Alfvén waves moving in the magnetosphere, and explain how field sheet Killing fields give rise to conserved quantities. We also note that the field equations of FFE amount to the conservation of two 'Euler currents', which have not been explicitly discussed before.

Section 4 is devoted to presenting several exact solutions to FFE involving outgoing electromagnetic energy flux (Poynting flux) in flat, Schwarzschild, and Kerr spacetimes. These include a solution in Kerr recently found by Menon & Dermer (2007), a time-dependent and non-axisymmetric generalization of that (Brennan et al. 2013), as well as a solution sourced from an arbitrary accelerated world line in flat spacetime (Brennan & Gralla 2014). We showcase the remarkably simple expression of these solutions in the language of differential forms, as well as the efficient computational techniques we can use to check that they are force free. The solutions illustrate how force-free fields can transport energy via Poynting flux in ways that are unfamiliar in (but not completely absent from) ordinary electrodynamics. Appendix B is devoted to examples that further develop insight into the physical nature of this energy transport.

Turning next to the physics of magnetospheres, Section 5 builds on the Poynting flux solutions to present several exact solution models with a monopolar central rotating source. (The more realistic case of a split monopole is deferred for clarity to the next section.) We begin with a discussion of the classic Michel solution, which illustrates the basic mechanism of electromagnetic extraction and transport of the rotational energy of a conducting magnetized star. We obtain this solution as a superposition of a monopole and an outgoing Poynting flux solution satisfying the perfect conductor boundary condition, and use it to illustrate the nature of field sheet geometry. We next show how our time-dependent generalizations can be used to model dynamical pulsar magnetospheres. In particular, we debut the 'whirling monopole', which is the exact monopolar magnetosphere of a conducting star undergoing *arbitrary* time-dependent rigid body motion with a fixed centre. Finally, we discuss the monopolar approximate solution of BZ for a rotating black hole. We obtain their solution to first order in the spin by promoting the Michel solution to Kerr in a simple way. The result is an exceptionally simple expression for the BZ field in terms of differential forms, from which its force-free nature as well as basic properties (such as its 'rotation frequency' of one half the horizon frequency) are easily seen.

In Section 6, we discuss the role of current sheets in force-free magnetospheres and provide a simple invariant criterion for the shape and time evolution of a current sheet across which the electromagnetic field flips sign. We use this criterion to efficiently reproduce the standard aligned and inclined split-monopole solutions and discuss generalizations, such as a glitching split-monopole pulsar. We also discuss a more general, reflection split construction in which the magnetic field has a component normal to the current sheet.

Section 7 is devoted to the general theory of stationary, axisymmetric, force-free magnetospheres in stationary, axisymmetric spacetimes. We make extensive use of the natural 2 + 2 decomposition into 'toroidal' submanifolds spanned by the angular and time-translation Killing vectors and the orthogonal 'poloidal' submanifolds. The Uchida (1997b) method of determining the general form of Euler potentials for fields with symmetry is presented using differential forms in Appendix D. We explain how and why the field is characterized by three quantities: the 'magnetic flux function' ψ, the 'angular velocity of field lines' $\Omega_F(\psi)$, and the 'polar current' $I(\psi)$, derive the general force-free 'stream equation' relating these quantities, and discuss approaches to solving it. Expressions for the energy and angular momentum flux are derived, using the corresponding Noether current 3-forms whose derivation is given in Appendix E. We explain how the 'light surfaces' (where the field rotation speed is that of light) are causal horizons for particles and Alfvén waves, and derive the relationship between the particle and angular momentum flow directions. We discuss general restrictions on the topology of poloidal field lines, presenting a new result that smooth closed loops cannot occur and clarifying the circumstances under which field lines cannot cross a light surface twice. Finally, we present the stream equation for the special case where there is no poloidal magnetic field, which has been largely overlooked in previous work.

Section 8 discusses basic properties of the pulsar magnetosphere in the case of aligned rotational and magnetic axes, using the stationary, axisymmetric formalism of the previous section. We discuss the corotation of the field lines with the star as well as the dichotomy between *closed* field lines that intersect the star twice and *open* field lines that proceed from the star to infinity. We clarify the precise circumstances under which closed field lines must remain within the light cylinder, and discuss other circumstances in which they may extend outside.

Section 9 addresses black hole magnetospheres, focusing on stationary, axisymmetric fields in the Kerr geometry. We derive the so-called Znajek horizon regularity condition and identify an additional condition required for regularity in the extremal case. We discuss the status of energy extraction as a Penrose process and discuss the nature of the two light surfaces. Finally, we present the 'no-ingrown-hair' theorem of MacDonald & Thorne (1982), showing that a black hole cannot have a force-free zone of closed poloidal field lines. We discuss the types of closed field lines that can in fact occur.

We adopt the spacetime signature $(-, +, +, +)$, choose units with the speed of light $c = 1$ and Newton's constant $G = 1$, and use Latin letters $a, b, c, \ldots$ for abstract tensor indices (there is no use of coordinate indices in the paper). For Maxwell's equations, we use Heaviside–Lorentz units.

2 ASTROPHYSICAL SETTING

Force-free plasmas exist naturally in pulsar magnetospheres, and possibly in several other astrophysical systems. Goldreich & Julian (1969) pointed out that the rotation of a magnetized conducting star in vacuum induces an electric field, with the Lorentz scalar $\boldsymbol{E} \cdot \boldsymbol{B}$ non-zero outside the star. Undeflected acceleration of charges along the direction of the magnetic field will thus occur. For typical pulsar parameters, the electromagnetic force is large enough to overwhelm gravitational force and strip charged particles off the star. Even if strong material forces retain the particles, the large $\boldsymbol{E} \cdot \boldsymbol{B}$ outside the star will create particles in another way (Ruderman & Sutherland 1975): any stray charged particle will be accelerated to high energy along curved magnetic field lines, leading to curvature radiation and a cascade of electron–positron pair production. These mechanisms act to fill the pulsar magnetosphere with plasma.

To estimate the density of plasma, note that produced charges act to screen the component of $\boldsymbol{E}$ along $\boldsymbol{B}$, eventually shutting off production when $\boldsymbol{E} \cdot \boldsymbol{B}$ becomes small enough. The number of particles created should thus roughly agree with the minimum amount required to ensure $\boldsymbol{E} \cdot \boldsymbol{B} = 0$. If the particles corotate with the star, the required charge density is the so-called Goldreich–Julian charge density $\rho \propto \Omega B$, where Ω is the stellar rotation frequency. The minimum associated particle density occurs for complete charge separation (one sign of charge only at each point), which for typical pulsar parameters corresponds to a plasma rest mass density that is *16–19 orders of magnitude* (for protons or electrons, respectively) smaller than the electromagnetic field energy. Even if particle production mechanisms significantly overshoot this density, the criterion for the force-free description is easily satisfied.[1] Detailed calculations support these simple arguments, finding an overshoot of a few orders of magnitude (e.g. Beskin 2010).

Force-free models of the pulsar magnetosphere provide a foundation on which studies of pulsar emission processes may be based. Models of pulsed emission generally involve particles or plasma instabilities streaming outwards along the magnetic field lines of the magnetosphere (e.g. Beskin 2010). Pulsed emission is observed in radio, optical, X-ray, and gamma-ray, with some pulsars active only in a subset of these bands, and with a variety of pulse profiles. The challenge of modelling these complex features remains an active field of research.

The force-free model has also been applied to black holes, beginning with the work of BZ. Following the observation of Wald (1974) that immersing a spinning black hole in a magnetic field gives rise to electric fields with non-zero $\boldsymbol{E} \cdot \boldsymbol{B}$, BZ argued that a pair-production mechanism could also operate to produce a force-free magnetosphere near a spinning black hole with a magnetized accretion disc. If the whole system is simulated using magnetohydrodynamics (MHD; e.g. McKinney, Tchekhovskoy & Blandford 2012 and references therein), it is generally found that the plasma density is very low away from the disc (and especially in any jet region), so that the dynamics there is effectively force free. Finally, the last few years has seen work on force-free magnetospheres of binary black hole and neutron star systems (e.g. Palenzuela, Lehner & Liebling 2010; Alic et al. 2012; Palenzuela et al. 2013; Paschalidis, Etienne & Shapiro 2013), motivated in part by the possibility of observing electromagnetic counterparts to gravitational-wave observations of binary inspiral. These simulations have shown energy extraction and jet-like features, even in the case of *non-spinning* (but moving) black holes.

3 FORCE-FREE ELECTRODYNAMICS

In this section, we introduce the essential properties of FFE in an arbitrary curved spacetime background and its description in the language of differential forms.

An electromagnetic field F_{ab} normally exchanges energy and momentum when interacting with charged matter. The energy–momentum tensor for the field is given by

$$T^{\mathrm{EM}}_{ab} = F_{ac}F_b{}^c - \frac{1}{4}F_{cd}F^{cd}g_{ab}, \tag{1}$$

and Maxwell's equations imply that the exchange is expressed by the equation $\nabla^b T^{\mathrm{EM}}_{ab} = -F_{ab}j^b$, where j^b is the electric four-current density. $F_{ab}j^b$ is the four-force density, describing the rate of transfer of energy and momentum between the field and the charges. FFE describes the electromagnetic field interacting with a plasma in a regime in which the transfer of energy and momentum from the field to the plasma can be neglected, not because the current is unimportant, but because the field energy–momentum overwhelms that of the plasma. FFE is thus governed by Maxwell's equations together with the force-free condition

$$F_{ab}j^b = 0. \tag{2}$$

[1] While the bulk of the magnetosphere should be force free, small violating regions of two types can exist. First, regions where particles are produced may expel those particles with high velocity, so that plasma density high enough to achieve $\boldsymbol{E} \cdot \boldsymbol{B} = 0$ is never attained in those regions. Such regions are called *gaps*, and may provide a source of the high-energy particles observed in the pulsar wind. Secondly, as we discuss in some detail later, force-free fields tend to produce thin sheets of current where the field is not force free.

In this regime, remarkably, the field can be evolved autonomously, without keeping track of any plasma degrees of freedom, as we now explain.

Maxwell's equations take the form

$$\nabla_{[a}F_{bc]} = 0, \tag{3}$$

$$\nabla_b F^{ab} = j^a, \tag{4}$$

where the square brackets denote antisymmetrization of the indices, ∇_b is the spacetime covariant derivative, and we use Heaviside–Lorentz units. The first equation is equivalent to the statement that F_{ab} is (at least locally) derivable from a potential, $F_{ab} = 2\nabla_{[a}A_{b]}$. The second equation relates the field to the electric four-current density. In the force-free setting, this second equation is simply used to identify the four-current, and so the equation imposes no condition on the field. We may thus eliminate j^a from the description, and FFE becomes the pair of equations

$$\nabla_{[a}F_{bc]} = 0, \quad F_{ab}\nabla_c F^{bc} = 0. \tag{5}$$

Note that vacuum Maxwell fields trivially satisfy these equations. In this paper, a 'force-free solution' will always mean a non-vacuum solution of equations (5), i.e. one with $\nabla_b F^{ab} \neq 0$. This is the case of relevance to plasma magnetospheres, and it has a rich structure distinct from that of the vacuum case.

3.1 Determinism

The FFE equations determine the evolution of the field given initial data, provided the field is magnetically dominated i.e. $F_{ab}F^{ab} = 2(B^2 - E^2) > 0$. To see how this could be, one can make a 3+1 decomposition in flat spacetime. The force-free condition (2) then takes the form

$$\boldsymbol{E} \cdot \boldsymbol{J} = 0, \qquad \rho \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} = 0, \tag{6}$$

stating that the work done on the charges and the momentum transfer to the charges both vanish. These equations imply the (Lorentz invariant) condition

$$\boldsymbol{E} \cdot \boldsymbol{B} = 0, \tag{7}$$

unless both the charge and three-current densities vanish. Provided $|\boldsymbol{B}| \neq 0$ (which holds in all frames if the field is magnetically dominated), equation (6) determines $\boldsymbol{J}_\perp = |B|^{-2}\rho \boldsymbol{E} \times \boldsymbol{B}$, the component of the three-current perpendicular to the magnetic field. Moreover, Gauss' law $\nabla \cdot \boldsymbol{E} = \rho$ determines the charge density in terms of spatial derivatives at one time. To determine the component of $\boldsymbol{J}$ parallel to the magnetic field, consider Maxwell's time evolution equations

$$\partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E} \tag{8}$$

$$\partial_t \boldsymbol{E} = \nabla \times \boldsymbol{B} - \boldsymbol{J}. \tag{9}$$

The time derivative of the orthogonality condition (7) implies that $\boldsymbol{E} \cdot$ (equation 8) + $\boldsymbol{B} \cdot$ (equation 9) vanishes, which determines $\boldsymbol{J} \cdot \boldsymbol{B}$. Thus the force-free condition implies

$$\boldsymbol{J} = \frac{1}{B^2}\left[(\nabla \cdot \boldsymbol{E})\boldsymbol{E} \times \boldsymbol{B} + (\boldsymbol{B} \cdot \nabla \times \boldsymbol{B} - \boldsymbol{E} \cdot \nabla \times \boldsymbol{E})\boldsymbol{B}\right]. \tag{10}$$

With this substitution, equations (8) and (9) determine the time derivatives of the fields in terms of the field values at one time, and the initial value constraints $\nabla \cdot \boldsymbol{B} = 0$ and $\boldsymbol{E} \cdot \boldsymbol{B} = 0$ are preserved by the time evolution. The equations are therefore potentially deterministic. It turns out that they are indeed deterministic (i.e. hyperbolic), provided the (Lorentz invariant) scalar $B^2 - E^2$ is positive (Komissarov 2002; Palenzuela et al. 2011; Pfeiffer & MacFadyen 2013). That restriction is not surprising, since when this scalar is negative, there exists at each point a Lorentz frame in which $\boldsymbol{B} = 0$. In such a frame, one cannot solve for $\boldsymbol{J}$ at that point in terms of the fields and their spatial derivatives only. This shows that the character of the equations is different in the electrically dominated case.

There is no a priori reason to expect that the condition $B^2 > E^2$ is preserved under time evolution. In fact, it is seen numerically that the condition is *not* preserved. When the condition is violated, some other physics must determine the evolution, which is modelled via various prescriptions in numerical codes. It is generally found that violation occurs only in regions that are stable under the associated prescriptions, and that these regions tend to be compressed and of high current density: they are the *current sheets* discussed below in Section 6.

3.2 Degenerate electromagnetic fields

In this subsection, we discuss electromagnetic fields satisfying $F_{[ab}F_{cd]} = 0$ (equivalently $\boldsymbol{E} \cdot \boldsymbol{B} = 0$ in flat spacetime), which are called *degenerate*. All force-free fields are degenerate, but degeneracy can occur more generally, as explained below.

3.2.1 Field tensor

The force-free condition (2) implies that $F_{[ab}F_{cd]}j^d = 0$. Since every totally antisymmetric four-index tensor (in four dimensions) is proportional to the volume element ϵ_{abcd}, this implies the degeneracy condition

$$F_{[ab}F_{cd]} = 0, \tag{11}$$

which is equivalent to equation (7) in flat spacetime. This in turn implies that F_{ab} itself can be written as the antisymmetrized product of two rank-1 covectors[2]

$$F_{ab} = 2\alpha_{[a}\beta_{b]}. \tag{12}$$

To see this, consider the contraction $F_{[ab}F_{cd]}v^b w^d$ with two vector fields v^a and w^a such that $F_{ab}v^a w^b \neq 0$. Expanding out the antisymmetrization produces an expression for F_{ab} of the form (12), where the factors α_a and β_a are proportional to $F_{ab}v^b$ and $F_{ab}w^b$.

An electromagnetic field can be degenerate without being force free. Degeneracy occurs any time there is some vector field v^b such that $F_{ab}v^b = 0$. For instance, in the presence of a 'perfect' conductor (like a metal or a suitable plasma), the electric field in the local rest frame of the conductor vanishes:

$$F_{ab}U^b = 0, \tag{13}$$

where U^a is the unit timelike four-velocity of the conductor's rest frame. Thus fields in perfect conductors are degenerate. For an ionic plasma described by ideal MHD, U^a might be the four-velocity of the ion 'fluid', but degeneracy does not require a unique rest frame to be singled out. As long as there is enough free charge to screen the component of the electric field in the direction of

[2] This factorization property holds at each point, but it can happen that there is no pair of smooth tensor fields α_a and β_b such that equation (12) holds everywhere (for an example, see the end of section 3.5 of Penrose & Rindler 1984).

the magnetic field, $\boldsymbol{E}\cdot\boldsymbol{B}$ will vanish and hence the field will be degenerate.

Conversely, a degenerate field $F_{ab} = 2\alpha_{[a}\beta_{b]}$ always admits at each point a two-dimensional space of vectors that annihilate it (in the sense that $F_{ab}v^a = 0$). This space, called the *kernel* of F_{ab}, consists of the intersection of the three-dimensional kernels of α_a and β_a. The covectors α_a and β_a themselves span a (co)plane, and any two linearly independent, suitably scaled covectors in the coplane may be chosen. Taking α_a and β_a to be orthogonal, the square of the field tensor is then

$$F^2 = F_{ab}F^{ab} = 2(B^2 - E^2) = 2\alpha^2\beta^2. \tag{14}$$

The sign of this Lorentz scalar determines whether the field is magnetically dominated, electrically dominated, or null. Since there do not exist two orthogonal timelike vectors, this is positive if and only if both α and β are spacelike.

For a magnetically dominated field, the α–β plane is thus spacelike and the kernel, which is orthogonal to α and β, is timelike. There is a one-parameter family of four-velocities U^a lying in this timelike kernel, each of which defines a Lorentz frame in which the electric field vanishes (equation 13). The orthogonal projection of a preferred frame t^a into the kernel of F selects one of these, U^a_t, whose velocity relative to t^a is known as the *drift velocity*. This relative velocity is the minimum for all U^a in the kernel of F, and is given by $\boldsymbol{E}\times\boldsymbol{B}/B^2$ in the frame t^a.

For a field with $E^2 = B^2$, either α or β must be null, so the α–β plane is null and so is the kernel (with the same null direction). For an electrically dominated field, the α–β plane is timelike, so the kernel is spacelike, and there is always a Lorentz frame in which the magnetic field vanishes (since the kernel of $*F$ is timelike).

3.2.2 Stress tensor

For a non-null degenerate field, one can decompose the spacetime metric into a metric h_{ab} on the kernel of F_{ab} that vanishes on vectors orthogonal to the kernel (so $h_{ab}\alpha^a = h_{ab}\beta^b = 0$) and a metric $h^{\perp}_{ab}$ that vanishes on vectors in the kernel, $g_{ab} = h_{ab} + h^{\perp}_{ab}$. Using these, the stress tensor (equation 1) can be expressed as

$$T_{ab} = \frac{1}{4}F^2(h^{\perp}_{ab} - h_{ab}). \tag{15}$$

This can be quickly verified by noting that the right-hand side is the only symmetric tensor built from the available ingredients that is traceless and satisfies $T_{ab}h^{ab} = -\frac{1}{2}F^2$, which holds because $F_{bc}h^{ab} = 0$.

In the magnetic case and in a 3 + 1 decomposition, equation (15) may be interpreted in terms of the standard concepts magnetic pressure and magnetic tension. Choose any frame in which there is no electric field i.e. any unit timelike U^a in the kernel of F. Let s^a be the unit orthogonal spacelike vector in the kernel. The magnetic field in this frame is directed along s^a, and we denote its magnitude by B. If γ_{ab} is the spatial metric orthogonal to U^a, then the stress tensor (equation 15) may be written $T_{ab} = \frac{1}{2}B^2(U_aU_b + \gamma_{ab} - 2s_as_b)$. From each term, respectively, we identify the energy density of $\frac{1}{2}B^2$, an isotropic magnetic pressure of $\frac{1}{2}B^2$, and a magnetic tension of B^2 along the magnetic field lines.

3.2.3 Field sheets

When a degenerate field F_{ab} satisfies the Maxwell equation $\nabla_{[c}F_{ab]} = 0$ (3), the kernels of F_{ab} are *integrable*, i.e. tangent to two-dimensional submanifolds. (A proof of this will be given in the next subsection.) In the magnetic case ($F^2 > 0$), these submanifolds are timelike, and their intersection with a spacelike hypersurface gives the magnetic field lines defined by the observers orthogonal to the hypersurface.[3] Each submanifold can thus be thought of as the *spacetime evolution* of a field line, which we will call a *field sheet*.[4] While the field lines depend on the arbitrary choice of spacelike hypersurface or observers, the field sheets are an intrinsic aspect of the degenerate structure of the field. The force-free condition (2) amounts to the statement that the current four-vector j^a is tangent to the field sheets. This generalizes to dynamical fields in curved spacetime the statement that, in a force-free plasma with zero electric field in flat spacetime, the current is tangent to the magnetic field lines.

The field sheets can be used to understand and describe particle and wave motion in the underlying plasma in a manner that does not require choosing an arbitrary frame. In that application the *field sheet metric*, induced by the spacetime metric, plays a central role. We now discuss two examples of this viewpoint: the propagation of charged particles and Alfvén waves.

In a collisionless plasma, viewed (locally) in a frame with zero electric field, a charged particle will spiral around a magnetic field line, executing cyclotron motion while the centre of the transverse circular orbit is 'guided' along the field line. Ignoring the cyclotron motion and the drift away from the field line, the particle is thus 'stuck' on the field line (e.g. Northrop & Teller 1960). The manifestly frame-invariant version of this statement is that the particle's world line is stuck on the field sheet. That is, its possible motions are the timelike trajectories on the sheet.

When one can furthermore neglect radiation reaction from 'curvature radiation' due to the bending of the field sheet, then the motion of the particle is in fact *geodesic* on the field sheet. This follows simply from the fact that the Lorentz force $qF_{ab}U^b$ vanishes for a four-velocity U^a tangent to the sheet. This viewpoint makes it easy to exploit symmetries. For example, in a stationary, axisymmetric magnetosphere, each field sheet will have a helical symmetry under a combined time-translation and rotation. The field sheet particle motion, being one-dimensional, will thus be integrable using the associated conserved quantity (see Section 7.2.6).

Field sheet geometry also governs the propagation of Alfvén waves, which are transverse oscillations of the magnetic field lines embedded in a plasma (Alfvén 1942). In a force-free plasma, these are characterized by a wave four-vector whose pullback to the field sheet is null with respect to the field sheet metric (Uchida 1997d), which implies that their group four-velocity is null and tangent to the field sheet. Thus wavepackets propagate at the speed of light along the field sheets.

3.2.4 Degenerate fields and differential forms

The mathematical language of *differential forms* is ideally suited to working with degenerate fields, and we shall make extensive use of

[3] Curiously, for any degenerate field, the magnetic field defined by *arbitrary* observer, $B^d = \frac{1}{2}\epsilon^{abcd}F_{ab}U_c$, lies in the kernel of F, since $F_{[ab}F_{d]e} = F_{[ab}F_{de]} = 0$.

[4] The submanifolds were called 'flux surfaces' by Carter and Uchida. We prefer the name field sheet because of the connotation of time evolution suggested by the similarity with the term 'worldsheet', which is appropriate in the magnetic case. We note also that the term 'flux surface' is commonly used in another sense, to describe a spacelike 2-surface to which the magnetic field is everywhere tangent.

it in this paper. The basic properties of differential forms are summarized in Appendix A, to which we refer for all definitions. One of the reasons it is so convenient is that electromagnetism in general, and especially when fields are degenerate, has a rich differential and algebraic structure that is in fact independent of the spacetime metric. By using the (metric-independent) exterior derivative, and (metric-independent) wedge products rather than covariant derivatives and inner products, we avoid unnecessary appearance of the metric and thus keep the formalism as close as possible to the structure inherent in the field itself. The metric does of course play a role, but for the most part we can sequester that in the Hodge duality operator (which is especially simple to work with in stationary axisymmetric spacetimes).

The field strength tensor is a 2-form, denoted simply by F, and the source-free Maxwell equation (3) corresponds to the statement that the 2-form F is 'closed', i.e.

$$dF = 0, \tag{16}$$

where d is the exterior derivative. This equation, which we dub the *covariant Faraday law*, encompasses both the absence of magnetic monopoles and the 3+1 Faraday law (equation 8). The degeneracy condition (11) and decomposition (12) are expressed using the wedge product $\wedge$ as

$$F \wedge F = 0 \tag{17}$$

and

$$F = \alpha \wedge \beta \tag{18}$$

for some pair of 1-forms α and β. A 2-form F with this property is sometimes called *simple*.

To prove that the field sheets exist, one can invoke a version of the Frobenius theorem: it follows from $dF = 0$ and $F = \alpha \wedge \beta$, together with the antiderivation property of d and antisymmetry of $\wedge$, that $d\alpha \wedge \alpha \wedge \beta = d\beta \wedge \alpha \wedge \beta = 0$. This guarantees complete integrability of the Pfaff system $\alpha = \beta = 0$ (Choquet-Bruhat & Dewitt-Morette 1982), which means that the vectors annihilating both α and β are tangent to submanifolds. A more intuitive argument for integrability will be given in the next subsection.

3.2.5 *Frozen flux theorem*

If the electric field vanishes in the local rest frames defined by a timelike vector field U,

$$U \cdot F = 0, \tag{19}$$

then the magnetic flux is 'frozen in' along the flow of U. [The dot in (19) is our notation for the contraction of a vector with the first slot of the adjacent form, here $U^a F_{ab}$.] More precisely, the flux through a loop is conserved if the loop is flowed along U. In ideal MHD, the fluid four-velocity satisfies equation (19). The frozen flux theorem (also known as the frozen-in theorem or Alfvén's theorem) is the source of much insight into the behaviour of such plasmas.

To prove the theorem, consider a loop flowed along U to create a timelike tube, and form a closed 2-surface by capping the ends of the tube with topological discs bounded by the initial and final loops. The integral of F over any closed 2-surface vanishes since $dF = 0$. The difference of the fluxes through the initial and final caps is therefore equal to the integral of F on the tube wall, which vanishes because the vector U that annihilates F (equation 19) is tangent to the wall. Using the language of differential forms, Alfvén's theorem is thus recovered immediately, with no calculation, in an arbitrary curved spacetime. It is interesting to contrast the simplicity of this completely general derivation with the usual one using electric and magnetic fields in flat spacetime.

A differential version of the statement of flux freezing may be obtained from the relation between the Lie derivative and the exterior derivative, sometimes called 'Cartan's magic formula' as

$$\mathcal{L}_v \, \omega = v \cdot d\omega + d(v \cdot \omega). \tag{20}$$

Here v is any vector field, ω is any differential form, and $\mathcal{L}$ is the Lie derivative. Applying the magic formula to $\mathcal{L}_U F$, the $U \cdot dF$ term vanishes by the covariant Faraday law (3), and the $d(U \cdot F)$ term vanishes simply by the defining property (equation 19) of U. Thus we obtain

$$\mathcal{L}_U F = 0, \tag{21}$$

stating that the field strength is preserved along the flow of U. In ideal MHD, the magnetic field is thus 'frozen into the fluid'.

The frozen flux theorem is closely related to the integrability property that implies the existence of the field sheets. In fact we can use it to give a simple proof of integrability as follows. Recall that if F is degenerate, there is a two-dimensional space of vectors annihilating F at each point. To prove these are surface forming, let u be any vector field such that $u \cdot F = 0$ everywhere. As above, Cartan's magic formula implies $\mathcal{L}_u F = 0$. Now choose a second vector field b such that $b \cdot F = 0$ on one 3-surface transverse to the flow of u, and extend b along the flow by requiring $\mathcal{L}_u b = 0$, which implies that u and b are surface forming. The Leibniz rule for Lie derivatives implies $\mathcal{L}_u(b \cdot F) = 0$, so also $b \cdot F = 0$ everywhere. The integral surfaces of u and b are therefore the field sheets.

3.2.6 *Euler potentials*

The covariant Faraday law (equation 16) is equivalent, at least locally, to the statement that F derives from a potential 1-form, i.e. $F = dA$ for some 1-form A. For closed, simple 2-forms (such as degenerate EM fields), thanks to the existence of the field sheets, a much more restrictive statement holds: a pair of scalar 'Euler potentials' $\phi_{1,2}$ can be introduced such that (Carter 1979; Uchida 1997a)

$$F = d\phi_1 \wedge d\phi_2. \tag{22}$$

The field sheets are the intersections of the hypersurfaces of constant ϕ_1 and ϕ_2. To establish the (local) existence of the Euler potentials, note that coordinates (x^A, y^i), $A, i = 1, 2$ can be chosen such that y^i are constant on the field sheets, in which case we have $F = f(x^A, y^i)\, dy^1 \wedge dy^2$, for some function f. Then $dF = 0$ implies that $f = f(y^i)$. Defining a new coordinate $\tilde{y}^1 = \int f \, dy^1$, we thus have $F = d\tilde{y}^1 \wedge dy^2$.

The Euler potentials capture the freedom in a closed, simple 2-form, hence in any degenerate electromagnetic field. Rather than the four components of a (co)vector potential, there are just two scalar fields. Even so, the potentials are not uniquely determined. F defines an 'area element' on the field sheets, which is preserved under any replacement $(\phi_1, \phi_2) \to (\phi_1'(\phi_1, \phi_2), \phi_2'(\phi_1, \phi_2))$ with unit Jacobian determinant. This is a field redefinition, not a dynamical gauge freedom. In fact, the second time derivatives of both potentials are determined at each point by their value and first derivatives (Uchida 1997a).

3.3 Euler-potential formulation of FFE

Since all force-free fields are degenerate, we may formulate FFE as a theory of two scalar fields by plugging the Euler-potential form of a degenerate field strength (equation 22) in to the force-free condition (2). Rather than developing this technique in tensor language, we will instead discuss the differential forms version, which we find very useful in calculations. We also discuss an action principle for the equations.

3.3.1 Force-free condition and Euler currents

The differential forms approach to Maxwell's theory – using the current 3-form instead of the four-vector – is reviewed in Appendix A3. The force-free condition (2) can be expressed directly in terms of the current 3-form as

$$F_{a[b}J_{cde]} = 0. \tag{23}$$

To see the equivalence with equation (2), contract equation (23) with ϵ^{bcde} and use $j^b = \frac{1}{3!}\epsilon^{bcde}J_{cde}$. In terms of the 1-form factors of F (equation 18), this corresponds to the two conditions

$$\alpha \wedge J = 0 = \beta \wedge J. \tag{24}$$

The vanishing of these two 4-forms is an extremely simple and convenient characterization of the force-free condition. Given Maxwell's equation $d*F = J$, it amounts to the conditions

$$\alpha \wedge d*F = 0 = \beta \wedge d*F. \tag{25}$$

When the Euler potentials are used to express α, β, and F as in equation (22), equations (25) become

$$d\phi_i \wedge d*F = 0, \quad i = 1, 2 \tag{26}$$

which comprises the full content of FFE.

Note that these equations are equivalent to the statement that two currents are conserved:

$$d(d\phi_i \wedge *F) = 0. \tag{27}$$

The currents $d\phi_i \wedge *F$ deserve a name; we propose to call them *Euler currents*. That the force-free equations amount to the conservation of these two Euler currents is a trivial but useful observation which does not appear to have been made previously. In tensor notation, the Euler currents are given (up to a coefficient) by $F^{ab}\nabla_b\phi_i$. Note that we could have alternatively defined the Euler currents to be $\phi_i J$, which differs from the previous definition by the identically conserved 3-form $d(\phi_i *F)$.

3.3.2 Action

One can arrive directly at the force-free condition (27) starting from the usual Maxwell action $-\frac{1}{2}\int F \wedge *F$, expressed as a functional of the potentials,

$$S^{\rm FF} = -\frac{1}{2}\int d\phi_1 \wedge d\phi_2 \wedge *(d\phi_1 \wedge d\phi_2). \tag{28}$$

Variation of this action with respect to ϕ_1 and ϕ_2 yields conservation of the Euler currents (equation 27) as a pair of Euler–Lagrange equations.[5] This action, and the Hamiltonian formulation derived from it, was given by Uchida (1997a).[6] Note that the Lagrangian is quadratic in time derivatives, so the equations of motion are second order in time derivatives.

The action is a scalar, so the stress–energy tensor is conserved when the equations of motion are satisfied. This is to be expected, since our starting point was the force-free condition which implies that the field transfers no energy or momentum to the charges. Moreover, the dynamics share the symmetries possessed by the $*$ operator on 2-forms, namely symmetries and Weyl rescalings of the metric. This implies, for instance, that in a stationary axisymmetric spacetime there are conserved Killing energy and axial angular momentum currents, and that FFE shares with vacuum electrodynamics the property of depending only on the conformal structure of the spacetime. The potentials can also be restricted by a symmetry ansatz before variation, to directly obtain the equations governing the symmetric solutions.

3.3.3 Complex Euler potential

Finally, it seems worth noting that the two Euler potentials can be combined into one complex potential $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$. Then the field 2-form is given by $F = i\, d\phi \wedge d\bar{\phi}$, the force-free field equations correspond to the single complex equation $d\phi \wedge d*F = 0$, and the action is $\frac{1}{2}\int d\phi \wedge d\bar{\phi} \wedge *(d\phi \wedge d\bar{\phi})$. Whether this complex formulation is useful remains to be seen.

4 POYNTING FLUX SOLUTIONS

In this section, we recover and discuss a number of exact solutions to the force-free field equations (25) using the method of exterior calculus. In addition to introducing some important properties of force-free physics, we hope that this section will serve as a tutorial on computing with differential forms, for readers unfamiliar with that approach. The most unfamiliar element is perhaps the use of the Hodge dual in place of the metric. In Appendix A2, we review this operator and develop some computational techniques. With the aid of these techniques, computations using forms can be remarkably simple, as we demonstrate below. We begin by discussing the magnetic monopole, then cover solutions describing purely outgoing (or ingoing) Poynting flux, and finally superpose these to obtain the general solution used to construct monopole magnetospheres in the following section.

4.1 Vacuum monopole

To warm up, we begin with the magnetic monopole in the Schwarzschild background (which of course includes flat spacetime as a special case). It is a vacuum solution, and monopoles do not exist in nature, yet it has played an important role in the analytical modelling of force-free magnetospheres since the earliest years of the subject. The field strength 2-form is given by

$$F^{\rm mon} = q \sin\theta\, d\theta \wedge d\varphi. \tag{29}$$

[5] These Euler currents are the Noether currents associated with the global symmetries $\phi_1 \to \phi_1 + f_1(\phi_2)$ and $\phi_2 \to \phi_2 + f_2(\phi_1)$ of the action (28).

[6] An alternate approach (Thompson & Blaes 1998; Buniy & Kephart 2014) is to supplement the usual Maxwell action for the vector potential with a Lagrange multiplier term enforcing the degeneracy condition, $S = -\frac{1}{2}\int F \wedge *F - \lambda F \wedge F$. The resulting Euler–Lagrange equations are $d*F = d\lambda \wedge F$ and $F \wedge F = 0$, another formulation of FFE. We learn from this that $J = d\lambda \wedge F = d\lambda \wedge d\phi_1 \wedge d\phi_2$ for some scalar field λ, which immediately implies the force-free condition (26). (Conversely, it is possible to show that J has this directly from the force-free conditions 24 and $dF = 0$.)

This is proportional to the area element on the sphere, and has the same flux integral ($4\pi q$) for any radius, so it is clearly the monopole field.[7] But to illustrate the exterior calculus, let us check that the field equations are satisfied. We have $dF^{\rm mon} = q\cos\theta\, d\theta \wedge d\theta \wedge d\varphi$, which vanishes because $d\theta \wedge d\theta = 0$. As for the other field equation, according to equation (A12), the dual of the monopole 2-form is $*F^{\rm mon} = qr^{-2}dt \wedge dr$, so $d * F^{\rm mon} = -2qr^{-3}dr \wedge dt \wedge dr$. This too vanishes, because $dr \wedge dr = 0$. The 3 + 1 version of the magnetic monopole field in flat spacetime is $\boldsymbol{B} = (q/r^2)\hat{\mathbf{r}}$, $\boldsymbol{E} = 0$.

The 2-form (equation 29) is simple, i.e. the monopole field is degenerate. In particular, this implies that it can be expressed in terms of Euler potentials, which can be taken as $\phi_1 = -q\cos\theta$ and $\phi_2 = \varphi$. Note that the discontinuity of φ at 2π means that the Euler potential is not globally smooth. This presents no problem; moreover, were it not for this discontinuity, the field would be an 'exact form' $dA^{\rm mon}$, with $A^{\rm mon} = q\varphi\, {\rm d}(\cos\theta)$, so the total magnetic flux through the closed surface of the two-sphere would necessarily vanish.[8]

4.2 Outgoing Poynting flux

The next solution we consider is genuinely force free ($j^a \neq 0$) and remarkably simple and general. The solution is on Schwarzschild (and flat) spacetime and has no symmetries at all, being given in terms of a free function of three variables as

$$F^{\rm out} = d\zeta \wedge du, \tag{30}$$

where $\zeta = \zeta(\theta, \varphi, u)$ is a function of retarded time (outgoing Eddington–Finklestein time) u, and the sphere angles (θ, φ). (In flat spacetime, $u = t - r$.) This solution was first found in Brennan et al. (2013) using a Newman–Penrose formalism, but here we analyse it in the simpler language of differential forms. Comparison with equation (22) shows that ζ and u are Euler potentials for this solution. Since du is null and orthogonal to $d\zeta$, $F^{\rm out}$ is a null 2-form. The flat spacetime electric and magnetic fields are given below in equation (39). It is evident from equation (30) that $dF^{\rm out} = 0$. To check the force-free condition, we use equation (A14) for the dual of a null 2-form, giving $*F^{\rm out} = *(d\zeta \wedge du) \sim \star d\zeta \wedge du$, where $\star$ indicates dual on the sphere. The current is $J = d * F^{\rm out} \sim d\theta \wedge d\varphi \wedge du$, showing that $d\zeta \wedge J = du \wedge J = 0$, i.e. the force-free equations (25) are satisfied.

The electromagnetic stress–energy tensor $T^{\rm EM}_{ab}$ (equation 1) associated with equation (30) is given by

$$T^{\rm out}_{ab} = |d\zeta|^2 (du)_a (du)_b, \tag{31}$$

where $|d\zeta|^2$ denotes $g^{ab}(d\zeta)_a(d\zeta)_b$. Thus the solution represents a flow of electromagnetic energy along the outgoing radial null direction $(du)^a$. Because of this flux, we refer to $F^{\rm out}$ as the outgoing flux solution. The net flux of Killing energy leaving the system at retarded time u, calculated at $r = \infty$, is given by

$$\mathcal{P}^{\rm out}(u) \equiv \lim_{r\to\infty} \int T^{\rm out}_{ab} (\partial_t)^a (dr)^b d\Omega = \int |d\zeta|^2 d\Omega, \tag{32}$$

where $d\Omega$ is the area element on the unit sphere. Since the Killing energy is conserved as it propagates, this is also the Killing flux per Killing time through a sphere at any radius.[9]

The energy flow in the field (30) is unlike ordinary electromagnetic radiation in that the flux persists for stationary fields, i.e. energy is carried away even if ζ is independent of u. In this case, the solution has more the character of a flow than a wave, and such flows are sometimes called 'electromagnetic winds' or 'Poynting winds'. For vacuum fields, this situation is impossible with isolated sources, but it does occur in waveguides and in planar symmetry. In fact, these scenarios admit vacuum solutions that are highly analogous to equation (30). In Appendix B, we explore these examples as context for understanding the outgoing flux solution.

By itself, the outgoing flux solution is unphysical, since it describes energy emerging from the origin of coordinates in flat spacetime (where the solution is singular), or from the past horizon on the analytic extension of the Schwarzschild spacetime.[10] Additionally, as a null field, it lies on the threshold of the electrically dominated regime, and thus might be unstable to non-force-free processes. However, as described below, the solution is physically realized as part of magnetically dominated field configurations associated with a rotating star or black hole, which sources the outflow of energy.

The current $J \sim d\theta \wedge d\varphi \wedge du$ of the outgoing flux solution is a null 3-form. The dual of such a form is proportional to the null factor du (see Appendix A2.4), so we have $j^a \sim (du)^a$. That is, the current four-vector is null and radial. If the charges all have the same sign, they must be moving at the speed of light, but a null current can also be composed of charges of opposite sign moving such that the net charge density is equal to the magnitude of the net three-current in any Lorentz frame. The force-free equations are sensitive only to the *net* charge-current.

Using the standard orientation $dt \wedge dr \wedge d\theta \wedge d\varphi$, the current for this solution is given explicitly by

$$\begin{aligned} J &= (d \star d\zeta) \wedge du \\ &= (\Delta_2 \zeta) \sin\theta\, d\theta \wedge d\varphi \wedge du, \end{aligned} \tag{33}$$

where Δ_2 is the Laplacian on the unit sphere. This expression reveals two important points. First, the integral of the current over angles vanishes, so there is no net current entering or leaving the system. Since the current is null (equal magnitude of charge and current), this also indicates that there is no net charge.[11] Secondly, there is no vacuum solution of this sort that is everywhere regular on the sphere. In vacuum the current vanishes, which would require that

[7] Monopole charge is conventionally defined to equal the flux integral. Our q is thus $1/4\pi$ times the usual notion; we nevertheless refer to q as the monopole charge.

[8] The discontinuity could be avoided by using instead the potential $A^{\rm mon} = -q\cos\theta\, d\varphi$. However, the norm of the 1-form $d\varphi$ is $(g^{\varphi\varphi})^{1/2} = 1/(r\sin\theta)$, which blows up at the poles. This can be fixed at the north pole by using instead $A^{\rm mon,\,N} = -q(\cos\theta - 1)\, d\varphi$ and at the south pole by using $A^{\rm mon,\,S} = -q(\cos\theta + 1)\, d\varphi$, which differs from the northern potential by the pure gauge piece $-2q\, d\varphi$. The discontinuity of φ implies that this gauge transformation is not trivial however, which accounts for the existence of a non-zero magnetic flux through the sphere.

[9] The concept of Killing time applies to an individual integral curve of the Killing field ξ^a, and is given by the lapse of λ along the curve, where λ is any function satisfying $\xi^a \nabla_a \lambda = 1$ on the curve. In Schwarzschild, possible choices for λ include the usual time coordinate t as well as the outgoing and ingoing Eddington–Finklestein coordinates u and v. The Killing time may equivalently be defined as the lapse of parameter along the curve, when parametrized so that the tangent vector equals the Killing vector.

[10] Note also that the solution is not regular on the future horizon unless $d\zeta$ vanishes as $u \to \infty$.

[11] The reason for this can be traced to force-free condition $\boldsymbol{E} \cdot \boldsymbol{j} = 0$ (equation 6). Since the current is radial, this condition implies that $\boldsymbol{E}$ has no radial component, which implies that the flux of $\boldsymbol{E}$ through a sphere vanishes, so there can be no net charge inside the sphere.

ζ be a harmonic function on the sphere, $\Delta_2 \zeta = 0$. Other than a constant (which yields zero field), no such functions exist.

It is rather curious that a purely outgoing solution exists on a Schwarzschild background. One would expect that waves would backscatter from the effective potential caused by the spacetime curvature. The existence of non-scattering solutions like these was discovered by Robinson (1961). He showed that, associated with any shear-free null geodesic congruence, there is a family of null, non-scattering *vacuum* solutions to Maxwell's equations. For the radial outgoing null congruence in the Schwarzschild spacetime, the Robinson solutions are exactly the fields (equation 30) with $\Delta_2 \zeta = 0$. These are in some sense illusory solutions, since they are not globally regular on the sphere. However, they are resurrected as *bona fide*, regular solutions in the force-free context.

4.3 Outgoing flux from an arbitrary world line

In flat spacetime, the energy flux of equation (30) emerges from the origin of coordinates, which may be identified with a stationary world line. In fact, the solution generalizes readily to an *arbitrary* timelike world line, where u is taken to be the associated retarded time. That is, on the future light cone of any point p on the world line, u is the proper time at p. Then precisely the same expression (30) is a solution, if (r, θ, φ) are any coordinates such that $d\theta$ and $d\varphi$ are orthogonal to du, for example, global inertial spherical angles. This follows from the same computation used to check that equation (30) is a solution. This solution was first found in Brennan & Gralla (2014) using the Newman–Penrose formalism.

4.4 Outgoing flux in Kerr

We next present the generalization of the outgoing flux solution (30) to a rotating black hole background, i.e. the Kerr spacetime. The stationary axisymmetric version of this solution was found by Menon & Dermer (2007, 2011), and it was generalized to the non-stationary, non-axisymmetric case in Brennan et al. (2013) using the Newman–Penrose formalism. Here we recover that generalized solution using the exterior calculus.

It is simple to describe the solution using outgoing Kerr coordinates $(u, \bar{\varphi}, \theta, r)$, which are defined in Appendix C. A first guess would be that the field (30) is a solution, with the substitution $\varphi \to \bar{\varphi}$. However, that is not correct, because in Kerr the 1-form du is timelike rather than null, and the null property of du played a critical role in establishing that equation (30) is a solution in Schwarzschild. To motivate a modification, and to proceed with the calculations, we need the following properties of the Kerr metric in these coordinates (see Appendix C): (i) the 1-form $du - a\sin^2\theta\, d\bar{\varphi}$ is null and orthogonal to the 1-forms du, $d\bar{\varphi}$, and $d\theta$; and (ii) $d\theta$ and $d\bar{\varphi}$ are orthogonal to each other, and the ratio of their norms is $\sin\theta$.

The analogy with the case of the Schwarzschild metric now motivates the initial ansatz

$$F^{\mathrm{out,Kerr}} = d\zeta \wedge (du - a\sin^2\theta\, d\bar{\varphi}), \qquad (34)$$

where as before $\zeta = \zeta(\theta, \bar{\varphi}, u)$. However, note that $dF^{\mathrm{out,\,Kerr}}$ will be non-zero if ζ depends on u. Hence, let us assume that $\zeta = \zeta(\theta, \bar{\varphi})$ is independent of u, and check the force-free equations. (We will generalize this to non-stationary solutions momentarily.) The dual of equation (34) is given by $*F^{\mathrm{rad,Kerr}} = -\star d\zeta \wedge (du - a\sin^2\theta\, d\bar{\varphi})$, where $\star d\zeta$ is an r-independent linear combination of $d\theta$ and $d\bar{\varphi}$. The current is thus proportional to $d\theta \wedge d\bar{\varphi} \wedge du$, which has vanishing wedge product with the two factors of equation (34), so indeed the force-free field equations (24) hold.

Now to allow for u dependence, we must generalize the ansatz to

$$F^{\mathrm{rad,Kerr}} = (A\, d\theta + B\, d\bar{\varphi}) \wedge (du - a\sin^2\theta\, d\bar{\varphi}), \qquad (35)$$

where $A = A(\theta, \bar{\varphi}, u)$ and $B = B(\theta, \bar{\varphi}, u)$. This is not necessarily a closed 2-form so we must impose the covariant Faraday law

$$dF^{\mathrm{out,Kerr}} = [A_{,\bar{\varphi}} - B_{,\theta} + (a\sin^2\theta) A_{,u}]\, d\bar{\varphi} \wedge d\theta \wedge du = 0, \qquad (36)$$

which results in the differential equation

$$A_{,\bar{\varphi}} - B_{,\theta} + (a\sin^2\theta) A_{,u} = 0. \qquad (37)$$

In the non-spinning ($a = 0$) or stationary cases, the last term vanishes and we find $A = \zeta_{,\theta}$ and $B = \zeta_{,\bar{\varphi}}$ for some ζ as before. In the spinning, non-stationary case, we could for example choose any A, and define B by integration with respect to θ (although only for some A will the solution be smooth at the poles). Once we have solved equation (37), all that remains is to impose the force-free conditions, but these hold by exactly the same reasoning just used for the stationary solutions.

Note that, like in the Schwarzschild case, this solution has the remarkable property that the radiation has no backscattering. This is again directly linked to Robinson's theorem: the congruence tangent to the null vector obtained by contraction of $du - a\sin^2\theta\, d\bar{\varphi}$ with the inverse metric is geodesic and shear free. [It is the outgoing principal null congruence of the Kerr metric (e.g. Poisson 2004).] And again, there is no globally regular vacuum solution of this type, but in the presence of non-zero current there are regular force-free solutions. These solutions were first found by assuming that the current is along the principal null congruence (Brennan et al. 2013). That analysis also shows that there are no other solutions with such a current.

4.5 Ingoing flux

By taking the time-reverse[12] of the outgoing flux solution, one obtains an ingoing flux solution. This solution represents energy emerging from a distant region and converging on the origin of flat spacetime, or entering the horizon of a black hole. In the black hole case, the ingoing flux is regular at the future horizon and totally absorbed by the black hole, with no backscattering.

4.6 Superposed monopole and flux

Since FFE is non-linear, in general the superposition of two solutions does not yield a third solution. However, the vacuum monopole field (29) has no current, and exerts no force on the current of the radial flux solution (30) (i.e. $F^{\mathrm{mon}}_{ab}\, j^{\mathrm{out}\,a} = 0$), so their superposition yields a solution,

$$F^{\mathrm{sup}} = q\sin\theta\, d\theta \wedge d\varphi + d\zeta \wedge du. \qquad (38)$$

The field in equation (38) is magnetically dominated when $q \neq 0$, and is otherwise null. It is in fact the general force-free solution with radial, null current in Schwarzschild (and flat) spacetime (Brennan et al. 2013).

[12] In Kerr, the time-reverse refers to sending $t \to -t$ and $\varphi \to -\varphi$.

Unfortunately, this simple construction does not generalize to the Kerr background. Although exact monopole[13] and outgoing flux solutions on Kerr are known, the monopole field exerts a force on the null current. [This obstruction is a special case of a general theorem: a solution with current along a null geodesic twisting congruence cannot be magnetically dominated (Brennan et al. 2013).]

An interesting generalization applies however to a monopole moving along an arbitrary world line in Minkowski space: the dual of the Lienard–Wiechert vacuum field can be superposed with the outgoing flux solution described in Section 4.3 (Brennan & Gralla 2014). This yields a magnetically dominated solution, in which normal radiation (in the dual Lienard–Wiechert field) coexists with current-supported Poynting flux.

It is instructive to write the 3 + 1 version of the superposed solution for a stationary world line in flat spacetime. The electric and magnetic fields associated with equation (38) in flat spacetime are given by the orthonormal frame components

$$E_{\hat\theta} = B_{\hat\varphi} = \frac{1}{r}\partial_\theta \zeta, \tag{39a}$$

$$E_{\hat\varphi} = -B_{\hat\theta} = \frac{1}{r\sin\theta}\partial_\varphi \zeta, \tag{39b}$$

$$B_{\hat r} = \frac{q}{r^2}. \tag{39c}$$

The outgoing flux part of the solution (second term in equation 38) corresponds to equations (39a) and (39b), while the monopole corresponds to equation (39c). The radial Poynting flux is carried by orthogonal E and B fields tangent to the sphere and equal in magnitude, while the magnetic monopole gives the magnetic field lines a radial component and ensures magnetic domination. The 3 + 1 version of the statement that this solution has a null, radial four-current is that the three-current density is radial and equal in magnitude to the charge density,

$$\rho = -\frac{\Delta_2\zeta}{r^2}, \quad \boldsymbol{J} = \frac{\Delta_2\zeta}{r^2}\hat{\mathbf{r}}, \tag{40}$$

where we remind the reader that Δ_2 is the Laplacian on the unit sphere. The monopole field is a vacuum solution and the charge and current come entirely from the outgoing flux solution. The fact that these solutions may be superposed can be understood by noting that the magnetic monopole field is in the same (radial) direction as the current of the outgoing flux solution, so that the addition of the monopole yields no Lorentz force. As described below, different choices of q and ζ give rise to different solutions relevant to the exterior of different rotating bodies.

5 MONOPOLE MAGNETOSPHERES

In this section, we apply the solutions discussed in the previous section to model magnetospheres external to rotating stars and black holes with monopole charge. These models present basic physical properties of force-free magnetospheres in a simple setting, most importantly the conversion of rotational kinetic energy to Poynting flux. Using the same solutions, a closer approximation to real magnetospheres is obtained by 'splitting' the monopole, as discussed in Section 6.

[13] The exact magnetic monopole solution on Kerr is obtained by taking the dual of the solution generated by the vector potential $A_a = \xi_a$, where ξ^a is the (asymptotic) time-translation Killing field.

5.1 Rotating monopole (Michel solution)

The Michel (1973) rotating monopole solution has served for decades as a starting point for analytical modelling of pulsar and black hole magnetospheres. Michel found his solution using an early version of the stationary, axisymmetric framework that we treat in Section 7. Here we instead recover the solution as a special case of the monopole/flux solution (38). Specifically, the Michel solution is given by specializing to flat spacetime and choosing $\zeta = q\Omega\cos\theta$ with constant Ω,

$$F^{\rm Michel} = -q\, d(\cos\theta)\wedge(d\varphi - \Omega\, du). \tag{41}$$

This solution can be terminated on the surface of a perfectly conducting star rotating with angular velocity Ω. The 1-forms $d\theta$ and $d\varphi - \Omega\, du$ both vanish when contracted with the four-velocity of any point comoving with the surface (which is proportional to $\partial_t + \Omega\partial_\varphi$), so that the electric field vanishes in the conductor rest frame. The conducting boundary conditions only require the tangential components to vanish; the fact that also the perpendicular component also vanishes is a consequence of the force-free magnetosphere outside, and would not hold for the (non-degenerate) exterior field of a rotating magnetized conductor in vacuum (see discussion at the end of Section 8.1). For comparison with the more realistic cases of higher multipoles, it is conventional for a spherical star to rewrite the monopole charge q in terms of the magnetic field strength B_0 at the surface of radius R, $q = B_0R^2$.

The current 3-form for the Michel solution is given by equation (33), which evaluates to $J = -q\Omega\sin 2\theta\, d\theta\wedge d\varphi\wedge du$. Equivalently, the current four-vector is equal to the radial null vector $j^a = -2q\Omega(\cos\theta/r^2)(\partial_r)^a$.[14] In the northern hemisphere, this is a radial ingoing three-current and a negative charge density of the same magnitude as the three-current, while in the southern hemisphere, it is a radial outgoing three-current and a positive charge density.

The energy flux away from the rotating monopole comes only from the radiation part of the field, and is given by equation (32), which evaluates here to

$$\mathcal{P}^{\rm Michel} = \frac{8\pi}{3}q^2\Omega^2 = \frac{8\pi}{3}B^2R^4\Omega^2. \tag{42}$$

This outflow of energy is transferred from the rotational kinetic energy of the conductor, which is possible because the field is not force free in the conductor. The physics of the transfer can be understood as follows. Free charges in the conductor are carried by the rotational motion and hence feel a Lorentz force that drives a current in the surface from north to south. This current in turn feels a Lorentz force opposite to the motion, producing a reaction torque on the conductor, which acts to slow the rotation.

The Michel monopole illustrates a key physical effect of pulsar physics: a rotating, magnetized conductor generates an outgoing energy flux, even when stationary.

5.1.1 *Field sheet geometry of the Michel monopole*

The field of a monopole rotating in vacuum would of course be identical to that of a static monopole, but the Michel solution is in a certain sense 'really rotating'. The structure of this field can be elucidated via the geometry of its field sheets. The Euler potentials can

[14] The vector ∂_r is defined in the (u, r, θ, φ) coordinate system, so it corresponds to translation of r at fixed u, θ, φ, and hence is a future pointing, outgoing null vector.

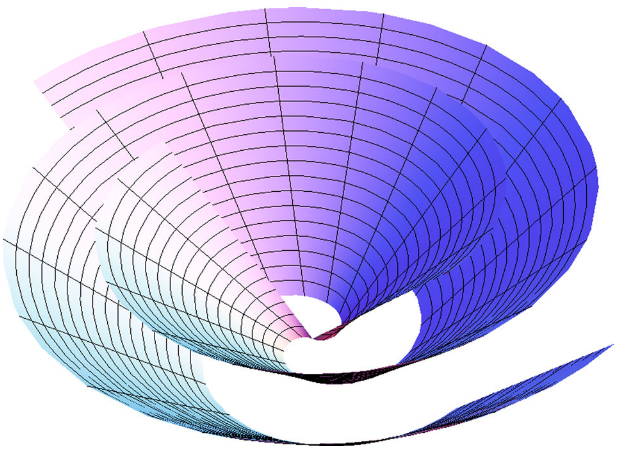

Figure 1. Two equatorial field sheets of the Michel monopole.

be taken as $\phi_1 = -q\cos\theta$ and $\phi_2 = \varphi - \Omega(t - r)$, so the field sheets are the surfaces where θ and $\varphi - \Omega(t - r)$ are constant. Lab frame field lines (intersections of the sheets with constant t planes) form Archimedean spirals in the equatorial plane, and conical helices for other values of θ (see Fig. 2). At successive times t and $t + \Delta t$, these lines are rotated relative to each other by an angle $\Omega\Delta t$, so one may think of the lines as rotating with angular velocity Ω. They are also related by $r \to r + \Delta t$, however, so one may equivalently think of them as expanding outwards at the speed of light. The field sheet is independent of which way one thinks of field line evolution (and also of the choice of frame used to define field lines). A spacetime plot of two equatorial field sheets is given in Fig. 1.

The field sheet metric is obtained by imposing the conditions $d\theta = 0$ and $d\varphi = \Omega\, du$ in the Minkowski metric, which yields $ds^2 = -(1 - r^2\Omega^2\sin^2\theta)du^2 - 2du\, dr$. Amusingly, this is nothing but 1+1 dimensional de Sitter spacetime in 'Eddington–Finkelstein' form. The de Sitter horizon corresponds to the light cylinder $r\sin\theta = 1/\Omega$ where a corotating observer would move with the velocity of light. The 'Hubble constant' is $\Omega\sin\theta$, which is also the surface gravity of the horizon. This Killing horizon interpretation of the light cylinder extends to general stationary axisymmetric magnetospheres, as we discuss in Section 7.2.5.

5.1.2 *Differential rotation*

Equation (41) remains a solution when Ω is promoted to an arbitrary function of θ. This corresponds to a conducting star that rotates at latitude-dependent speed. This generalization of the Michel monopole was first noted by BZ (see equation 6.4 therein). It corresponds to choosing $\zeta = -q\int \sin\theta\, \Omega(\theta) d\theta$ in the superposed solution (38). On account of the θ-dependence of ζ, the power radiated is modified from the Michel form (42).

5.1.3 *Variable rotation rate*

Equation (41) remains a solution when Ω is promoted to an arbitrary function of u, or more generally of u and θ. This corresponds to a conducting star whose rotational speed changes with time. The changes propagate outwards into the magnetosphere at the speed of light. This generalization of the Michel monopole was first noted by Lyutikov (2011). It corresponds to choosing $\zeta = -q\int \sin\theta\, \Omega(\theta, u)\, d\theta$ in the superposed solution (38). The flux at each retarded time u is given by the instantaneous value of the associated stationary solution.

5.2 Whirling monopole

As a final generalization of the Michel monopole equation (41) terminated on a conducting star, we take the star to be a sphere, and allow it to undergo *arbitrary* time-dependent rigid rotation with fixed centre. The Michel solution corresponds to the choice $\zeta(\theta, \varphi, u) = q\Omega\cos\theta$ in the superposed solution (38). To produce the whirling monopole, we replace the constant Ω by $\Omega(u)$, and we replace θ, the angle between the field point and the fixed rotation axis of the rotating monopole, by $\Theta = \Theta(\theta, \varphi, u)$, the angle between the field point and the rotation axis at the retarded time. In terms of the angular velocity vector $\mathbf{\Omega}(t)$ and the radial unit vector $\hat{\mathbf{r}}(\theta, \varphi)$, we have $\Omega(u)\cos\Theta = \mathbf{\Omega}(u)\cdot\hat{\mathbf{r}}(\theta, \varphi)$. This defines a suitable $\zeta(\theta, \varphi, u)$ for the monopole/flux solution (38), and yields the whirling solution

$$F^{\text{whirl}} = q\sin\theta\, d\theta\wedge d\varphi + q\, d[\mathbf{\Omega}(u)\cdot\hat{\mathbf{r}}(\theta, \varphi)]\wedge du. \tag{43}$$

At any retarded time, this agrees with the Michel solution corresponding to the instantaneous angular velocity vector; hence, it satisfies the conducting boundary condition on the surface of the whirling sphere. Furthermore, the flux of the whirling monopole at retarded time u agrees with the flux (42) of the corresponding Michel solution. Thus even if a pulsar undergoes a dramatic whirl (as could occur during a glitch), then the monopole model predicts no additional associated energy losses.

5.3 Black hole monopole (BZ solution)

As described in Section 4.6, the procedure of superposing monopole and outgoing flux solutions fails to produce a solution in Kerr. However, long ago BZ found a perturbative monopolar solution describing a stationary, axisymmetric outgoing flux of energy from a Kerr black hole to second order in the black hole spin parameter a. This solution may be recovered to first order in a simply by promoting the Michel monopole solution to Kerr, as we now explain. Recovering the second-order perturbations is also straightforward, though more involved. We focus on the first-order piece, which provides the leading outgoing energy flux.

Although we explicitly considered flat spacetime when discussing the Michel monopole (41), it is also a valid solution in Schwarzschild, with u the Schwarzschild retarded time (outgoing Eddington–Finklestein time). (Both follow from equation 38 with $\zeta = q\Omega\cos\theta$.) Through first order in a, the BZ solution is just the Michel monopole, exported from Schwarzschild to Kerr by identifying the Schwarzschild coordinates with the Boyer–Lindquist (BL) coordinates. Thus the ansatz for the solution is

$$F^{\text{BZ}}_{\text{ansatz}} = q\sin\theta\, d\theta\wedge(d\varphi - \Omega du). \tag{44}$$

Expecting Ω to be controlled by the spin of the black hole, we regard this quantity as $O(a)$. Note that the background solution for this perturbation analysis is then the vacuum monopole in Schwarzschild, equation (29).[15] We may take u to be the outgoing Kerr coordinate

[15] When perturbing a vacuum solution $F^{(0)}$ to a force-free solution $F^{(0)} + F^{(1)}$, the first-order force-free condition is simply $F^{(0)}\cdot j^{(1)} = 0$. Thus one may choose *any* conserved current $j^{(1)}$ transverse to the background field and then construct an associated Maxwell field $d * F^{(1)} = *j^{(1)}$. BZ eliminated this freedom by demanding that perturbative solutions approach a genuine non-linear force-free solution (in this case the Michel monopole) at large r. Here we simply promote the Michel solution to Kerr and note that the field equations hold to $O(a)$.

(C13), since that differs from the Schwarzschild one only at $O(a^2)$ when expressed in terms of t and r. This ansatz obviously satisfies $dF = 0$, so it remains only to check the force-free conditions.

Since there is no current in the monopole solution, the second factor in the force-free conditions (24) vanishes at $O(a^0)$; hence, in the $O(a)$ equation, the first factor (α or β) may be taken to be the zeroth-order parts $d(-q\cos\theta)$ and $d\varphi$. Thus up through $O(a)$, the force-free conditions amount to $d\theta \wedge J = d\varphi \wedge J = 0$, i.e. the statement that both $d\theta$ and $d\varphi$ are factors in the current 3-form $d * F$. The $O(a)$ terms in $d * F$ have two origins: the Ω term in equation (44) and the $O(a)$ part of the action of $*$ on the zeroth-order (monopole) solution. The contribution of the Ω term to the current is $\Omega d * (\sin\theta\, d\theta \wedge du) = \Omega d(\sin^2\theta\, d\varphi \wedge du) \sim d\theta \wedge d\varphi \wedge du$, which has both $d\theta$ and $d\varphi$ as factors. The $O(a)$ part of $*(d\theta \wedge d\varphi)_{\mu\nu} = 2\epsilon^{\theta\varphi r t} g_{r[\mu} g_{\nu]t}$ comes from $g_{\varphi t}$, the only $O(a)$ part of the Kerr metric in BL coordinates. Since $g_{r\mu} \propto \delta^r_\mu$, this $O(a)$ contribution has the form $C(r,\theta) dr \wedge d\varphi$ for some function C. It therefore contributes to the current $d * F$, a 3-form $\sim d\theta \wedge dr \wedge d\varphi$, which also has both $d\theta$ and $d\varphi$ as factors. Hence, the force-free condition is satisfied at $O(a)$.

Up to this point, the derivation would have also worked beginning with general superposed solution (38), provided the outgoing flux part is treated as $O(a)$. However, there is one more, crucial, consideration regarding this rotating black hole solution: it should be regular on the future event horizon. It is easy to see that this requirement can be met within this class *only* by the Michel monopole solution, with a specific value of Ω. The 1-forms $d\varphi$ and du are singular on the Kerr horizon, but there is a value of Ω for which their singularities cancel in $d\varphi - \Omega du$. To see this, we use ingoing Kerr coordinates v and $\tilde{\varphi}$, which are related to u and φ by equations (C8), (C9), and (C13). Using $a = \Omega_{\rm H}(r_+^2 + a^2)$, it follows that

$$d\varphi - \Omega du = d\tilde{\varphi} - \Omega dv + \frac{2\Omega(r^2 + a^2) - \Omega_{\rm H}(r_+^2 + a^2)}{\Delta} dr. \quad (45)$$

Thus the singularity at the horizon is avoided if and only if Ω is *one-half* the horizon angular velocity,

$$\Omega = \frac{1}{2}\Omega_{\rm H}. \quad (46)$$

Since $\Omega_{\rm H} \sim a$, it was indeed consistent to treat Ω as an $O(a)$ quantity when verifying that the force-free conditions are met. We may thus write the BZ solution, to $O(a)$, in the exceptionally simple form

$$F^{\rm BZ} = q \sin\theta\, d\theta \wedge \left(d\varphi - \frac{1}{2}\Omega_{\rm H} du\right). \quad (47)$$

This may of course also be written in a way that is manifestly regular on the horizon. Using equation (45), the second factor in equation (47) may be replaced by $d\tilde{\varphi} - \frac{1}{2}\Omega_{\rm H} dv + \Omega_{\rm H}(1 + 2M/r) dr$, dropping terms of $O(a^2)$.

The net energy flux can be computed far from the black hole where the metric is flat, hence the flux associated with equation (47) is given by the same expression (equation 42) as for the Michel monopole,

$$\mathcal{P}^{\rm BZ} = \frac{8\pi}{3} q^2 \left(\frac{1}{2}\Omega_{\rm H}\right)^2 \approx \frac{\pi}{24} q^2 \frac{a^2}{M^4} = \frac{1}{24\pi} B_0^2 a^2. \quad (48)$$

(Here $B_0 = 4\pi q/(2M)^2$ is defined as the magnetic flux through the horizon, divided by the horizon area.) The energy–momentum tensor (equation 1) contains cross-terms between the monopole and the $O(a^2)$ part of F, which we have not computed. However, since the monopole field has only a $\theta\varphi$ component, no $T^r{}_t$ component of the stress tensor arises in this way, so equation (48) is the full flux at this order. Note that, unlike with the rotating monopole terminated on a star, the energy carried by this flux does not appear in the field by violation of the force-free condition. Rather, the conserved Killing energy on the rotating black hole background is locally *momentum* in the ergosphere, hence can be negative there. A flux of negative Killing energy crosses the horizon, balancing the outward positive flux. The nature of this process is discussed more fully at the beginning of Section 9.

Rotating stars

To first order in the spin, the exterior metric of a rotating star is given by the Kerr metric linearized in a (Hartle & Thorne 1968). We may thus also use equation (44) to model stellar magnetospheres, including the leading gravitational effects of spin. As in the previous subsection, imposition of conducting boundary conditions at the star will fix Ω to equal the rotational velocity of the star. It is interesting to compare this with the black hole case (46), where there is an additional factor of one-half. As will be seen in Section 9, the energy flux from any axisymmetric black hole magnetosphere would vanish if the angular velocity of the field were equal to that of the black hole horizon.

6 CURRENT SHEETS AND SPLIT MONOPOLES

We have seen that the superposition of monopole and outgoing radiation solutions provides a simple analytic solution describing energy flux from rotating stars and black holes. The catch, of course, is that real stars and black holes do not have monopoles inside them! A cheap trick for addressing this last point is to artificially split the monopole in two or more parts, reversing the sign of the monopole charge (and perhaps also rescaling the charge) when passing from one region to the next. A crude model of a dipole can be constructed in this way, for example, while still using only the monopole solution. However, this splitting of the field has a dramatic consequence that must be confronted: since the field changes direction discontinuously across the splitting surface, Maxwell's equations imply the presence of a surface current and surface charge. Fortuitously, rather than being an unphysical embarrassment, this current sheet actually *enhances* the correspondence of the solution with a pulsar magnetosphere. We discuss the general necessity of such current sheets in Section 8.

6.1 Split monopole

To illustrate the basic idea of a split monopole, consider first the field of a point magnetic monopole in vacuum, $\boldsymbol{B} = (q/r^2)\hat{\mathbf{r}}$, and modify it by reversing the sign of the charge across the equatorial plane, yielding $\boldsymbol{B}^{\rm split} = {\rm sgn}(\cos\theta)(q/r^2)\hat{\mathbf{r}}$. In order for this to remain a solution to Maxwell's equations, there must be a surface layer of azimuthal current on the equatorial plane, i.e. a current sheet. Taking this solution to extend inwards only to some radius $r = R$, one may regard it as the exterior of a star that has been magnetized in a peculiar split-monopole pattern. Since the magnetic flux through closed surfaces vanishes, no monopole is required and ordinary currents flowing in the star can generate the field.

6.2 Generalized split field construction

We may split a field configuration across more general surfaces as follows. Begin with any force-free solution F, and replace it with a new solution

$$F^{\rm split} = \sigma F, \tag{49}$$

where σ is a 'step function' on spacetime: constant except where it has a jump across a timelike three-volume $\mathcal{S}$, the world volume of the current sheet. In the case of the vacuum monopole discussed above, $\mathcal{S}$ would be the equatorial plane, extended in time, but in general it is a dynamical submanifold whose motion must be determined.

The jump conditions implied by Maxwell's equations must hold at the current sheet. As explained in Appendix A3, these are that (i) the pullback to $\mathcal{S}$ of the jump in F must vanish (implying no monopole surface charge or current), and (ii) the pullback to $\mathcal{S}$ of the jump in $*F$ is the surface current 2-form K (which describes both charge and two-current densities). The jump in $F^{\rm split}$ is $[F^{\rm split}] = [\sigma]F$, so the jump conditions are

$$F|_{\mathcal{S}} = 0, \qquad *F|_{\mathcal{S}} = K/[\sigma], \tag{50}$$

where the bar notation $|_{\mathcal{S}}$ denotes the pullback to $\mathcal{S}$. The first of these conditions implies that the current sheet fully contains any field sheet that intersects it at a point where $F \neq 0$: at a point where a field sheet intersects but is not contained in $\mathcal{S}$, there exists a basis consisting of three vectors tangent to $\mathcal{S}$ and a fourth tangent to the field sheet, and F vanishes when contracted with any pair of these, so $F = 0$. It follows that the three-dimensional current sheet world volume must be foliated by field sheets. Equivalently, the current sheet must be given by an equation $f(\phi_1, \phi_2) = 0$, where f is some function depending only on the two Euler potentials. This criterion is necessary and sufficient for a valid split of the form (49).

In terms of a 3 + 1 split, these considerations tell us the possible shapes of current sheets of the form (49) and specifies their unique time evolution: an initial configuration for a current sheet must be a two-dimensional surface tangent to magnetic field lines, and the time evolution is that of the field lines. In a spacetime sense, the world volume $\mathcal{S}$ of a current sheet may be generated by selecting a single 'seed curve' γ, transverse to field sheets, and flowing to all points on the field sheets intersecting γ. Put differently, it is just the bundle of field sheets over γ.

So far we have treated current sheets as infinitesimally thin regions where the field has a discontinuity. A physical sheet would have a finite thickness determined by its internal structure and the forces confining it. A simple model for a finite-thickness current sheet is obtained by using a smooth transition function $\sigma(x)$ instead of the step function of equation (49), yielding a degenerate, but not force-free, field $\tilde{F} \equiv \sigma(x)F$. Provided F is magnetic, this leads to opposing, compressional Lorentz forces as follows. Like all electromagnetic fields, $\tilde{F}$ must satisfy Faraday's law $d\tilde{F} = 0$, which implies $d\sigma \wedge F = 0$. Thus σ must be constant on the field sheets. The divergence of the stress tensor $\tilde{T}_{ab} = \sigma^2 T_{ab}$ is equal to $T^{ab}\nabla_b \sigma^2$ since the original field was force-free ($\nabla_a T^{ab} = 0$). Using equation (15) for the stress tensor of a degenerate field, only the $h^{\perp}$ term contributes (since σ is constant on the field sheets) and we find that the Lorentz force $-\nabla^a \tilde{T}_{ab}$ is equal to $-\frac{1}{4}F^2\nabla_b\sigma^2$. For magnetically dominated fields, this force is towards the centre of the sheet on both sides, i.e. compressional. A more complete model would account for the opposing force establishing equlibrium; for example, thermal pressure provides the support in a Harris current sheet (Harris 1962).

6.3 Rotating split monopole

We now apply the splitting procedure to the Michel monopole (41) and discuss its application to the BZ black hole monopole (47) and the whirling monopole (43).

6.3.1 Aligned split monopole

We first perform the split in the equatorial plane. Since all field lines in the equatorial plane remain in the plane (see Fig. 2), it is clear that this plane is a valid location for a current sheet. This original Michel split monopole is simply

$$F^{\rm aligned} = {\rm sgn}(\cos\theta) F^{\rm Michel}, \tag{51}$$

where as before the solution should be terminated on a rotating, conducting star. We label this field as 'aligned' because the magnetic axis is aligned with the spin axis. The surface current for the resulting equatorial current sheet is given by equation (50) with $[\sigma] = 2$, i.e. $K^{\rm splitMichel} = 2 * F^{\rm Michel}|_{\mathcal{S}}$. Taking the dual of equation (41), we thus have

$$K^{\rm splitMichel} = \frac{2q}{r^2} dt \wedge dr + 2q\Omega d\varphi \wedge du. \tag{52}$$

The first term is an azimuthal current density that falls off like r^{-2}, while the second term is a radial null current density that falls off like r^{-1} (because $|d\varphi| = 1/r\sin\theta$). The latter is the 'return current' in the complete circuit: whereas the non-split monopole has a current flowing in from infinity in the northern hemisphere and out to infinity in the southern hemisphere, the split monopole has current flowing inwards in both hemispheres, and outwards in the current sheet.

6.3.2 Inclined split monopole

We may equally well consider the inclined case, with the star magnetized in a split-monopole pattern with the split along an equator inclined at an angle α to the rotation axis $\hat{\mathbf{z}}$ and corotating with the star. This provides a model for a pulsar with inclined magnetic axis.

Recall that the Michel field sheets are specified by the values of the two Euler potentials, $-q\cos\theta$ and $\varphi - \Omega u$, so that the current sheet must be given by an equation of the form $f(\theta, \varphi - \Omega u) = 0$. To produce an inclined sheet, we choose f to vanish on the corotating inclined circle. This circle at one time plays the role of the curve γ that generates the current sheet. A function satisfying this requirement is $f(\theta, \varphi - \Omega u) = \hat{\mathbf{m}}(u) \cdot \hat{\mathbf{r}}(\theta, \varphi)$, where $\hat{\mathbf{m}}(u)$ is the rotating split-magnetization axis inclined at the angle α to $\hat{\mathbf{z}}$), and $\hat{\mathbf{r}}(\theta, \varphi)$ is the angle-dependent radial unit vector. Since $\hat{\mathbf{m}}$ uniformly rotates

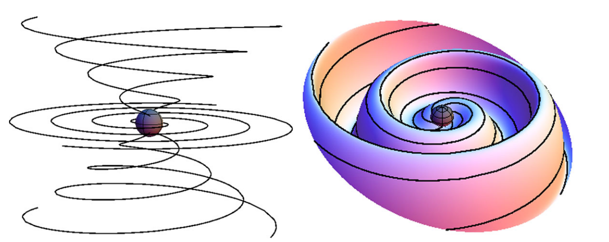

Figure 2. The Michel monopole, with central star drawn in. On the left, some representative lab frame magnetic field lines. On the right, the current sheet in the inclined case, with tangent field lines drawn in black. The pattern of the sheet rotates rigidly with the star, or equivalently moves radially outwards at the speed of light.

with angular velocity Ω about $\hat{\mathbf{z}}$, $\hat{\mathbf{m}} \cdot \hat{\mathbf{r}}$ actually depends on u and φ only through $\varphi - \Omega u$; explicitly,

$$\hat{\mathbf{m}}(u) \cdot \hat{\mathbf{r}}(\theta, \varphi) = \cos\alpha \, \cos\theta + \sin\alpha \, \sin\theta \, \cos(\varphi - \Omega u). \tag{53}$$

The inclined split monopole is thus given by

$$F^{\text{inclined}} = \text{sgn}[\hat{\mathbf{m}}(u) \cdot \hat{\mathbf{r}}(\theta, \varphi)] F^{\text{Michel}}. \tag{54}$$

The rather intricate shape of this current sheet is shown in Fig. 2. The complete field configuration, where the field changes sign on either side of the dynamical current sheet, is sometimes known as a 'striped wind'. Equation (54) was derived by Bogovalov (1999) in 3 + 1 language. A current sheet of nearly identical shape and dynamics is observed outside the light cylinder in simulations of inclined dipolar magnetospheres (Spitkovsky 2006; Kalapotharakos, Contopoulos & Kazanas 2012).[16]

The dipolar split monopole is the most relevant split configuration for emulating a dipole pulsar, but a variety of other configurations are possible. For example, one may split the solution on cones of fixed latitude, as is clear from the field lines shown in Fig. 2. Having two such cones, say at latitudes where $\cos\theta = \pm 1/\sqrt{3}$, provides a rough imitation of a quadrupole pulsar. In this aligned case, the conical sheets are stationary, but it would be straightforward to determine the more complicated shapes and dynamics in the inclined case. Most generally, one may use *any* seed curve on the sphere at one time to construct a sheet, since the monopolar (radial) component of the Michel monopole ensures that all such curves are transverse to field lines. In this sense, one may consider a sphere of arbitrary split-monopolar magnetization.

6.3.3 *Black hole split monopole*

One may also split the black hole version of the Michel monopole, as done by BZ in their original paper. The procedure is precisely analogous to the case of flat spacetime discussed above. The BZ model involves splitting in the equatorial plane, equation (51) with F^{Michel} replaced with F^{BZ} (equation 47). The sheet extends all the way to the event horizon. In nature, a magnetized accretion disc could source a field, and the current sheet becomes a crude model of such a disc. However, Lyutikov has raised the interesting possibility that the gravitational collapse of a pulsar could form a split-monopole black hole magnetosphere, where the current sheet originally present outside the light cylinder (e.g. Fig. 4) meets the horizon. If so, then the split BZ model would directly describe an astrophysical magnetosphere, if only for a brief time before magnetic reconnection destroys the sheet.

Although only the equatorial splitting has been explicitly considered in the black hole context, the more general splits discussed in the previous section are also possible. In particular, the inclined equatorial split also yields equation (54), with F^{Michel} replaced with F^{BZ} (47). As argued by Lyutikov in the aligned case, it is conceivable that this solution could model a black hole newly formed from the gravitational collapse of an inclined pulsar.

6.3.4 *Whirling split monopole*

In the whirling case (43), as in the simple rotating case, any curve tangent to the sphere is transverse to field lines, and so is a valid seed curve for a splitting. Thus while we do not construct explicit examples, our results do cover the magnetosphere, including current sheet dynamics, of an arbitrarily whirling, arbitrarily split-monopole-magnetized, conducting sphere. Astrophysically, the whirling split monopole could be helpful for modelling emission (or lack thereof) associated with pulsar glitches, including the case where in addition to the magnitude the direction of angular velocity is modified.

[16] Such sheets are supported in simulations by (non-force-free) prescriptions that enforce magnetic domination.

6.4 Reflection split

The preceding picture of current sheet behaviour applies to sheets produced by simple *rescalings* of the field strength across the sheet (equation 49). While this type of sheet commonly appears (for example in the outer region of many pulsar magnetosphere models), Maxwell's equations admit other types of field discontinuities supported by a current sheet $\mathcal{S}$, provided only that the pullback to $\mathcal{S}$ of the jump in F vanishes. For this type of discontinuity, the magnetic field is not necessarily tangent to the sheet, and there is no simple story regarding the location and dynamics of the current sheets.

A common example of such a discontinuity occurs in reflection-symmetric magnetospheres, where F is reflected across the equatorial plane, entailing an equatorial current sheet. The field at $\mathcal{S}$, the world volume of the equatorial plane, can be decomposed as $F = F_{\parallel} + F_{\perp}$, where $F_{\parallel}$ is the projection of F into $\mathcal{S}$ and is invariant under reflection, and $F_{\perp}$ flips sign under reflection. $F_{\parallel}$ comprises the magnetic field normal to and the electric field tangent to the symmetry plane, and $F_{\perp}$ comprises the tangent magnetic field and the normal electric field. The jump in F across $\mathcal{S}$ is thus $[F] = 2F_{\perp}$, and the pullback of this vanishes, so the jump condition on F is satisfied. For the other jump condition, note that the dual $*F_{\perp}$ is entirely 'parallel', so the jump in $*F$ is $[*F] = 2 * F_{\perp} = K$, which determines the surface current.

The aligned split monopole discussed in Section 6.3.1 is a special case of this construction. In that example, $F_{\parallel}$ vanishes, so the effect of the reflection is an overall sign change. Examples with a non-zero normal component of the magnetic field (and tangential component of the electric field) are the pulsar magnetospheres considered in Gruzinov (2011) and Contopoulos, Kalapotharakos & Kazanas (2014), the black hole magnetospheres of Uzdensky (2005), and the paraboloidal magnetospheres of Blandford (1976) and BZ.

7 STATIONARY, AXISYMMETRIC MAGNETOSPHERES

We turn now from specific analytical models to a general treatment of stationary, axisymmetric force-free magnetospheres, relevant both to spinning stars and to black holes. This section consists primarily of a systematic review and derivation of the standard mathematical and physical results, but using new computational techniques and the conceptual framework developed in Section 3. It can be seen as a spacetime counterpart to the 3+1 presentation of MacDonald & Thorne (1982), using an extension to curved spacetimes of the Euler-potential methods developed by Uchida (1997b). With our systematic use of differential forms, the efficiency and elegance of Uchida's approach is fully realized. In the following sections, we apply these results to pulsar and black hole magnetospheres.

Our treatment is spacetime geometrical in the sense that we do not decompose tensors into spatial components and temporal components with respect to a time foliation. However, we make heavy use of the existence of a coordinate system in which the metric components are block diagonal and do not depend on the two 'symmetry coordinates'. This hybrid technique of using spacetime objects, specifically differential forms, in concert with special coordinates, is both remarkably efficient for computations and revealing about the structure of the theory. Another source of the efficiency and simplicity is the avoidance of unnecessary introduction of metric dependence into the calculations, and of confining what metric dependence there is to the action of the Hodge dual operator and metric determinants. This is achieved by using the exterior derivative rather than covariant derivatives, integrating p-forms on p-surfaces, and using the Hodge dual operator.

7.1 2+2 decomposition of spacetime

In this section, we set up the decomposition of spacetime that is central to our treatment. We assume that the spacetime is stationary and axisymmetric, and that these two symmetries commute, so that there exist coordinates t, φ such that ∂_t and ∂_φ are the time-translation and axial-rotation Killing fields, respectively. Moreover, we assume that these Killing fields are orthogonal to two-dimensional surfaces. These assumptions should hold to a very good approximation in most astrophysically relevant settings.[17] We refer to these surfaces as the 'poloidal subspaces', and to the surfaces generated by the Killing vectors ∂_t, ∂_φ as the 'toroidal subspaces'. This is the standard usage of the word poloidal, while it generalizes toroidal to refer to the t–φ sector, rather than just the spatial φ-direction. We will label the poloidal subspaces with coordinates (r, θ) that are constant along the integral curves of the Killing fields, so that the metric components in these coordinates are block diagonal,

$$g_{..}(r,\theta) = \begin{pmatrix} g^{\mathrm{T}}_{..} & 0 \\ 0 & g^{\mathrm{P}}_{..} \end{pmatrix}. \tag{55}$$

We refer to $g^{\mathrm{T}}_{..}$ and $g^{\mathrm{P}}_{..}$ as the toroidal and poloidal metrics, respectively. Although it will not be necessary for all considerations, we further assume that $g^{\mathrm{T}}_{..}$ is Lorentzian, while $g^{\mathrm{P}}_{..}$ is Riemannian. These metrics depend only on the point in the poloidal surface, i.e. their components are functions only of r, θ. We emphasize that here r and θ are just names for arbitrary poloidal coordinates.

We adopt the orientation of $dt \wedge d\varphi \wedge dr \wedge d\theta$ for all integrals and dualization. The corresponding metric volume elements ϵ, ϵ^{T}, and ϵ^{P} on full, toroidal, and poloidal subspaces (respectively) are given by

$$\epsilon = \sqrt{-g}\, dt \wedge d\varphi \wedge dr \wedge d\theta, \tag{56}$$

$$\epsilon^{\mathrm{T}} = \sqrt{-g^{\mathrm{T}}}\, dt \wedge d\varphi, \quad \epsilon^{\mathrm{P}} = \sqrt{g^{\mathrm{P}}}\, dr \wedge d\theta, \tag{57}$$

where g, g^{T}, and g^{P} are the determinants of the corresponding metrics in these coordinates. These satisfy the identities

$$\epsilon = \epsilon^{\mathrm{T}} \wedge \epsilon^{\mathrm{P}}, \quad *\epsilon^{\mathrm{T}} = -\epsilon^{\mathrm{P}}, \quad *\epsilon^{\mathrm{P}} = \epsilon^{\mathrm{T}}. \tag{58}$$

[17] There is no loss of generality in assuming that the symmetries commute (Carter 1970), and for asymptotically flat solutions to Einstein's equation in vacuum or with a circularly rotating fluid source, the 2-surface orthogonality property necessarily holds (Wald 1984). Non-circular spacetimes result from gravitational effects of meridional matter flow or toroidal magnetic fields (Gourgoulhon & Bonazzola 1993). A fully geometrical treatment of ideal MHD in stationary, axisymmetric spacetimes, allowing for non-circularity, is given in Gourgoulhon et al. (2011).

We use ϵ^{T} and ϵ^{P} to define the Hodge dual operator on toroidal and poloidal forms, and we denote this operator by $\star$, reserving $*$ for the spacetime dual. Specifically, if ω^{P} is a poloidal differential form (a form made from poloidal cotangent vectors), then $\star\omega^{\mathrm{P}}$ denotes its Hodge dual on the poloidal space with respect to the poloidal metric, and similarly for $\star\omega^{\mathrm{T}}$. On toroidal 1-forms, $\star\star = 1$, while on poloidal 1-forms, $\star\star = -1$. The signs are opposite to these on 0-forms and 2-forms. The dual operators satisfy the following useful identities:

$$*(\omega^{\mathrm{T}} \wedge \omega^{\mathrm{P}}) = -\star\omega^{\mathrm{T}} \wedge \star\omega^{\mathrm{P}}, \tag{59}$$

$$*\omega^{\mathrm{P}} = \star\omega^{\mathrm{P}} \wedge \epsilon^{\mathrm{T}}, \tag{60}$$

where ω^{T} and ω^{P} are toroidal and poloidal 1-forms, respectively. More discussion of orthogonal subspaces and duality is given in Appendix A.

7.2 Degenerate, stationary, axisymmetric fields

A stationary, axisymmetric electromagnetic field satisfies $\mathcal{L}_{\partial_t} F = \mathcal{L}_{\partial_\varphi} F = 0$. In this subsection, we assume that the electromagnetic field is degenerate, $F \wedge F = 0$, but not necessarily force free. Thus it can be expressed in terms of Euler potentials as $F = d\phi_1 \wedge d\phi_2$. The Euler potentials need not share the symmetry of F, but their dependence on the ignorable coordinates t and φ is very restricted. Their form is worked out in Appendix D, following Uchida (1997b). Apart from the special case of purely toroidal magnetic field ($\partial_\varphi \cdot F = 0$), one may always choose Euler potentials given by equation (D14),

$$\phi_1 = \psi(r,\theta), \quad \phi_2 = \psi_2(r,\theta) + \varphi - \Omega_{\mathrm{F}}(\psi)t. \tag{61}$$

We focus on this generic case in the following. The special case $\partial_\varphi \cdot F = 0$ is treated briefly in Section 7.6 below.

Field sheets are surfaces of constant ϕ_1 and ϕ_2, hence are labelled by a value of ψ and a value of ϕ_2. If the field is magnetically dominated, as we assume from now on unless otherwise stated, the field sheets are timelike. A magnetic field line, defined with respect to t, is the intersection of a field sheet with a surface of constant t. Besides the 'true' magnetic field lines, one can also define *poloidal field lines*, which are just the ψ contours in the poloidal space. The bending of a true field line in the azimuthal direction, i.e. the variation of its φ coordinate at fixed t, is determined by ψ_2. As explained below, ψ determines the polar magnetic flux, and the function $\Omega_{\mathrm{F}}(\psi)$ determines the angular velocity of the field lines.

The field strength corresponding to the general Euler potential of the form (61) is

$$F = d\psi \wedge d\psi_2 + d\psi \wedge \eta, \tag{62}$$

where $\eta \equiv d\varphi - \Omega_{\mathrm{F}}(\psi)dt$. (The properties of this useful 1-form are discussed below in Section 7.2.4.) Note that there is no term proportional to $d\varphi \wedge dt$, i.e. no 'toroidal electric field'. This is a consequence of Faraday's law for stationary, axisymmetric fields, since the toroidal line integral of the electric field must be equal to minus the vanishing time derivative of the magnetic flux through the loop. It does not depend on the field being force free or even degenerate. Using equation (59), the dual of F is given by

$$*F = \frac{I}{2\pi}\, dt \wedge d\varphi - \star d\psi \wedge \star\eta, \tag{63}$$

where, for the moment, I is simply defined by

$$*(d\psi \wedge d\psi_2) = \frac{I}{2\pi}\, dt \wedge d\varphi, \tag{64}$$

but it will be interpreted below as the polar current. We can express F in terms of I instead of ψ_2, by taking the dual of equation (64), using equation (58) and $** = -1$ on 2-forms, as

$$F = \frac{I}{2\pi(-g^{\mathrm{T}})^{1/2}}\epsilon_{\mathrm{P}} + d\psi \wedge \eta. \tag{65}$$

This displays the field as a sum of its poloidal and toroidal parts. Note the potentially confusing fact that because the magnetic field vector is defined via the *dual* of the field strength 2-form F, the poloidal part of F (the first term in equation 65) actually corresponds to the *toroidal* magnetic field, i.e. the magnetic field component in the ∂_φ-direction according to an observer at rest in the poloidal subspace. The notation commonly used for this toroidal field is $B_{\mathrm{T}} = I/2\pi$. Note that the proper magnitude of the toroidal field is thus not B_{T} but rather $B_{\mathrm{T}}/\sqrt{-g_{\mathrm{T}}}$.

The invariant $F^2 = F_{ab}F^{ab}$ is the sum of the invariant squares of the toroidal and poloidal parts in equation (65),

$$F^2 = \frac{I^2}{2\pi^2(-g^{\mathrm{T}})} + |d\psi|^2|\eta|^2. \tag{66}$$

Here and below, we use the notation $|\eta|^2$ to denote $g^{ab}\eta_a\eta_b$. The first, poloidal term is always positive or zero, while the sign of the second, toroidal term is that of $|\eta|^2$, which is negative when η is timelike.

In the following subsections, we expand on the interpretation and properties of the quantities introduced here.

7.2.1 *Magnetic flux function* ψ

It was noted above that ψ labels magnetic field lines, but it is also directly related to the flux as follows. Fix a poloidal point (r, θ) and time t, denote by $\mathcal{C}$ the loop obtained by flowing along ∂_φ, and let $\mathcal{S}$ be any topological disc bounded by $\mathcal{C}$. The integral of F over $\mathcal{S}$ is the magnetic flux through $\mathcal{C}$. (Integration of differential forms is reviewed in Appendix A1.) Writing F as an exact differential $F = d(\psi\, d\phi_2)$ and using Stokes' theorem, we find $\int_{\mathcal{S}} F = \psi \int_{\mathcal{C}} d\phi_2 = 2\pi\psi$, so

$$\psi(r, \theta) = \frac{1}{2\pi}\int_{\mathcal{S}(r,\theta)} F. \tag{67}$$

That is, $2\pi\psi(r, \theta)$ is the magnetic flux through the loop of revolution defined by the poloidal point (r, θ). This is why ψ is often called the magnetic flux function. Another common name is the stream function. We will use both of these names, depending on the context.

In deriving equation (67), we have chosen the orientation $d\varphi$ on the loop $\mathcal{C}$, which by Stokes' theorem fixes the orientation for the 2-surface $\mathcal{S}$ with respect to which the flux is defined. We will call this the flux in the 'upward' direction. To understand the name, consider flat spacetime in cylindrical coordinates (t, z, ρ, φ), and let $\mathcal{S}$ be a disc of constant z. Then the orientation $d\varphi$ on the boundary corresponds to the orientation $d\rho \wedge d\varphi$ for the disc. Given the spacetime orientation $dt \wedge dz \wedge d\rho \wedge d\varphi$, this corresponds to the flux of the magnetic field pseudo-vector using the surface-normal $+\partial_z$.

The potential ψ is also related to the electrostatic potential as follows. A particle of mass m and charge e in stationary gravitational and electromagnetic fields has a conserved energy $\xi \cdot (mU + eA)$, where ξ is the stationary Killing vector, U is the particle four-velocity, and A is a vector potential that is invariant under the symmetry, $\mathcal{L}_\xi A = 0$. Then it is natural to define $\xi \cdot A$ as the 'electrostatic potential'. Although not gauge invariant, under a gauge transformation $A \to A' = A + d\lambda$ this changes by $\xi \cdot d\lambda$, which must be a constant if $\mathcal{L}_\xi A'$ is to vanish. Hence, the electrostatic potential *difference* between two points is gauge invariant. For the degenerate fields discussed here, we may use $A = \psi\, d\phi_2$, so that the electrostatic potential is $-\Omega_{\mathrm{F}}(\psi)\psi$. This determines the 'potential drop' between magnetic field lines.

7.2.2 *Polar current* I

The integral of the charge-current 3-form $J = d * F$ over the 3-surface $\mathcal{S} \times \Delta t$, formed by flowing $\mathcal{S}$ along ∂_t for a coordinate distance Δt, is (by Stokes' theorem) equal to the integral of $*F$ over the boundary. The contributions from the initial and final copies of $\mathcal{S}$ cancel out by stationarity, leaving $\int J = \int_{\mathcal{C}\times\Delta t} *F$. Since this surface extends only in t and φ, only the first term of equation (63) contributes, and we have simply

$$I(r, \theta) = \frac{1}{\Delta t}\int_{\mathcal{S}(r,\theta)\times\Delta t} J, \tag{68}$$

assuming the orientation $dt \wedge d\varphi$ on $\mathcal{C} \times \Delta t$, which by Stokes' theorem fixes the 'upward' orientation on $\mathcal{S} \times \Delta t$.[18] Thus $I(r, \theta)$ is equal to the electric current, with respect to Killing time, flowing in the upward direction through the loop of revolution defined by the poloidal point (r, θ). We will call I the *polar current*. Another common name is the poloidal current; however, we reserve that name for the current density flowing in the poloidal subspace, as distinguished from the net current through a loop. Besides its interpretation as a current, recall (cf. discussion below equation 65) that $I/2\pi$ is equal to B_{T}, the toroidal magnetic field times $\sqrt{-g_{\mathrm{T}}}$, which controls the bending of field lines in the φ-direction. (This relation between B_{T} and I is an instance of Ampère's law.) In addition to these roles, I gives the angular momentum flux per unit ψ, equation (80) below.

7.2.3 *Angular velocity of field lines* $\Omega_{\mathrm{F}}(\psi)$

The stationary axisymmetry implies that the field F is unchanged by a shift in φ and/or t; however, the potential ϕ_2 is in general unchanged only by a combined, helical shift $(\Delta t, \Delta\varphi = \Omega_{\mathrm{F}}(\psi)\Delta t)$. Under such a helical shift, the two Euler potentials are both unchanged, so a field sheet maps into itself. We may therefore interpret $\Omega_{\mathrm{F}}(\psi)$ as the angular velocity of the field line, the latter being defined by the intersection of the field sheet with a surface of constant t.

7.2.4 *Corotation 1-form* η

It is already apparent that the 1-form

$$\eta = d\varphi - \Omega_{\mathrm{F}}(\psi)dt \tag{69}$$

plays an important role in characterizing stationary, axisymmetric magnetospheres. In light of $(\partial_t + \Omega\partial_\varphi) \cdot \eta = \Omega - \Omega_{\mathrm{F}}$, η measures the extent to which a trajectory corotates with field lines. We refer to η as the *corotation 1-form*. Defining the *corotation vector* $\chi_{\mathrm{F}} = \partial_t + \Omega_{\mathrm{F}}\partial_\varphi$, we have $\chi_{\mathrm{F}} \cdot \eta = 0$, so η and χ_{F} are orthogonal as vectors (using the inverse metric to convert η to a vector). Both vectors lie in the two-dimensional, timelike toroidal subspace so,

[18] In cylindrical coordinates in flat spacetime for a disc of constant z, this corresponds to the orientation $dt \wedge d\varphi \wedge d\rho$ on $\mathcal{S} \times \Delta t$, which, given the spacetime orientation $dt \wedge d\varphi \wedge d\rho \wedge dz$, corresponds to flux along the $+\partial_z$-direction.

being orthogonal, they evidently have opposite timelike/spacelike causal character. Explicitly,

$$|\chi_F|^2 = g^T|\eta|^2, \tag{70}$$

where the determinant g^T of the toroidal metric is negative (since that metric is Lorentzian). Observers corotating with the magnetic field therefore exist only where χ_F is timelike and η is *spacelike*.

7.2.5 Light surfaces

At a point where η and χ_F are null, the corotating observer would need to travel at the speed of light. For this reason, a surface composed of such points is generally called a *light surface*, other names being critical surface, singular surface, velocity-of-light surface, or light cylinder.[19] The latter name stems from the fact that in flat spacetime, with $\Omega_F = \text{const}$, there is one light surface located where the cylindrical radius is equal to $1/\Omega_F$.

Light surfaces in magnetospheres play a significant role for two reasons. One is that the equation satisfied by the magnetic flux function (the so-called stream equation, cf. Section 7.4) has a critical point at a light surface. The implications of this for solutions of the equation are described briefly in Section 7.4.2.

The other role of light surfaces is that they determine causal boundaries of propagation of charged particle winds and Alfvén waves. As explained in Section 3.2.3, the field sheet metric governs such transport. Where the corotation vector χ_F is null, it coincides with one of the two field sheet light rays delineating the light cone on the sheet. Since χ_F is strictly toroidal, the light surface is evidently a causal boundary (at least locally) for either ingoing or outgoing motion on the sheet. In the case of the Michel monopole solution (41), for example, outside the light cylinder particles can propagate only to larger radii. For field sheet modes, the light cylinder is thus a horizon, beyond which influences cannot affect the interior.

In a general stationary, axisymmetric magnetosphere, the allowed direction of particle flow across the light surface, i.e. the direction of the other future pointing light ray on the field sheet, is the same as the direction of positive angular momentum flow in the field if Ω_F is greater than Ω_Z, the angular velocity of the local zero angular momentum observer (ZAMO). If instead $\Omega_F < \Omega_Z$, these directions are opposite. This is demonstrated in Section 7.3.1.

7.2.6 Field sheet Killing vector

As noted in Section 7.2.3, the Euler potentials are unchanged under a combined time-translation and rotation $(\Delta t, \Delta\varphi = \Omega_F(\psi)\Delta t)$. The field sheets and the field strength are preserved under this transformation, which is generated by the flow of the corotation vector field

$$\chi_F = \partial_t + \Omega_F(\psi)\partial_\varphi, \tag{71}$$

introduced in Section 7.2.4.[20] This is not only a symmetry of the electromagnetic field; it is also a symmetry of the intrinsic geometry of the field sheets. That is, although χ_F is not a spacetime Killing vector if $\Omega_F(\psi)$ is not constant, it is always a Killing vector of the induced metric on the field sheets, because ψ is constant on a field sheet. We therefore refer to χ_F also as the *field sheet Killing vector*.

As explained in Section 3.2.3, the field sheet metric governs the propagation of collisionless charged particles and Alfén waves in a certain approximation. The field sheet Killing vector thus provides conservation laws for these sorts of transport. In particular, there is a conserved quantity $\chi \cdot p = p_t + \Omega_F p_\varphi$ associated with each particle or wavepacket trajectory, where p is the four-momentum or wave four-vector, respectively. Since the field sheet metric is two-dimensional, the single conserved quantity is enough to completely determine the motion from a choice of initial position and velocity. In applications, such initial conditions may be provided e.g. by particle injection velocities at non-degenerate gaps in an otherwise force-free magnetosphere. The four-velocity u of a particle at any point is then determined by the equations $u^2 = -1$, $u \cdot F = 0$, and $u_a\chi_F^a = \text{const}$.

In flat spacetime, the conserved quantity for particles moving along field lines is $u_a\chi_F^a = \gamma(-1 + \Omega_F\rho\, v_\varphi)$, where γ is the Lorentz factor of the trajectory (in the rest frame defined by ∂_t), ρ is the cylindrical radius, and $v_\varphi = \rho\, d\varphi/dt$ is the azimuthal three-velocity. This quantity is sometimes used to determine outflow velocities from force-free solutions (e.g. Contopoulos, Kazanas & Fendt 1999, equation 16). We have obtained the conserved quantity as a simple consequence of the existence of a Killing vector on the field sheets, a formulation that generalizes to arbitrary circular spacetimes.

We note that the intersection of a light surface with a given field sheet is a *Killing horizon* for the field sheet Killing vector. That is, it is a null curve to which the Killing vector is tangent. As mentioned in Section 5.1.1, in the case of the Michel monopole, the field sheets are isometric to two-dimensional de Sitter space, and the light cylinder horizon is a de Sitter horizon.

7.3 Energy and angular momentum currents

A physical system governed by a Lagrangian on a spacetime with a symmetry generated by a Killing field ξ^a has an associated conserved Noether current, $\mathcal{J}_\xi$. In Appendix E, we review how this comes about using the language of differential forms. The electromagnetic field contribution to the Noether current 3-form (neglecting couplings) is given by

$$\mathcal{J}_\xi = -(\xi \cdot F) \wedge *F + \tfrac{1}{4}F^2\xi \cdot \epsilon. \tag{72}$$

This is the dual of $-T^a{}_b\xi^b$, where T^{ab} is the Maxwell stress–energy tensor (1). The current is conserved if and only if $F_{ab}J^b\xi^a = 0$, i.e. when the component of four-force in the ξ-direction vanishes. As explained in Appendix E, the second term of equation (72) is conserved automatically when the electromagnetic field also shares the symmetry, $\mathcal{L}_\xi F = 0$.

In terms of the Euler potentials for a degenerate field, we have

$$\xi \cdot F = \xi \cdot (d\phi_1 \wedge d\phi_2) = (\xi \cdot d\phi_1)d\phi_2 - (\xi \cdot d\phi_2)d\phi_1. \tag{73}$$

The first term on the right vanishes for stationary, axisymmetric fields characterized by the Uchida potentials (equation 61), while the second term is simply $-d\psi$ for the angular Killing field ∂_φ and $+\Omega_F\, d\psi$ for the time-translation Killing field ∂_t. Thus the angular momentum and energy currents are given by

$$\mathcal{J}_L = -d\psi \wedge *F - \frac{1}{4}F^2\,\partial_\varphi \cdot \epsilon, \tag{74}$$

$$\mathcal{J}_E = -\Omega_F(\psi)\,d\psi \wedge *F + \tfrac{1}{4}F^2\,\partial_t \cdot \epsilon. \tag{75}$$

[19] We caution the reader that some authors reserve the term 'light surface' for a place where F^2 vanishes. These two notions of light surface agree only when $I = 0$ (see equation 66).

[20] The existence of this symmetry of the Euler potentials is an example of a general property, discussed in Appendix D, which holds for degenerate fields with two commuting symmetry vectors X and Y, provided the (constant) quantity $X \cdot Y \cdot F$ is non-vanishing.

The angular momentum current is *minus* the Noether current (72).[21] The conserved quantity associated with the asymptotic time-translation symmetry is sometimes called *Killing energy*, or *energy at infinity*, to distinguish it from energy as defined by local observers. We will often simply call it 'energy', when the meaning is clear from the context.

When the electromagnetic field is coupled to charges, the energy and angular momentum currents (74) and (75) are not conserved, unless the four-force vanishes along ∂_φ and ∂_t, respectively. Since the second terms in equations (74) and (75) are automatically conserved for stationary axisymmetric fields (see note below equation 72), conservation of energy and angular momentum amounts to the condition $d\psi \wedge d * F = 0$. (In particular, if a stationary, axisymmetric, degenerate field conserves one of these, it also conserves the other.) This is equivalent to the first of the two force-free equations (26) or, equivalently, conservation of the first Euler current (27).[22]

Suppose that energy and angular momentum are conserved, either because the field is fully force free or because the dissipation vanishes in symmetry directions, $F_{ab}J^b\xi^b = 0$. We have shown above that this is equivalent to $d\psi \wedge d * F = 0$. Using equation (63) for $*F$, we have

$$\begin{aligned} 0 &= d\psi \wedge d * F \\ &= \frac{1}{2\pi} d\psi \wedge dI \wedge d\varphi \wedge dt - d\psi \wedge d(\star d\psi \wedge \star\eta). \end{aligned} \tag{76}$$

The second term vanishes because when factored it contains three poloidal 1-forms, while the poloidal space is only two-dimensional. It follows that $d\psi \wedge dI = 0$, which implies that

$$I = I(\psi). \tag{77}$$

Thus for stationary, axisymmetric, degenerate energy- and angular-momentum-conserving fields, the polar current I, like the angular velocity of field lines Ω_{F}, is a function of the stream function alone. The physical interpretation is that the poloidal current flows along poloidal magnetic field lines, so that the Lorentz force along ∂_φ vanishes.[23]

The angular momentum and energy currents (74) and (75) both contain a $d\psi$ factor and therefore vanish when integrated on a surface of constant ψ. This means that there is no flux of angular momentum or energy across such a surface; put differently, these quantities flow along the poloidal field lines, as well as in toroidal directions. (The vectorial characterization of this property is that poloidal part of the vector current $(*\mathcal{J})^a$ is tangent to poloidal field lines.) To evaluate the flux, let $\mathcal{P}$ be a poloidal curve, and consider the 3-surface $\mathcal{S} = \mathcal{P} \times S^1 \times \Delta t$ generated by rotating $\mathcal{P}$ all the way around the axis, and extended in Killing time by an amount Δt. The total flux of angular momentum across $\mathcal{S}$ is the integral of $\mathcal{J}_L$ over that surface. The $\partial_\varphi \cdot \epsilon$ term does not contribute, since its pullback to a surface including the ∂_φ-direction vanishes. The $\partial_t \cdot \epsilon$ term similarly vanishes for the total energy flux, since the surface also includes the ∂_t-direction. The total fluxes are therefore given by

$$\int_{\mathcal{S}} \mathcal{J}_L = -\int_{\mathcal{S}} d\psi \wedge *F, \tag{78}$$

$$\int_{\mathcal{S}} \mathcal{J}_E = -\int_{\mathcal{S}} \Omega_F \, d\psi \wedge *F. \tag{79}$$

Since the surface $\mathcal{S}$ extends in both φ and t, the integral vanishes unless the integrand contains a toroidal 2-form. Since $d\psi$ is poloidal, only the pure toroidal part of $*F$ (equation 63), i.e. $(I/2\pi) dt \wedge d\varphi$, contributes, and so the flux rates through $\mathcal{P} \times S^1$ are given by[24]

$$d\mathcal{L}/dt = -\int_{\mathcal{P}} I \, d\psi, \tag{80}$$

$$d\mathcal{E}/dt = -\int_{\mathcal{P}} \Omega_{\mathrm{F}} I \, d\psi. \tag{81}$$

If the poloidal curve $\mathcal{P}$ is a line of constant ψ, i.e. a poloidal field line, these integrals obviously vanish, illustrating the point made above that these currents 'flow along the poloidal field lines'. When energy and angular momentum are conserved (such as when the fields are force free), then we have $I = I(\psi)$ (equation 77) and equations (80) and (81) become ordinary one-dimensional integrals over a coordinate ψ, with limits corresponding to the value of ψ at the start and end of the curve $\mathcal{P}$.

7.3.1 Direction of particle flow at a light surface

We now establish the result mentioned in Section 7.2.5 that the direction of particle flow across a light surface is the same or opposite to the direction of positive angular momentum flow, according to whether $\Omega_{\mathrm{F}} - \Omega_{\mathrm{Z}}$ is positive or negative. Here Ω_{Z} is the ZAMO angular velocity discussed beginning with equation (87) below.

In the physical setting discussed in Section 3.2.3, charged particle motion is effectively tangent to the field sheets. The four-velocity u of such a particle thus satisfies

$$0 = u \cdot F = \frac{I}{2\pi(-g^{\mathrm{T}})^{1/2}} \, u \cdot \epsilon^{\mathrm{P}} - (u \cdot \eta) \, d\psi + (u \cdot d\psi) \, \eta \tag{82}$$

using equation (65). The last term is toroidal, and vanishes identically since ψ is constant on the field sheet. The poloidal angular momentum current 3-form (i.e. the part of $\mathcal{J}_L$ containing ϵ^{T} as a factor) is given by

$$\begin{aligned} \mathcal{J}_L^{\mathrm{P}} &= [-d\psi \wedge *F]^{\mathrm{P}} = -\frac{I}{2\pi(-g^{\mathrm{T}})^{1/2}} d\psi \wedge \epsilon^{\mathrm{T}} \qquad (83) \\ &= \frac{-1}{u \cdot \eta} \frac{I^2}{4\pi^2(-g^{\mathrm{T}})} (u \cdot \epsilon^{\mathrm{P}}) \wedge \epsilon^{\mathrm{T}}, \qquad (84) \end{aligned}$$

using equation (63) in the second step and equation (82) in the final step. The particle current is a positive number times $u \cdot \epsilon = (u \cdot \epsilon^{\mathrm{P}}) \wedge \epsilon^{\mathrm{T}} + \epsilon^{\mathrm{P}} \wedge u \cdot \epsilon^{\mathrm{T}}$, whose poloidal part is the first term, $(u \cdot \epsilon^{\mathrm{P}}) \wedge \epsilon^{\mathrm{T}}$. The relative sign of the poloidal angular momentum current and particle current in the direction u is thus equal to the

[21] The total four-momentum vector P^a is defined by $\int \mathcal{J}_\xi = -\eta_{ab} P^a \xi^b_\infty$, so that the time and space components of P^a, which define the energy and translational momentum, have opposite signs in relation to the corresponding 'Hamiltonian' $\int \mathcal{J}_\xi$.

[22] This property also holds for configurations with a single symmetry: for a degenerate EM field that is Lie derived by a Killing field ξ of the background spacetime, conservation of the current conjugate to ξ is equivalent to the force-free condition involving the potential that is invariant under the symmetry. This follows from equation (106) and the analysis of Appendix D1.

[23] In the electrically dominated case, we instead have that poloidal current flows along poloidal equipotentials, i.e. perpendicular to electric field lines.

[24] With the outward orientation for $\mathcal{P} \times S^1$, equations (80) and (81) give the *outward* flux of angular momentum and energy, respectively, so they give *minus* the angular momentum and energy change, respectively, of the system inside the surface.

sign of $-u \cdot \eta$. Since η is null at the light surface, the sign of its contraction with all future pointing vectors is the same. In particular, the ZAMO four-velocity $u_Z = \partial_t + \Omega_Z \partial_\varphi$ is a future pointing timelike vector everywhere,[25] so $\mathrm{sgn}(u \cdot \eta) = \mathrm{sgn}(u_Z \cdot \eta) = \mathrm{sgn}(\Omega_Z - \Omega_F)$. We conclude that particle flow and positive angular momentum flow have the same direction if $\Omega_F > \Omega_Z$, and opposite direction if $\Omega_F < \Omega_Z$. In flat or Schwarzschild spacetime, $\Omega_Z = 0$, so all that matters is the sign of the angular velocity of the field line, and the direction of particle flow agrees with the direction of energy flow.

7.4 Stream equation

Up to now our discussion of stationary, axisymmetric fields has assumed degeneracy of the field, but has not assumed that it is force free. For force-free fields, the stream function ψ satisfies a nonlinear partial differential equation which is known by many names: stream equation, Grad–Shafranov equation, transfield equation, and, in flat spacetime, pulsar equation (Michel 1973; Scharlemann & Wagoner 1973; Okamoto 1974; BZ). We will call this equation the *stream equation*, and we now derive it in the case of a general stationary, axisymmetric metric of the block diagonal form (55). A similar equation can be derived in the presence of other sorts of symmetries.

The stream equation follows directly from the two force-free equations (26). As already demonstrated above equation (77), the first force-free condition implies that $I = I(\psi)$, which is equivalent to conservation of energy and angular momentum. The second force-free condition yields

$$\begin{aligned} 0 &= d\phi_2 \wedge d * F \\ &= (d\psi_2 + \eta - \Omega_F' t\, d\psi) \wedge \left(\frac{I'}{2\pi} d\psi \wedge dt \wedge d\varphi - d(\star d\psi \wedge \star\eta)\right) \\ &= \frac{I'}{2\pi} d\psi_2 \wedge d\psi \wedge dt \wedge d\varphi - \eta \wedge d(\star d\psi \wedge \star\eta) \\ &= \frac{II'}{4\pi^2 g^{\mathrm{T}}} \epsilon + d(\eta \wedge \star d\psi \wedge \star\eta) - d\eta \wedge \star d\psi \wedge \star\eta \\ &= \frac{II'}{4\pi^2 g^{\mathrm{T}}} \epsilon - d(|\eta|^2 \star d\psi \wedge \epsilon^{\mathrm{T}}) + \Omega_F' d\psi \wedge dt \wedge \star d\psi \wedge \star\eta \\ &= \frac{II'}{4\pi^2 g^{\mathrm{T}}} \epsilon - d(|\eta|^2 * d\psi) - \Omega_F' |d\psi|^2 \langle dt, \eta\rangle\, \epsilon. \end{aligned} \tag{85}$$

In the second line, prime denotes a ψ derivative, and we use equations (61), (69), (63), and (77). Of the six cross terms, only two survive in the third line; two vanish because they contain three poloidal 1-forms, one vanishes because it contains three toroidal 1-forms, and one vanishes because it contains the same 1-form twice. In the fourth line, in the first term, we use the dual of equation (64), together with equations (57) and (58) (alternatively, equation A9). The other two terms arise from 'integration by parts' of the second term in the previous line, using the antiderivation property of d. To obtain the fifth line, we use equation (A9) in the second term, and the definition of η (equation 69) in the third term. In the last line, we use equation (60) in the second term, and equations (A9) and (58) in the third term.

[25] u_Z is future pointing timelike at infinity, is timelike everywhere (since it is orthogonal to the spacelike vector ∂_φ and lies in the toroidal plane), and is nowhere zero. Hence, it is future timelike everywhere.

Finally, since $d * \omega = \nabla_a \omega^a \epsilon$ for any 1-form ω, the last line of equation (85) yields

$$\nabla_a(|\eta|^2 \nabla^a \psi) + \Omega_F' \langle dt, \eta\rangle |d\psi|^2 - \frac{II'}{4\pi^2 g^{\mathrm{T}}} = 0, \tag{86}$$

where ∇_a is the covariant derivative determined by the spacetime metric. This is the stream equation, in a form that holds for any metric of the form (55). If Ω_F and I are specified as given functions of ψ, then equation (86) becomes a quasi-linear elliptic equation for ψ, with critical points where the 1-form η is null, i.e. at light surfaces (see Section 7.2).

The stream equation (86) involves the quantities $|\eta|^2$ and $\langle dt, \eta\rangle$, which depend on Ω_F and the toroidal metric. Without loss of generality, we may write this metric in the common form

$$(ds^{\mathrm{T}})^2 = -\alpha^2 dt^2 + \rho^2 (d\varphi - \Omega_Z dt)^2, \tag{87}$$

where α, ρ, and Ω_Z are functions of the poloidal coordinates (r, θ). The quantity Ω_Z is the angular velocity of ZAMOs, who follow the (non-gedoesic) toroidal curves orthogonal to the angular Killing field ∂_φ, while α is the rate of ZAMO time with respect to t, sometimes called the redshift factor (MacDonald & Thorne 1982). In terms of these quantities, those appearing in the stream equation are given by

$$|\eta|^2 = \rho^{-2} - \alpha^{-2}(\Omega_F - \Omega_Z)^2 \tag{88}$$

$$\langle dt, \eta\rangle = \alpha^{-2}(\Omega_F - \Omega_Z) \tag{89}$$

$$-g^{\mathrm{T}} = \alpha^2 \rho^2. \tag{90}$$

In particular, the light surfaces are located where $\rho = \pm\alpha/(\Omega_F - \Omega_Z)$, and $\langle dt, \eta\rangle$ vanishes where $\Omega_Z = \Omega_F$.

For comparison with other treatments, note that the four-dimensional determinant g can also be expressed as $-\alpha\rho g^{\mathrm{P}}$ or as $g^{\mathrm{T}} g^{\mathrm{P}}$. We may thus write the first term in equation (86) using the covariant derivative $\boldsymbol{D}_a$ on the three-dimensional surfaces of constant t or the two-dimensional poloidal covariant derivative D_a, giving

$$\nabla_a(|\eta|^2 \nabla^a \psi) = \alpha^{-1/2} \boldsymbol{D}_a[\alpha^{1/2} |\eta|^2 \boldsymbol{D}^a \psi] \tag{91}$$

$$= (-g^{\mathrm{T}})^{-1/2} D_a[(-g^{\mathrm{T}})^{1/2} |\eta|^2 D^a \psi]. \tag{92}$$

The RHS of equation (91) is the standard 3 + 1 form (MacDonald & Thorne 1982), while equation (92) gives a 2 + 2 form.

It is worth mentioning that the stream equation can apply more generally than in the stationary axisymmetric case. In particular, for any 2 + 2 metric, if the field is symmetric under one of the factors of the 2 + 2, and falls into Uchida's case 1 (Appendix D2), then the same manipulations above will give rise to a stream equation that differs only in minor details. For example, a stream equation applies to the case where the field is plane symmetric, i.e. x and y are the ignorable coordinates, while the fields depend on z and t, in flat spacetime.

7.4.1 *Action derivation of stream equation*

The stream equation can also be efficiently derived directly from the action (28), with the symmetric form (61) for the potentials. Uchida (1997a) worked this out and explained the relation to the Scharlemann–Wagoner action (Scharlemann & Wagoner 1973) from which the derivation is even simpler. Here we will briefly summarize Uchida's analysis using our methods.

The action (28) takes the form

$$S^{\text{sym}} = -\tfrac{1}{2}\int (d\psi \wedge d\psi_2) \wedge *(d\psi \wedge d\psi_2) + |\eta|^2 |d\psi|^2 \epsilon. \qquad (93)$$

The quantities to be varied are ψ and ψ_2, while $\Omega_F(\psi)$ in η is treated as a fixed function. The variation of ψ_2 yields the equation $d\psi \wedge d*(d\psi \wedge d\psi_2) = 0$, which using equation (63) implies that $d\psi \wedge dI = 0$, and hence $I = I(\psi)$. (This is basically the same as the derivation of equation 77.) The variation of ψ in the second term of the action yields minus the first two terms of the stream equation (86) times ϵ, while variation in the first term yields

$$\begin{aligned} d\psi_2 \wedge d*(d\psi \wedge d\psi_2) &= \frac{1}{2\pi} d\psi_2 \wedge dI \wedge dt \wedge d\varphi \\ &= \frac{II'}{4\pi^2 g^{\text{T}}}\epsilon, \end{aligned} \qquad (94)$$

where in the last step we used the conclusion $I = I(\psi)$ from the ψ_2 variation, together with

$$d\psi_2 \wedge dI = I' d\psi_2 \wedge d\psi = -\frac{II'}{2\pi(-g^{\text{T}})^{1/2}}\epsilon^{\text{P}}. \qquad (95)$$

Hence, we recover the stream equation (86).

It is tempting, after having found that $I = I(\psi)$, to substitute $d\psi \wedge d\psi_2 = (I/2\pi\sqrt{-g^{\text{T}}})\epsilon^{\text{P}}$ back into the action, eliminating ψ_2 and yielding $(-I^2/4\pi^2 g^{\text{T}})\epsilon$ for the first term in the integrand, and then treating I as a fixed function. This is not correct: it would be like solving for a velocity component $\dot{q}(p, q^i, \dot{q}^i)$ in mechanics in terms of a conserved conjugate momentum p and the other coordinates and velocities, and substituting that back into the action. The resulting action would yield invalid equations of motion, because in the original action the conserved quantity was not held fixed. However, if at the same time one modifies the Lagrangian by addition of $-p\dot{q}(p, q^i, \dot{q}^i)$, the procedure is then correct. (This amounts to using the Hamiltonian formalism for q, and the Lagrangian formalism for the remaining coordinates.) Following an analogous procedure to trade the ψ_2 dependence of the action in favour of $I(\psi)$, Uchida shows that the net result is simply to flip the sign of the I^2 term, yielding the action

$$S^{\text{SW}} = -\tfrac{1}{2}\int \left(\frac{I^2}{4\pi^2 g^{\text{T}}} + |\eta|^2 |d\psi|^2\right)\epsilon. \qquad (96)$$

This is the Scharlemann–Wagoner action (Scharlemann & Wagoner 1973), from which the stream equation (86) follows immediately as the ψ stationarity condition when treating $I(\psi)$ as a fixed function.

7.4.2 *Solution of the stream equation*

The stream equation (86) for the stream function ψ has the peculiar feature that it contains unknown functions $\Omega_F(\psi)$ and $I(\psi)$ which must also be somehow determined. In this subsection, we briefly discuss the nature of this equation and mention several approaches to finding solutions.

If $\Omega_F(\psi)$ and $I(\psi)$ are specified, then the stream equation (86) becomes a quasi-linear equation for ψ. Where $|\eta| \neq 0$ (i.e. away from any light surfaces), the equation is second order, with elliptic principal part. Thus on a domain not containing light surfaces, one expects unique solutions given suitable boundary data for ψ.[26] At a light surface (where $|\eta| = 0$), the stream equation becomes first order,[27]

$$\nabla_a(|\eta|^2)\nabla^a\psi + \Omega_F'\langle dt, \eta\rangle |d\psi|^2 + \frac{II'}{4\pi^2 g^{\text{T}}} = 0. \qquad (97)$$

When $I(\psi)$ and $\Omega_F(\psi)$ are specified, this may be viewed as a Robin-type boundary condition for ψ at a new boundary, the light surface. If a single light surface cuts a domain in two, one expects a unique solution on either side, but the solutions will generally not match smoothly at the light surface. It is thus plausible that the requirement of smooth matching restricts the choice of $I(\psi)$ and $\Omega_F(\psi)$ to a single free function on field lines, at least on field lines (values of ψ) that cross the light surface. If a second light surface is crossed by the same field line, one expects both $I(\psi)$ and $\Omega_F(\psi)$ to be determined.

These expectations are borne out in numerical calculations that iteratively update guesses for the free functions until a sufficiently smooth match is achieved across all light surfaces. This approach to solving the stream equation in the presence of light surfaces was introduced by Contopoulos et al. (1999) and later used by several other authors (Uzdensky 2005; Gruzinov 2006; Timokhin 2006; Contopoulos, Kazanas & Papadopoulos 2013; Nathanail & Contopoulos 2014). For a pulsar magnetosphere, $\Omega_F(\psi)$ may be fixed in advance to be the (constant) angular velocity of the star (cf. Section 8.1), and the single free function $I(\psi)$ may be determined by matching across the single light surface. Black hole magnetospheres are qualitatively different in three respects: (i) the location of the light surfaces generally depends on $\Omega_F(\psi)$ and ψ, (ii) if the black hole is spinning, there can be two light surfaces (cf. Section 9.3), and (iii) at the horizon there is a fixed relation between ψ, I, and Ω_F, the Znajek condition (cf. Section 9.1). The Znajek condition can be viewed as determining ψ on the horizon, given $I(\psi)$ and $\Omega_F(\psi)$. On field lines that cross both light surfaces, the latter two functions would also be determined.

In order to find analytic solutions to the stream equation, one approach is to restrict the dependence of ψ to a one-dimensional subspace of the two-dimensional poloidal space, converting the stream equation into an ordinary differential equation (ODE). Then, for example if Ω_F is a fixed constant, the boundary condition on ψ can determine $I(\psi)$ locally, leaving just an ODE to be solved. This kind of tactic was used for example by Menon & Dermer (2007), who found a family of solutions in the Kerr spacetime where ψ, Ω_F, and I are independent of BL radial coordinate r.

Finally, stationary, axisymmetric force-free solutions can be generated by time-dependent evolution from non-force-free initial data, using numerical devices that short out electric fields and dissipate energy. Thus one effectively solves the stream equation through time-dependent evolution (e.g. Komissarov 2001; McKinney 2006; Spitkovsky 2006; Komissarov & McKinney 2007).

7.5 Field line topology

In this subsection, we establish two restrictions on the possible topology of magnetic field lines in stationary, axisymmetric force-free magnetospheres. In Sections 8 and 9, we apply the second of these results to pulsar and black holes magnetospheres.

[26] For Dirichlet data, choosing $I(\psi)$ and $\Omega_F(\psi)$ is equivalent to specifying I and Ω_F on the boundary. Thus the total boundary data are a component of the poloidal magnetic field (derived from ψ), a component of the poloidal electric field (obtained from Ω_F and ψ), and the toroidal magnetic field (proportional to I).

[27] If $|\eta|^2$ vanishes quadratically or faster, the equation would actually be zeroth order, i.e. algebraic. However, this situation does not arise in practice.

7.5.1 No closed loops

We begin with the simpler of the two restrictions:

A stationary, axisymmetric, force-free, magnetically dominated field configuration cannot possess a closed loop of poloidal field line.

By a closed loop of poloidal field line we mean a level set of ψ that forms a smooth closed curve, i.e. a closed set $\psi = \text{const}$ on which $d\psi \neq 0$. Note that such loops do not in general correspond to closed loops of 'true' field line, since those lines bend in the ∂_φ-direction. To establish the result, we employ the expression (26) of the force-free condition in terms of the conservation of the Euler currents. In particular, we use the fact that the Euler current $J_2 = d\phi_2 \wedge *F$ is a closed 3-form. By Stokes' theorem, this implies the vanishing of the integral of J_2 over any closed 3-surface bounding a force-free region of spacetime.

Suppose for contradiction that a closed loop of poloidal field line exists, i.e. that a smooth level curve $\mathcal{C}$ of ψ is closed. Flowing this loop along ∂_φ, one obtains a torus, and flowing that along ∂_t (by an amount Δt), one obtains a closed 3-surface, consisting of a timelike tube $\mathcal{S} = \mathcal{C} \times S^1 \times \Delta t$ and initial and final spacelike, solid torus caps $\mathcal{C} \times S^1$. Integrating J_2 on this surface, the contributions from the caps cancel. The timelike tube is a surface of constant ψ on which $d\psi \neq 0$, so we can express it using equation (A7), with v any vector field such that $v \cdot d\psi = 1$ on $\mathcal{S}$:

$$\begin{aligned} 0 &= \int_{\mathcal{S}} d\phi_2 \wedge *F \\ &= \int_{\mathcal{S}} v \cdot (d\psi \wedge d\phi_2 \wedge *F) \\ &= \int_{\mathcal{S}} v \cdot (F \wedge *F) \\ &= \frac{1}{2} \int_{\mathcal{S}} F^2\, v \cdot \epsilon \\ &= \pi \Delta t \oint_{\mathcal{C}} F^2 \sqrt{-g^{\mathrm{T}}} v \cdot \epsilon^{\mathrm{P}}. \end{aligned} \tag{98}$$

In the third line, we used $F = d\phi_1 \wedge d\phi_2$ (equation 22) and $\phi_1 = \psi$ (equation 61); in the fourth line, we used equation (A9); and in the last line, we used (58) and carried out the toroidal part of the integral. The contraction of the poloidal 1-form $v \cdot \epsilon^{\mathrm{P}}$ with a tangent vector to $\mathcal{C}$ vanishes only if v is also tangent to $\mathcal{C}$, which is excluded by $v \cdot d\psi = 1$. Thus if F is magnetically dominated ($F^2 > 0$) everywhere on $\mathcal{C}$, the integral (98) cannot vanish, so we have a contradiction. This establishes our 'no closed loops' result.

7.5.2 Light surface loop lemma

In this subsection, we prove a light surface loop lemma that will be useful when we treat pulsar and black hole magnetospheres. This lemma was part of the 'no-ingrown-hair' argument of MacDonald & Thorne (1982), but here we present it on its own and also discuss two related results. The lemma states that for stationary, axisymmetric, force-free fields (not necessarily magnetically dominated),

no poloidal field line may pierce a light surface twice in a contractible region where $\Omega_{\mathrm{F}}' = I' = 0$

The condition $\Omega_{\mathrm{F}}' = 0$ indicates that all field lines rotate with the same angular velocity, while $I' = 0$ implies that no poloidal current flows in this region (otherwise I, being the total current through the cap with boundary at ψ, would depend on ψ). Our hypotheses allow non-zero toroidal magnetic field I, supported by poloidal current flowing elsewhere in the magnetosphere.

The proof is based on the fact that when $\Omega_{\mathrm{F}}' = I' = 0$, conservation of the second Euler current is equivalent to conservation $|\eta|^2 * d\psi$,

$$0 = d(d\phi_2 \wedge *F) = d(|\eta|^2 * d\psi). \tag{99}$$

Equation (99) result follows from the derivation of the stream equation (85), and the vectorial version $\nabla_a(|\eta|^2 \nabla^a \psi) = 0$ can be seen directly from the final form of the stream equation (86). Suppose for contradiction that a smooth ($d\psi \neq 0$) poloidal field line intersects a light surface (where $|\eta| = 0$) twice, without intersecting another light surface, as depicted in Fig. 3(b). We may construct a closed 3-surface by considering the portions of the field line and light surface that form a closed poloidal loop, and flowing this loop along ∂_t (for time Δt) and along ∂_φ. Integrating $|\eta|^2 * d\psi$ over this 3-surface, the initial and final spacelike caps cancel by stationarity, while the timelike portion involving the light surface vanishes since $|\eta| = 0$ there. The remaining portion is that generated by the field line segment, which is a surface of constant ψ on which $d\psi \neq 0$. With the same notation as in the previous subsection, that integral is given by

$$\begin{aligned} 0 &= \int |\eta|^2 v \cdot (d\psi \wedge *d\psi) \\ &= \int |\eta|^2 |d\psi|^2 v \cdot \epsilon. \end{aligned} \tag{100}$$

By assumption, η is non-null everywhere on the field line segment, so cannot change sign. Since the poloidal subspace is Riemannian, we have $|d\psi|^2 > 0$. Thus $|\eta|^2 |d\psi|^2$ does not change sign, and the reasoning used below equation (98) implies that the integral (100) cannot vanish, a contradiction.

If more than one light surface is present, then, for any field line segment that pierces a single light surface twice (or more), there will also be a subsegment that pierces a (possibly different) light surface twice without encountering any other light surface. We may then run the above argument on that subsegment, again concluding that the field configuration is impossible. This establishes the lemma.

The conclusion of the light surface lemma also holds in two additional cases. First, since it is the product $\Omega_{\mathrm{F}}' \langle dt, \eta \rangle$ that appears in the stream equation (85) or (86), we may replace the assumption $\Omega_{\mathrm{F}}' = 0$ with $\langle dt, \eta \rangle = 0$. The interpretation of $\langle dt, \eta \rangle = 0$ is that field lines corotate with ZAMOs, cf. equation (89).

Secondly, we may drop the force-free assumption and instead assume that (i) angular momentum is conserved (first force-free condition is satisfied and hence $I = I(\psi)$) and (ii) there is a reflection

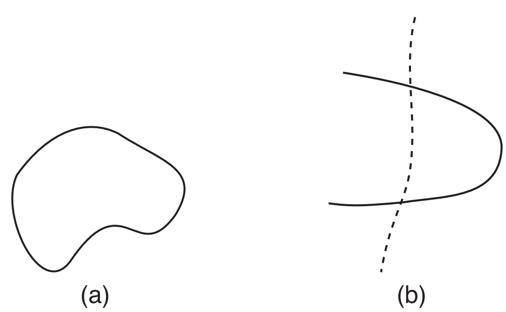

Figure 3. Disallowed topologies of force-free poloidal magnetic field lines. (a) No closed loops when magnetically dominated. (b) No light surface loops when $\Omega_{\mathrm{F}}' = I' = 0$.

isometry[28] (of the spacetime and the fields) about a spacelike 2-surface (poloidal curve flowed in toroidal directions) intersecting the light surface loop. The reflection isometry implies that I is odd under reflection (see equation 65, noting that F is even while $\epsilon_{\rm P}$ is odd), but $I = I(\psi)$ and the evenness of ψ implies that I is even. Thus we in fact have $I = 0$, and the assumption $I' = 0$ is superfluous. We must still include $\Omega'_{\rm F} = 0$ as an assumption, and in this case we still have the last equality in equation (99), $d\phi_2 \wedge d * F = d(|\eta|^2 * d\psi)$. These quantities are now *not* known to vanish, but we may use the reflection isometry to argue that their integrals do vanish. To do so note that $d\phi_2$ is even (first use the evenness of η and F to establish evenness of $d\psi$, and then $F = d\psi \wedge d\phi_2$ implies $d\phi_2$ is even), while $d * F$ is odd on account of the duality. Thus the form $d\phi_2 \wedge d * F$ is odd. Since the shape of the light surface loop is symmetric, the integral of this form over the interior of the loop (flowed in φ and t to form a four-volume) is vanishing. This establishes the first line of equation (100), and the same arguments establish a contradiction. To summarize, we have shown that for stationary, axisymmetric, reflection-symmetric, degenerate energy- and angular-momentum-conserving fields with $\Omega'_{\rm F} = 0$ (or $\langle dt, \eta\rangle = 0$), no light surface loops straddling the reflection surface may exist.

7.6 Special case: no poloidal field

As mentioned above, the form (61) of the Euler potentials of a stationary, axisymmetric solution is valid only when $F \cdot \partial_\varphi \neq 0$. The case $F \cdot \partial_\varphi = 0$ corresponds to a purely toroidal magnetic field, and may be useful in situations where the poloidal field is small (e.g. Contopoulos 1995). When $F \cdot \partial_\varphi = 0$ one may choose the form (D15),

$$\phi_1 = \chi(r, \theta), \quad \phi_2 = \chi_2(r, \theta) + t. \tag{101}$$

We may then write F as

$$F = \frac{I}{2\pi(-g^{\rm T})^{1/2}} \epsilon_{\rm P} + d\chi \wedge dt, \tag{102}$$

where I satisfies $*(d\chi \wedge d\chi_2) = (I/2\pi) dt \wedge d\varphi$, the analogue of equation (64). It is evident that χ is an electric potential for the (purely poloidal) electric field, while I is again equal to the electric current through a toroidal loop (Section 7.2.2). As in the generic case, the first force-free equation implies $I = I(\chi)$ (see discussion above equation 77). To derive the associated stream equation, we may follow the same steps of equations (85), finding

$$\nabla_a(|dt|^2 \nabla^a \chi) - \frac{II'}{4\pi^2 g^{\rm T}} = 0, \tag{103}$$

where $'$ is a χ derivative. Note that equations (101) and (103) can be obtained from the generic versions (61) and (86) (respectively) by the replacements $\psi \to \chi$, $\varphi \to t$, and $\Omega_{\rm F} \to 0$. However, since ψ determines the poloidal magnetic field (which vanishes in the special case), it is more physical to obtain them from the limit $\psi \to 0$ and $\Omega_{\rm F} \to \infty$ with the product $\Omega_{\rm F}\psi \to -\chi$ finite.

An example of this limit is provided by the Michel monopole solution $-qd(\cos\theta) \wedge (d\varphi - \Omega_{\rm F} du)$. In the limit $q \to 0$, $\Omega_{\rm F} \to \infty$, with $q\Omega_{\rm F}$ held fixed. The vacuum monopole term (which provides the poloidal magnetic field) vanishes, leaving just the stationary, axisymmetric outgoing Poynting flux solution (30), $F = q\Omega_{\rm F} d(\cos\theta) \wedge du$, which satisfies the stream equation (103) rather than the generic stationary axisymmetric stream equation. Euler potentials for this solution in the above notation are specified by $\chi = q\Omega_{\rm F}\cos\theta$ and $\chi_2 = -r$.

Another interesting example arises in the Menon–Dermer solution, i.e. the stationary axisymmetric case of the Poynting flux solution (34) in Kerr spacetime (35), $A(\theta) d\theta \wedge (du - a\sin^2\theta\, d\bar{\varphi})$. That solution falls in the generic class on account of the $d\bar{\varphi}$ term, but since that term vanishes along the axis, the axis limit lands on this special case. The angular velocity of the field lines in this solution is $\Omega_{\rm F} = 1/(a\sin^2\theta)$, whose limit indeed diverges as the axis is approached.

The case of no poloidal field does not appear to have been previously considered in the force-free context. However, Gourgoulhon et al. (2011) have given a completely general treatment of stationary, axisymmetric equilibria in the context of ideal MHD, which includes the magnetically dominated force-free case as a limit.

8 PULSAR MAGNETOSPHERE

This section addresses general features of magnetospheres around conducting, magnetized stars in the case of aligned rotation and magnetic axes. Such a configuration is stationary and axisymmetric, so does not pulse; however, it serves as a simple example of key properties of pulsar magnetospheres, and as an approximation for a nearly aligned pulsar. Specifically, we discuss the boundary condition at the stellar surface which determines the angular velocity $\Omega_{\rm F}(\psi)$ of the field, and the roles of the light cylinder and current sheet in delimiting the region of closed field lines.

The pulsar magnetosphere has mainly been studied in flat spacetime. We will discuss general features of pulsar magnetospheres based on the general metric (55), so our comments will hold when gravity is included. Our analysis also serves to identify precise circumstances under which each particular feature must hold.

8.1 Angular velocity of field lines

The angular velocity of field lines $\Omega_{\rm F}$ may be determined by the assumption of a perfectly conducting stellar surface, which should be a good approximation for neutron stars. If U is the four-velocity field of a perfectly conducting surface, then the contraction of $U \cdot F$ with any vector tangent to the surface vanishes. That is, the electric field in the rest frame of the conductor must have no component tangent to the surface. If the surface is that of an axisymmetric star with four-velocity $U \propto \partial_t + \Omega\partial_\varphi$, then for a stationary, axisymmetric degenerate field (62) we have $U \cdot F = -(U \cdot \eta) d\psi \propto (\Omega_{\rm F} - \Omega) d\psi$. Provided the poloidal magnetic field is not tangent to the stellar surface (i.e. provided there is a surface tangent vector v with $v \cdot d\psi \neq 0$), it follows that $\Omega = \Omega_{\rm F}$. We have thus shown that for stationary, axisymmetric, degenerate fields,

> *poloidal field lines that non-tangentially intersect a perfectly conducting star must have* $\Omega_{\rm F} = \Omega$

Thus the field lines corotate with the star. Note that when $\Omega = \Omega_{\rm F}$ we have $U \cdot F = 0$ (see expression in text above), implying that also the *normal* component of the electric field in the rotating frame must vanish at the surface of the conducting star. Thus there is no induced charge on the stellar surface, according to corotating observers. Static observers, on the other hand, will generically measure induced charge, depending on the assumptions for the field configuration within the star.

The lack of induced surface charge in the rotating frame is a direct consequence of degeneracy and the conducting boundary condition: since the tangential components must vanish on the conducting

[28] A definition of a reflection isometry is given in Section 8.2.

surface, the electric field is purely normal. But if the magnetic field has a normal component, $\boldsymbol{E} \cdot \boldsymbol{B} = 0$ implies that the electric field vanishes entirely, and there is no induced charge. If the star were instead surrounded by vacuum, the field would not be degenerate, and generically there would be a surface charge and a normal component of the electric field in the rotating frame.

8.2 Open and closed zones

Closed field lines are defined to be field lines that intersect the star twice, while open field lines intersect it once. In vacuum, the field lines of a monopole star would all be open, while those of a dipole are all closed. The standard aligned force-free pulsar magnetosphere (Fig. 4), on the other hand, is a mix: the field lines form a dipole pattern at the star, but only some of them return to close, with the rest opening up to infinity. This basic structure of closed and open zones was postulated in the earliest work on the subject (Goldreich & Julian 1969), and later work has confirmed that such solutions do exist.

An important feature of all configurations previously considered is that the closed field lines remain within the light cylinder,[29] unless they pass through a non-force free region such as a current sheet. While it is commonly asserted that closed force-free field lines *must* remain within the light cylinder, we are unaware of any explicit demonstration in the literature. In this section, we will critique the reasoning that one often hears or reads, and then demonstrate several related results based on various specific assumptions. We will conclude by explaining why, despite these results, the possibility that closed force-free field lines could venture outside the light cylinder has not (yet) been ruled out.

An argument one often hears or reads for the impossibility of closed magnetic field lines in a degenerate field outside the light cylinder is based on the notion that particles stuck on such field lines (in cyclotron motion) would have to be moving faster than the speed of light. It seems unsatisfactory to invoke 'particles' to determine something about force-free fields, however, since the matter plays no role in the dynamics of those fields (other than to carry the current). To the extent that such an argument is valid, it should be possible to reformulate it without reference to particles.

To identify such a reformulation, we note that (i) particles are stuck on field lines only if the field is magnetically dominated, and (ii) such field lines *always* admit subluminal particles stuck on them, because magnetically dominated, degenerate fields always have timelike field sheets. Hence, to rule out some behaviour of a degenerate field 'because particles stuck on field lines cannot go faster than light must be logically equivalent to ruling it out simply by the assumption that the field is magnetically dominated. In what follows, we will base our arguments on magnetic domination rather than considerations of particles. We will also give one argument that does not require magnetic domination.

A stationary, axisymmetric field cannot remain magnetically dominated outside the light cylinder if the field line has no toroidal component, because such a line sweeps out a spacelike field sheet outside the light cylinder. To reach this conclusion computationally, note that the absence of a toroidal magnetic field is equivalent to the condition $I = 0$ (cf. equation 65), which implies $F^2 = |d\psi|^2|\eta|^2$ (equation 66). The factor $|d\psi|^2$ is non-negative, since the poloidal subspace is Riemannian, and outside the light cylinder $|\eta|^2 < 0$ by definition, so if $I = 0$, then F^2 must be negative (or zero) outside the light cylinder, violating magnetic domination. This establishes a link between the absence of polar current and the confinement of magnetic field lines:

> *In a stationary, axisymmetric, degenerate, magnetically dominated field configuration, field lines with $I = 0$ must lie within the light cylinder.*

The idea that *closed* field lines must remain within the light cylinder may in part be due to an association between closed field lines and $I = 0$. The original Goldreich–Julian model postulated a corotating portion of the magnetosphere, where one sign of charge rotates rigidly with the star. This portion indeed has $I = 0$, since there is no poloidal current. In most self-consistent models constructed by solving the stream equation (e.g. Contopoulos et al. 1999), $I = 0$ is assumed in the closed zone. One possible reason for this assumption is the fact that

> *reflection symmetry and the force-free condition imply that closed field lines crossing the equatorial plane must have $I = 0$.*

By 'reflection symmetry' we mean an isometry of the spacetime and fields under which the volume element ϵ changes sign, and which leaves fixed a 3-surface composed of a spacelike 2-surface flowed along the timelike Killing field. In flat spacetime, this corresponds to the usual reflection symmetry about the plane $z = 0$, and in the general case, we refer to the spacelike 2-surface as the equatorial 'plane' although it would generally have intrinsic curvature. To establish the result, we note that $I(\psi)$ is constant on the field line, hence is even under reflection, but it is also required to be odd since F (equation 65) is even while the poloidal area element ϵ^{P} is odd. Since this result relies only on $I = I(\psi)$, it suffices to assume just that energy and angular momentum are conserved, rather than the full force-free condition (see discussion in Section 7.3 above).

Combining the previous two quoted statements, we have the following theorem on closed field lines:

> *In a stationary, axisymmetric, reflection-symmetric, force-free, magnetically dominated field configuration, field lines crossing the equatorial plane must have $I = 0$ and remain within the light cylinder.*

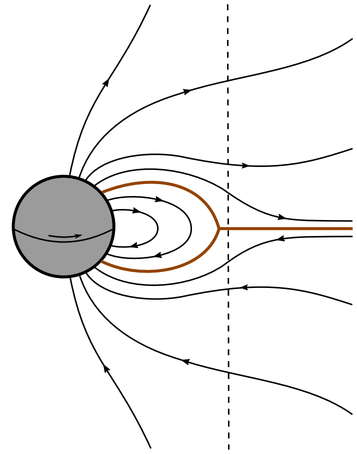

Figure 4. Diagram illustrating the poloidal structure of the standard aligned pulsar magnetosphere. A current sheet (thick brown line) separates poloidal field lines (black) into three zones, one closed and two open. The closed zone terminates at, or just within, the light cylinder (dashed line, shown artificially close to the star).

[29] To model pulsars, we focus on spacetimes for which degenerate field configurations will have a single light surface with the topology of a cylinder, outside of which corotating trajectories are spacelike. We refer to this surface as the light cylinder.

Note that reflection symmetry implies that field lines which come from the star and cross the equator are closed, so this result provides a precise set of assumptions under which the basic topology of the standard pulsar magnetosphere is to be expected. Again, the force-free assumption may be replaced by that of conservation of energy and angular momentum. The assumption of magnetic domination can be replaced by the condition that $\Omega_F' = 0$ which, as shown in Section 8.1, necessarily holds if the field terminates on a rigidly rotating, perfectly conducting star with non-tangential poloidal surface field. Then, together with the other assumptions, the light surface loop lemma of Section 7.5.2 applies and establishes that closed lines cannot extend outside the light cylinder. We thus have the following alternate result:

In a stationary, axisymmetric, reflection-symmetric, force-free, field configuration with $\Omega_F' = 0$, field lines crossing the equatorial plane must have $I = 0$ and remain within the light cylinder.

Given the above considerations about closed field lines, a natural (and so-far standard) configuration for the dipole pulsar magnetosphere is that shown in Fig. 4. Field lines near the equator of the star would, in vacuum, have returned more quickly to the star, so it stands to reason that the closed zone of the configuration will be near the equator. On the other hand, field lines near the poles would have extended far from the star in vacuum, so it makes sense that in the force-free case these field lines will open up to infinity. Outside the light cylinder, there are no closed field lines, and the reversal of the sign of the field across the equator implies the presence of a current sheet (see Section 6). A current sheet is also present at the boundary between closed and open zones, connecting to the star. This general structure of the aligned dipole pulsar force-free magnetosphere has become standard, but alternative models do exist (Gruzinov 2011; Contopoulos et al. 2014).

We conclude this section with a discussion of situations in which closed field lines could be present outside the light cylinder in a magnetically dominated field configuration. We first discuss fields that are assumed to be degenerate but are otherwise arbitrary. In this case, it is straightforward to construct closed zones that proceed outside the light cylinder, since ψ, $\Omega_F(\psi)$, and I may be chosen freely. In particular, we may take $\Omega_F = \text{const}$, choose ψ corresponding to the standard closed/open structure of the pulsar magnetosphere except with the closed zone penetrating the light cylinder, and then choose I large enough that F^2 is positive everywhere in the closed zone (see equation 66). This configuration must violate reflection symmetry, but by a similar construction we may form closed loops outside the light cylinder in a reflection-symmetric magnetosphere by confining those loops to a hemisphere.

A more interesting example is provided by the new pulsar magnetosphere of Contopoulos et al. (2014). There the field is magnetically dominated, reflection symmetric, satisfies $\Omega_F' = 0$, and is force free everywhere except in the equatorial plane where lies a current sheet. There are closed field lines with $I(\psi) \neq 0$ that extend outside the light cylinder and have a kink at the current sheet. This configuration escapes our no-go theorem because of the infinitesimally thin current sheet violating the force-free condition. Specifically, a non-zero current is consistent with reflection symmetry because $I(\psi)$ flips sign across the current sheet. That is, $I_{\rm below}(\psi) = -I_{\rm above}(\psi)$. This reversal of the sign of I on a field line would not be allowed if the field were force free everywhere.

Adopting the additional constraint that the fields be everywhere force free has the potential to eliminate the possibility of closed poloidal field lines venturing outside the light cylinder, but we have not found an argument that does so. Gruzinov (2006) has found force-free solutions with closed lines having $I \neq 0$, showing that there is no difficulty of the lines closing. He has chosen to study configurations where the closed lines remain within the light cylinder, but we see nothing preventing alternative configurations for which they do not. As far as we are aware, it is currently an open question whether magnetically dominated force-free fields can have closed field lines outside the light cylinder.

9 BLACK HOLE MAGNETOSPHERE

Force-free magnetospheres of spinning black holes differ qualitatively from those of stars because of the presence of the event horizon and ergosphere. Within a star, the force-free condition does not hold, so energy (and angular momentum) can be transferred from the star to the electromagnetic field. In the case of a stationary black hole, by contrast, the force-free condition may in principle hold up to and across the horizon, and conditions behind the horizon cannot affect the exterior, so there is no analogue of the star transferring energy to the field. On the other hand, the meaning of this conserved energy is modified because of the spacetime curvature: as discussed in Section 7.3, it is the integral of the Noether current associated with the ‘time-translation’ Killing vector. On a stationary spinning black hole spacetime, this Killing vector is timelike far from the black hole, but spacelike near the black hole in the *ergosphere*, due to the extreme ‘dragging of inertial frames’. In the ergosphere, therefore, Killing energy is actually spatial momentum as defined by local observers. Hence, the electromagnetic field can have a local negative Killing energy density. Killing energy can therefore be extracted from the black hole, despite its conservation, because a corresponding *negative* Killing energy can flow across the horizon into the black hole. A process in which rotational energy is extracted from a spinning black hole, with a negative Killing energy flux across the horizon balancing a positive Killing energy flux at infinity, is (or should be[30]) called a *Penrose process* (Penrose 1969, 2002). When the mechanism involves a force-free magnetosphere, it is generally known as the *BZ process* or mechanism. For further discussion of the relationship between the Penrose and BZ processes, see e.g. Komissarov (2009) and Lasota et al. (2014).

Another qualitative difference between stellar and black hole magnetospheres lies in the topology of their field lines. The main qualitative feature of the pulsar magnetosphere relative to vacuum is that some field lines are open, although some loop back on to the star. In the case of a spinning black hole, it turns out that *all* field lines extending from the horizon must be open, unless they enter or loop around a non-force-free region, such as an accretion disc. In reference to the ‘no-hair’ theorems on black hole uniqueness in vacuum, we name this result the *no-ingrown-hair* theorem. This illustrates the fact that while the magnetic field lines (hairs) may indeed emerge from the horizon of a black hole with a force-free magnetosphere, they may *not* return unless they encounter a non-force-free region. The no-ingrown-hair result was first derived by

[30] The term is sometimes reserved for a particular class of such processes. However, while Penrose introduced the idea with the example of lowering a mass into the ergosphere at the end of a rope, he concluded that ‘Thus, in a sense, we have found a way of extracting rotational energy from the “black hole”. Of course, this is hardly a practical method! Certain improvements may be possible, e.g. using a ballistic method.[7] But the real significance is to find out what can and what cannot be done in principle since this may have some indirect relevance to astrophysical situations.’. (His footnote 7 states ‘Calculations show that this can indeed be done’, and goes on to describe the particle splitting method, now usually called *the* Penrose process).

MacDonald & Thorne (1982); our proof is similar but uses some different arguments.

Finally, there is a tricky technical point that arises only in the treatment of black hole magnetospheres. The $2+2$ poloidal/toroidal decomposition of the spacetime, which is so useful in handling the stationary axisymmetric force-free equations, breaks down at the horizon. One issue is that the 1-form dr becomes null (it is normal to the horizon, which lies at constant r), so that the poloidal subspace becomes null, rather than spacelike. Another is that the 1-forms dt and $d\varphi$ diverge, both in a manner proportional to dr, so that the toroidal subspace also becomes null, with the same null direction as the poloidal subspace. One way to handle this is to instead use coordinates that are regular at the horizon. Alternatively, one can continue to use the $2+2$ decomposition, being careful to determine the appropriate conditions that must be imposed to ensure regularity at the horizon. Here we will do some of both.

In this section, we adopt the Kerr metric for the black hole; however, the main important property is the presence of a horizon-generating Killing vector $\partial_t + \Omega_{\rm H}\partial_\varphi$, so analogous results could be easily derived for other spinning black hole metrics of the form (55).

9.1 Znajek horizon condition

On the future horizon of the Kerr spacetime, the quantities $\Omega_{\rm F}$, I, and ψ are not independent, but instead obey the *Znajek condition* (Znajek 1977):

$$I = 2\pi(\Omega_{\rm F} - \Omega_{\rm H})\psi_{,\theta}\sqrt{\frac{g_{\varphi\varphi}}{g_{\theta\theta}}}. \tag{104}$$

This holds for any stationary axisymmetric degenerate solution of Maxwell's equations with $\partial_\varphi \cdot F \neq 0$. Here $\Omega_{\rm H}$ is the angular velocity of the horizon, defined by the condition that the Killing field

$$\chi = \partial_t + \Omega_{\rm H}\partial_\varphi \tag{105}$$

is tangent to the null horizon generators. Znajek obtained equation (104) by demanding that the corresponding electromagnetic field strength F (equation 65 in our notation) is regular on the horizon, and the condition is often employed to guarantee regularity in calculations involving irregular coordinates. However, as elaborated below, all quantities appearing in equation (104) have invariant geometrical status, making the condition independent of any coordinate concerns, including their regularity. To emphasize this point, we begin by providing a tensorial derivation of equation (104). We then present a derivation expressing F in regular coordinates, which shows how equation (104) guarantees regularity of F on the future horizon for non-extremal black holes. We find an additional condition required for regularity in the extremal case $a = M$. Finally, we reproduce the surprising fact, first pointed out by MacDonald & Thorne (1982), that the stream equation (86) in fact *implies* the Znajek condition up to sign, corresponding to regularity on either the future or past horizon.

To see that equation (104) is a relationship between invariants, recall that $\Omega_{\rm F}$ may be defined by the condition $(\partial_t + \Omega_{\rm F}\partial_\varphi)\cdot F = 0$ for degenerate fields F with $\partial_\varphi \cdot F \neq 0$, the polar current I is the upward flow of charge per unit Killing time through a polar cap bounded by an axial loop, $2\pi\psi$ is the upward flux of F through such a loop, and $2\pi\sqrt{g_{\varphi\varphi}}$ is its circumference. These notions are valid on the horizon, where the remaining ingredient $\psi_{,\theta}/\sqrt{g_{\theta\theta}}$ is the derivative with respect to proper length along the horizon in the direction orthogonal to the two Killing vectors, away from the upward pole.

The tensorial derivation makes use of the horizon-generating Killing field (105) which, in view of the defining property of $\Omega_{\rm F}$ and the structure of the field strength (62), has the useful property

$$\chi \cdot F = (\Omega_{\rm F} - \Omega_{\rm H})d\psi. \tag{106}$$

The derivation proceeds by evaluating the polar current on the horizon and performing a few manipulations:

$$I/2\pi = *F_{ab}\partial_t^a\partial_\varphi^b \tag{107}$$

$$= *F_{ab}\chi^a\partial_\varphi^b \tag{108}$$

$$= F_{ab}*(\chi^a\partial_\varphi^b) \tag{109}$$

$$= F_{ab}\chi^a\partial_\theta^b\sqrt{g_{\varphi\varphi}/g_{\theta\theta}} \tag{110}$$

$$= (\Omega_{\rm F} - \Omega_{\rm H})\psi_{,\theta}\sqrt{g_{\varphi\varphi}/g_{\theta\theta}}. \tag{111}$$

In the first line, we used equation (64), in the second line, we used antisymmetry of F_{ab} to replace ∂_t by χ, and in the third line, we shifted the duality operation from F_{ab} to $\chi^a\partial_\varphi^b$. In the fourth line, we used the fact that on the horizon χ is null and orthogonal to ∂_φ, so that the 2-form $\chi_{[a}(\partial_\varphi)_{b]}$ is null and therefore can be dualized as explained in Appendix A2.3. In this step, we also use the fact that ∂_θ is orthogonal to both ∂_φ and χ. We have chosen the sign appropriate for the future horizon.[31] Finally, in the last line, we use equation (106). Note that, other than stationary axisymmetry with commuting Killing fields, the only special property of the spacetime used in this derivation is the existence of a Killing horizon generated by χ.

When thinking of the intrinsic quantities I, $\Omega_{\rm F}$, and ψ individually, we are unaware of any reason to expect them to be related on the horizon. However, since the poloidal and toroidal subspaces become null, and with non-trivial intersection in the limit at the horizon, the corresponding parts of the 2-form F are not independent. This makes the existence of the Znajek condition less surprising.

To show that equation (104) guarantees regularity on the future horizon of Kerr (in the non-extremal case), we begin with equation (65), $F = I/(2\pi\sqrt{-g^{\rm T}})\epsilon_{\rm P} + d\psi \wedge \eta$. Using $\sqrt{-g^{\rm P}/g^{\rm T}} = (r^2 + a^2\cos^2\theta)/(\Delta\sin\theta)$, the first term can be written as

$$\frac{I}{2\pi(-g^{\rm T})^{1/2}}\epsilon_{\rm P} = \frac{I}{2\pi}\frac{r^2 + a^2\cos^2\theta}{\Delta\sin\theta}\, dr \wedge d\theta, \tag{112}$$

where we have used relations from Appendix C. Note that this term is singular on the horizon $r = r_+$, where $\Delta \equiv (r - r_+)(r - r_-) = 0$. The second term $d\psi \wedge \eta$ is also singular because η (equation 69) is composed of the singular 1-forms $d\varphi$ and dt.

To isolate the divergent behaviour, we define a 'regularized' corotation 1-form,

$$\tilde{\eta} = d\tilde{\varphi} - \Omega_{\rm F}\, dv, \tag{113}$$

where v and $\tilde{\varphi}$ are the regular ingoing Kerr coordinates (C8) and (C14). Then $\tilde{\eta}$ is regular on the future horizon, and differs from η

[31] To determine the sign of $\star\chi$, note that on the future horizon $\chi_a \sim (du)_a$ (both 1-forms are null and normal to the horizon). As shown at the end of Appendix A2.3, in the Schwarzschild case, $\star du = du$; hence, $\star\chi = \chi$. The Kerr case is related by a continuous deformation to Schwarzschild, so the sign is the same. On the past horizon, $\chi_a \sim (dv)_a$ and $\star dv = -dv$; hence, $\star\chi = -\chi$.

by a singular form proportional to dr,

$$\eta = \tilde{\eta} + [\Omega_F(r^2 + a^2) - a]\frac{dr}{\Delta}$$

$$= \tilde{\eta} + \left[\Omega_F(r^2 + a^2) - \Omega_H\left(r_+^2 + a^2\right)\right]\frac{dr}{\Delta}. \quad (114)$$

The η term in F can then be written as

$$d\psi \wedge \eta = d\psi \wedge \tilde{\eta}$$

$$- \psi_{,\theta}\frac{(\Omega_F(r^2 + a^2) - \Omega_H(r_+^2 + a^2)}{\Delta}\, dr \wedge d\theta. \quad (115)$$

The field strength is the sum of equations (112) and (115),

$$F = d\psi \wedge \tilde{\eta} + \frac{f(r)}{(r - r_+)(r - r_-)}\, dr \wedge d\theta, \quad (116)$$

with

$$f(r) = \frac{I(r^2 + a^2\cos^2\theta)}{2\pi\sin\theta}$$

$$- \psi_{,\theta}\left[(\Omega_F(r^2 + a^2) - \Omega_H\left(r_+^2 + a^2\right)\right]. \quad (117)$$

Regularity of the field at the horizon requires $f(r_+) = 0$, i.e.

$$I = 2\pi(\Omega_F - \Omega_H)\psi_{,\theta}\frac{\left(r_+^2 + a^2\right)\sin\theta}{r_+^2 + a^2\cos^2\theta} \quad (118)$$

evaluated at r_+. The rightmost factor agrees with $\sqrt{g_{\varphi\varphi}/g_{\theta\theta}}$ on the horizon, so we have recovered the Znajek condition (104) on the future horizon.

In the non-extremal case $r_+ \neq r_-$, this condition is also sufficient for regularity, and the horizon value of the field takes the form

$$F = d\psi \wedge \tilde{\eta} + \frac{f'(r_+)}{r_+ - r_-}\, dr \wedge d\theta. \quad (119)$$

In the extremal case, $f'(r_+) = 0$ is a *second* necessary condition, and together the two are sufficient. Using $r_+ = M = a$, which holds in the extremal case, this second condition becomes

$$0 = \frac{f'(r_+)}{2M^2} = \frac{I'\psi_{,r}(1 + \cos^2\theta)}{4\pi\sin\theta} + \frac{1}{M}\left(\frac{I}{2\pi\sin\theta} - \psi_{,\theta}\Omega_F\right)$$

$$- \psi_{,\theta r}(\Omega_F - \Omega_H) - \psi_{,\theta}\psi_{,r}\Omega_F', \quad (120)$$

where now $\Omega_H = 1/2M$, and the expression is evaluated at $r = r_+$. We are not aware of a previous derivation of this condition, although the analogous condition is known in the Reissner–Nordstrom case (Takamori et al. 2011). In the *near* extremal case, F would become large if $f'(r_+)$ does not go to zero as $r_+ - r_- = 2M\sqrt{1 - (a/M)^2}$. It seems reasonable to expect that F does not blow up as extremality is approached (for physically reasonable boundary conditions), so we expect equation (120) to be approximately satisfied for near-extremal black holes.

BZ regarded the Znajek condition (104) as a boundary condition at $r = r_+$ for the stream equation. However, MacDonald & Thorne (1982) pointed out that it in fact *follows* from the stream equation, up to a sign. To see this, first write the stream equation (86) near the horizon as

$$\frac{II'}{4\pi^2} = \frac{A}{\Sigma}\sin\theta\,\partial_\theta\left[\frac{\sin\theta}{\Sigma}(\Omega_F - \Omega_H)^2\,\partial_\theta\psi\right]$$

$$- \frac{A}{\Sigma^2}\sin^2\theta\,\Omega_F'(\Omega_F - \Omega_H)(\partial_\theta\psi)^2 + O(\Delta), \quad (121)$$

where A, Σ, and Δ are defined in Appendix C, and we have made use of equations (88)–(90). Here we assume that ψ, Ω_F, and I are smooth in r and θ. The fact that only θ-derivatives appear is related to the vanishing of g^{rr} on the horizon. On the horizon $r = r_+ =$ const, we may relate ψ and θ derivatives using the ordinary chain rule, $f' = \partial_\theta f/\partial_\theta\psi$ for functions f. Then we have

$$\partial_\theta\left(\frac{I^2}{8\pi^2}\right) = \frac{II'}{4\pi^2}\partial_\theta\psi \quad (122)$$

$$= \partial_\theta\left[\frac{A\sin^2\theta}{2\Sigma^2}(\Omega_F - \Omega_H)^2(\partial_\theta\psi)^2\right], \quad (123)$$

where the second line follows using equation (121) and the Leibniz rule. We may now integrate in θ along the horizon, yielding

$$I^2 = 4\pi^2\frac{A\sin^2\theta}{\Sigma^2}(\Omega_F - \Omega_H)^2(\partial_\theta\psi)^2 + C. \quad (124)$$

The integration constant C must vanish for the fields to be regular on the axis (otherwise a line current will be present there), so we conclude that the stream equation implies

$$I = \pm 2\pi\frac{\sqrt{A}\sin\theta}{\Sigma}(\Omega_F - \Omega_H)\,\partial_\theta\psi. \quad (125)$$

This is the Znajek condition (104), with an additional $\pm$ on the right-hand side.

If the minus sign is chosen, then the fields are regular on the past horizon instead of the future horizon and thus represent a white hole magnetosphere rather than a black hole magnetosphere. Note that it is impossible for the fields to be regular on both horizons, since all quantities appearing in equation (125) take the same value on both horizons. The fact that equation (125) is always satisfied by smooth solutions to the stream equation indicates that, when solving the stream equation on $r \geq r_+$, the only boundary condition at $r = r_+$ that one needs to impose is the sign choice corresponding to a black hole.

The Znajek condition has a number of practical applications. First, it can be used as a horizon boundary condition for the stream equation, as in the original BZ paper. (The BZ solution can equally well be derived without the use of the condition; cf. McKinney & Gammie 2004 and our Section 5.3.) The condition is also helpful in derivations of theoretical results. We use it to obtain the illustrative flux formulae (126) and (127) (following BZ) and to prove the no-ingrown-hair theorem (following MacDonald & Thorne 1982).

9.2 Energy and angular momentum flux

The general expressions (80) and (81) give the outward energy and angular momentum flux in terms of the invariants I and Ω_F. In the Kerr spacetime, it is instructive to push these integrals to the horizon. That is, let the poloidal curve $\mathcal{P}$ be the horizon $r = r_+$. Using the Znajek condition (104) and the fact that $dr = 0$ on the horizon, we have

$$d\mathcal{L}/dt = 2\pi\int_0^\pi(\Omega_H - \Omega_F)(\psi,_\theta)^2\sqrt{\frac{g_{\varphi\varphi}}{g_{\theta\theta}}}\, d\theta \quad (126)$$

$$d\mathcal{E}/dt = 2\pi\int_0^\pi\Omega_F(\Omega_H - \Omega_F)(\psi,_\theta)^2\sqrt{\frac{g_{\varphi\varphi}}{g_{\theta\theta}}}\, d\theta. \quad (127)$$

It follows that positive Killing energy flows outwards if and only if Ω_F is between 0 and Ω_H. Since no influence can emerge from behind the horizon, however, it is more natural to say that negative Killing energy flows inwards across the horizon, as explained at the beginning of this section. Note that the BZ solution (47) has

$\Omega_{\rm F} = \Omega_{\rm H}/2$, the value that maximizes the energy flux at fixed magnetic flux through the horizon.

It also follows from equations (126) and (127) that any outflow of energy is accompanied by an outflow of angular momentum. This is consistent with the fact that the source of the energy is the rotation of the black hole. A universal upper limit to the energy extracted for a given angular momentum extracted can be found using the null energy condition, $T_{ab}\ell^a\ell^b > 0$ for null vectors ℓ^a, which in particular is satisfied by the electromagnetic field stress tensor. Since the horizon-generating Killing field χ is null on the horizon, we have $T_{ab}\chi^a\chi^b \geq 0$ there. This expression is equal to the inward flux of energy $T_{ab}\partial_t^a\chi^b$ minus $\Omega_{\rm H}$ times the inward flux of angular momentum $-T_{ab}\partial_\varphi^a\chi^b$. It follows that the *outward* flux of energy is bounded by $\delta\mathcal{E} \leq \Omega_{\rm H}\delta\mathcal{J}$. The BZ process satisfies $\delta\mathcal{E} = \Omega_{\rm F}\delta\mathcal{J}$, so its efficiency is governed by the ratio $\Omega_{\rm F}/\Omega_{\rm H}$.

As noted in the original paper, the process can be characterized in thermodynamic terms. The BZ process is stationary, but when the back reaction on the geometry is taken into account it becomes a quasi-stationary process in which the first law of black hole mechanics (Bardeen, Carter & Hawking 1973; Bekenstein 1973) should apply, $\delta M - \Omega_{\rm H}\delta J = (\kappa/8\pi)\delta A$, where M and J are the mass and angular momentum of the black hole, κ is the surface gravity, and A is the horizon area. The second law of black hole mechanics (which follows from the null energy condition and cosmic censorship) states that the area cannot decrease, $\delta A \geq 0$ (Hawking 1972; Hawking & Ellis 1973; Penrose & Floyd 1971). In the BZ process, $\delta M = -\delta\mathcal{E}$ and $\delta J = -\delta\mathcal{J}$, so the first and second laws imply the same upper bound obtained above using the null energy condition directly. Perfect efficiency corresponds to the case in which the area of the horizon is unchanged. According to the second law of black hole mechanics, only in that limit is the process reversible.

9.3 Light surfaces in a black hole magnetosphere

Recall that a light surface is a hypersurface in spacetime where the field sheet Killing vector $\chi_{\rm F} = \partial_t + \Omega_{\rm F}(\psi)\partial_\varphi$ is null or, equivalently, where the corotation 1-form η is null. Light surfaces play a practical role in finding force-free solutions, since they correspond to singular points of the stream equation (see Section 7.4.2). They also act as horizons for the propagation of particles and Alfvén waves through the magnetosphere (see Section 7.2.5). In the Kerr spacetime, there are in general *two* light surfaces, an outer one qualitatively similar to the ordinary light cylinder, and an inner one within the ergosphere. Outside the outer light surface, a corotating curve with angular velocity $\Omega_{\rm F}$ is rotating too fast to be timelike, whereas inside the inner light surface, it is rotating too *slowly* to be timelike. The existence of the inner surface follows from the fact that within the ergosphere observers (i.e. timelike curves) must rotate with a minimum angular velocity, which approaches $\Omega_{\rm H}$ at the event horizon. Any field line with $\Omega_{\rm F} < \Omega_{\rm H}$ will therefore cross an inner light surface at some point sufficiently close to the horizon. The inner light surface meets the horizon at the poles.

The field sheet Killing vector is spacelike inside the inner light surface and outside the outer one, and timelike in between. For $\Omega_{\rm F} < \Omega_{\rm H}$, wind particles and Alfvén waves can travel only inwards across the inner light surface and only outwards across the outer light surface, as indicated in Fig. 5. This follows from the analysis of Section (7.3.1), which established that the particle wind direction relative to that of positive angular momentum flow (126) is determined by the sign of $\Omega_{\rm F} - \Omega_{\rm Z}$, which is positive at the outer light surface and negative at the inner one. This was shown using a different method by Komissarov (2004), who has given a detailed discussion of the properties of the light surfaces of Kerr.

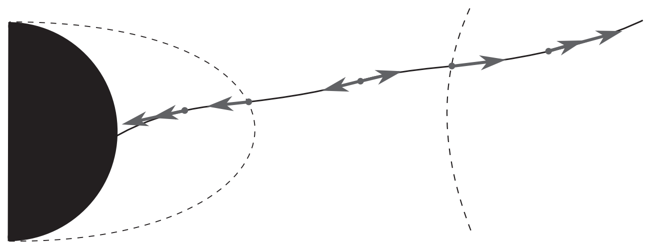

Figure 5. Diagram of black hole light surfaces (dashed lines) and wind propagation along a field line. The inner light surface is drawn exaggeratedly far from the black hole. Arrows indicate the projection of the field sheet null vectors on to the poloidal plane. These bound the possible poloidal velocities of particles moving on the field line. Particles may only move inwards inside of the inner surface, and only outwards outside of the outer surface.

9.4 No ingrown hair

We now prove the no-ingrown-hair theorem, which forbids regions of closed field lines for black hole magnetospheres (MacDonald & Thorne 1982). By a closed field line we mean a smooth poloidal field line that non-tangentially[32] intersects the horizon twice, i.e. a level set of ψ on which $d\psi \neq 0$, which intersects $r = r_+$ twice, each time with $\psi_{,\theta} \neq 0$. We first establish that $I = 0$ and $\Omega_{\rm F} = \Omega_{\rm H}$ for a closed field line in a force-free region. (We use only the first force-free condition $I = I(\psi)$ for this part of the argument.) The regularity condition (104) implies that the sign of I is determined by the product of $\Omega_{\rm F}(\psi) - \Omega_{\rm H}$, which is the same at the two ends (and everywhere on the line), with $\psi_{,\theta}$, which has opposite sign at the two ends (since loops with different values of ψ are nested and θ is monotonic along the horizon). Thus I has opposite sign at opposite ends; however, I is also constant on the line, and hence must vanish. The regularity condition (together with $\psi_{,\theta} \neq 0$) then implies that $\Omega_{\rm F} = \Omega_{\rm H}$ for the line.

In a force-free region of closed field lines, we thus have $\Omega_{\rm F}' = I' = 0$, so the conditions for the light surface loop lemma hold. Furthermore, since $\Omega_{\rm F} = \Omega_{\rm H}$, the horizon itself is a light surface [recall that the Killing vector (105) is null at the horizon], and the lemma applies to it. This proves the no-ingrown-hair theorem:

> *A contractible force-free region of closed poloidal field lines cannot exist in a stationary, axisymmetric, force-free Kerr black hole magnetosphere.*

Thus a black hole cannot have a 'closed zone' like a dipole pulsar does. Note that the theorem does not rely on reflection symmetry or magnetic domination.

Closed field lines may exist if they pass through, or loop around, non-force-free regions of the magnetosphere. For example, closed field lines may connect the black hole to an accretion disc. Or if a torus of material orbits the black hole, field lines may loop around the torus before returning to the horizon. (These looping field lines must still have $\Omega_{\rm F} = \Omega_{\rm H}$ and $I = 0$.) The no-ingrown-hair theorem is a natural generalization to the force-free setting of the no-hair idea that an astrophysical black hole cannot have its 'own' magnetic

[32] In earlier discussion, we did not include this additional proviso as part of the definition of 'closed field line'. Note, however, that non-tangential intersection is necessary for establishing that conductors determine the rotation frequency of their field lines (see Section 8.1).

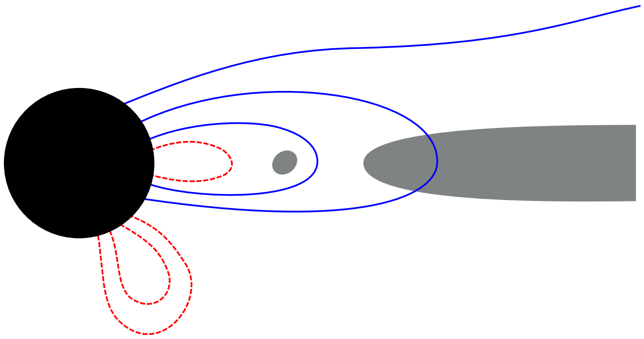

Figure 6. Allowed (blue solid) and disallowed (red dashed) topologies of poloidal field lines in a force-free black hole magnetosphere. Open lines are allowed. Closed lines must pass through, or loop around, a non-force-free region (grey).

field. In order for closed field lines to exist, non-force-free currents must flow to support them. These ideas are illustrated in Fig. 6.

Like its classical counterpart, the no-ingrown-hair theorem deals only with stationary situations, giving no insight into how any closed loop will be destroyed during the approach to stationarity. It seems likely that loops will either be absorbed by the black hole (Thorne, Price & MacDonald 1986) or opened up by non-force-free processes (Lyutikov & McKinney 2011). Qualitative discussions of different field line types in black hole magnetospheres can be found in Thorne et al. (1986), Blandford (2002), Hirose et al. (2004), and McKinney (2005).

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REFERENCES

Alfvén H., 1942, Nature, 150, 405
Alic D., Moesta P., Rezzolla L., Zanotti O., Jaramillo J. L., 2012, ApJ, 754, 36
Bardeen J. M., Carter B., Hawking S. W., 1973, Commun. Math. Phys., 31, 161
Bekenstein J. D., 1973, Phys. Rev. D, 7, 2333
Beskin V. S., 2010, MHD Flows in Compact Astrophysical Objects: Accretion, Winds and Jets. Springer-Verlag, Berlin
Blandford R. D., 1976, MNRAS, 176, 465
Blandford R. D., 2002, in Gilfanov M., Sunyeav R., Churazov E., eds, Lighthouses of the Universe: The Most Luminous Celestial Objects and Their Use for Cosmology To the Lighthouse. Springer-Verlag, Berlin, p. 381
Blandford R. D., Znajek R. L., 1977, MNRAS, 179, 433 (BZ)
Bogovalov S. V., 1999, A&A, 349, 1017
Brennan T. D., Gralla S. E., 2014, Phys. Rev. D, 89, 103013
Brennan T. D., Gralla S. E., Jacobson T., 2013, Class Quantum Gravity, 30, 195012
Buniy R. V., Kephart T. W., 2014, Ann. Phys., 344, 179
Carter B., 1970, Commun. Math. Phys., 17, 233
Carter B., 1979, in Hawking S. W., Israel W., eds, General Relativity: An Einstein Centenary Survey. Cambridge Univ. Press, Cambridge, p. 294
Choquet-Bruhat Y., Dewitt-Morette C., 1982, Analysis, Manifolds and Physics, Part I: Basics, Revised Edition. North Holland, Amsterdam
Contopoulos J., 1995, ApJ, 450, 616
Contopoulos I., Kazanas D., Fendt C., 1999, ApJ, 511, 351
Contopoulos I., Kazanas D., Papadopoulos D. B., 2013, ApJ, 765, 113
Contopoulos I., Kalapotharakos C., Kazanas D., 2014, ApJ, 781, 46
Galili I., Goihbarg E., 2005, Am. J. Phys., 73, 141
Gold T., 1968, Nature, 218, 731
Goldreich P., Julian W. H., 1969, ApJ, 157, 869
Gourgoulhon E., Bonazzola S., 1993, Phys. Rev. D, 48, 2635
Gourgoulhon E., Markakis C., Uryū K., Eriguchi Y., 2011, Phys. Rev. D, 83, 104007
Gruzinov A., 2006, preprint (astro-ph/0604364)
Gruzinov A., 2011, preprint (arXiv:1101.3100)
Harris E. G., 1962, Il Nuovo Cimento Ser. 10, 23, 115
Hartle J. B., Thorne K. S., 1968, ApJ, 153, 807
Hawking S. W., 1972, Commun. Math. Phys., 25, 152
Hawking S. W., Ellis G. F. R., 1973, The Large-Scale Structure of Space-Time. Cambridge Univ. Press, Cambridge
Hewish A., Bell S. J., Pilkington J. D. H., Scott P. F., Collins R. A., 1968, Nature, 217, 709
Hirose S., Krolik J. H., De Villiers J.-P., Hawley J. F., 2004, ApJ, 606, 1083
Kalapotharakos C., Contopoulos I., Kazanas D., 2012, MNRAS, 420, 2793
Komissarov S. S., 2001, MNRAS, 326, L41
Komissarov S. S., 2002, MNRAS, 336, 759
Komissarov S. S., 2004, MNRAS, 350, 427
Komissarov S. S., 2009, J. Korean Phys. Soc., 54, 2503
Komissarov S. S., McKinney J. C., 2007, MNRAS, 377, L49
Lasota J.-P., Gourgoulhon E., Abramowicz M., Tchekhovskoy A., Narayan R., 2014, Phys. Rev. D, 89, 024041
Lyutikov M., 2011, Phys. Rev. D, 83, 124035
Lyutikov M., McKinney J. C., 2011, Phys. Rev. D, 84, 084019
MacDonald D., Thorne K. S., 1982, MNRAS, 198, 345
McKinney J. C., 2005, ApJ, 630, L5
McKinney J. C., 2006, MNRAS, 368, L30
McKinney J. C., Gammie C. F., 2004, ApJ, 611, 977
McKinney J. C., Tchekhovskoy A., Blandford R. D., 2012, MNRAS, 423, 3083
Menon G., Dermer C. D., 2007, Gen. Relativ. Gravit., 39, 785
Menon G., Dermer C. D., 2011, MNRAS, 417, 1098
Michel F. C., 1973, ApJ, 180, 207
Nathanail A., Contopoulos I., 2014, ApJ, 788, 186
Northrop T. G., Teller E., 1960, Phys. Rev., 117, 215
Okamoto I., 1974, MNRAS, 167, 457
Pacini F., 1968, Nature, 219, 145
Palenzuela C., Lehner L., Liebling S. L., 2010, Science, 329, 927
Palenzuela C., Bona C., Lehner L., Reula O., 2011, Class. Quantum Gravity, 28, 134007
Palenzuela C., Lehner L., Ponce M., Liebling S. L., Anderson M., Neilsen D., Motl P., 2013, Phys. Rev. Lett., 111, 061105
Paschalidis V., Etienne Z. B., Shapiro S. L., 2013, Phys. Rev. D, 88, 021504
Penrose R., 1969, Riv. Nuovo Cimento, 1, 252
Penrose R., Floyd R. M., 1971, NPhS, 229, 177
Penrose R., 2002, Gen. Relativ. Gravit., 34, 1141
Penrose R., Rindler W., 1984, Spinors and Space-Time. Volume 1: Two-Spinor Calculus and Relativistic Fields. Cambridge Univ. Press, Cambridge
Pfeiffer H. P., MacFadyen A. I., 2013, preprint (arXiv:1307.7782)
Poisson E., 2004, A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics. Cambridge Univ. Press, Cambridge
Punsly B., 2008, Black Hole Gravitohydromagnetics. Springer, New York
Robinson I., 1961, J. Math. Phys., 2, 290

Ruderman M. A., Sutherland P. G., 1975, ApJ, 196, 51
Scharlemann E. T., Wagoner R. V., 1973, ApJ, 182, 951
Schmidt M., 1963, Nature, 197, 1040
Spitkovsky A., 2006, ApJ, 648, L51
Takamori Y., Nakao K.-I., Ishihara H., Kimura M., Yoo C.-M., 2011, MNRAS, 412, 2417
Thompson C., Blaes O., 1998, 57, 3219
Thorne K. S., Price R. H., MacDonald D. A., 1986, Black Holes: The Membrane Paradigm. Yale Univ. Press, New Haven, CT
Timokhin A. N., 2006, MNRAS, 368, 1055
Uchida T., 1997a, Phys. Rev. E, 56, 2181
Uchida T., 1997b, Phys. Rev. E, 56, 2198
Uchida T., 1997c, MNRAS, 286, 931
Uchida T., 1997d, MNRAS, 291, 125
Uchida T., 1998, MNRAS, 297, 315
Uzdensky D. A., 2005, ApJ, 620, 889
Wald R. M., 1974, Phys. Rev. D, 10, 1680
Wald R. M., 1984, General Relativity. Univ. Chicago Press, Chicago, IL
Znajek R. L., 1977, MNRAS, 179, 457

APPENDIX A: DIFFERENTIAL FORMS

A differential form is an antisymmetric, covariant tensor. Under the operations of scalar multiplication, addition, and wedge product $\wedge$, differential forms comprise a graded algebra. The correspondence with tensor index notation is given by

$$(\alpha \wedge \beta)_{a_1\ldots a_p b_1\ldots b_q} = \frac{(p+q)!}{p!q!}\alpha_{[a_1\ldots a_p}\beta_{b_1\ldots b_q]}, \tag{A1}$$

where the square brackets denote antisymmetrization. Note that for any 1-form α we thus have $\alpha \wedge \alpha = 0$. The exterior (antisymmetrized) derivative d is a graded derivation on the algebra, and satisfies $dd = 0$. The graded derivation property is that for any p-form α and q-form β, we have

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge d\beta. \tag{A2}$$

In tensor notation,

$$(d\alpha)_{aa_1\ldots a_p} = (p+1)\nabla_{[a}\alpha_{a_1\ldots a_p]}, \tag{A3}$$

where ∇_a is any torsion-free derivative operator, e.g. coordinate partial derivatives. A form α is *closed*, if $d\alpha = 0$ and *exact* if $\alpha = d\gamma$ for some γ. An exact form is always closed, since $dd = 0$. In a contractible region, a closed form is always exact.

A p-form can be contracted with any number of vectors up to a maximum of p. Given a p-form α and a vector v, the contraction on the first 'slot' or first 'index' of α will be denoted here by a dot:

$$(v \cdot \alpha)_{a\ldots b} = v^m \alpha_{ma\ldots b}. \tag{A4}$$

A more common notation for this operation is $i_v\alpha$. The *pullback* of a form β to a p-dimensional submanifold $\mathcal{S}$ is just β, considered as a form on $\mathcal{S}$. That is, the contraction of the pullback of β with any set of p vectors tangent to $\mathcal{S}$ is, by definition, the contraction of those vectors with β.

A1 Integration of forms

A p-form α can be integrated on a p-dimensional surface, or submanifold $\mathcal{S}$. Intuitively, one chops up $\mathcal{S}$ into infinitesimal parallelopipeds each generated by p infinitesimal vectors, evaluates α on these p vectors, and adds the resulting numbers. Because of the multilinearity and antisymmetry of α, the result is independent of how the chopping is done. Note, however, that the *sign* of the result depends on the order taken for the edge vectors of the parallelopipeds. Thus the integral is well defined only once an *orientation* for $\mathcal{S}$ is specified, i.e. an equivalence class of continuous, nowhere vanishing p-forms on $\mathcal{S}$ related by positive multiples. The edge vectors are ordered so that the result is positive on members of this class. (For brevity, we will sometimes refer to a continuous, nowhere vanishing p-form ω as an 'orientation' for a p-surface, meaning actually that ω determines the orientation.) An ordered coordinate system $(y^1, \ldots, y^p)$ on $\mathcal{S}$ determines the orientation $dy^1 \wedge \cdots \wedge dy^p$, with respect to which the integral of $\alpha = \alpha_{1\ldots p} dy^1 \wedge \cdots \wedge dy^p$ is given by an ordinary multiple integral:

$$\int_{\mathcal{S}} \alpha = \int \alpha_{1\ldots p}\, dy^1 \cdots dy^p. \tag{A5}$$

In this paper, we make use of two properties of p-form integrals. One is Stokes' theorem, which relates the integral of $d\omega$ on $\mathcal{S}$ to the integral of ω on the boundary $\partial\mathcal{S}$:

$$\int_{\mathcal{S}} d\omega = \int_{\partial\mathcal{S}} \omega, \tag{A6}$$

where the orientation on $\partial\mathcal{S}$ is the one induced by contracting an outward-pointing vector on the first slot of the orientation form on $\mathcal{S}$. In particular, note that if $d\omega = 0$, then $\int_{\partial\mathcal{S}} \omega = 0$.

The other property pertains to the integral of a 3-form (more generally, to the integral of an $(n-1)$-form over a hypersurface in an n-dimensional space). Suppose $\mathcal{S}$ is a level hypersurface of the function y, i.e. it is defined by the equation $y = y_0$ for some constant y_0, and let v be any vector field such that $v \cdot dy = 1$ on $\mathcal{S}$. Then the pullback of a 3-form ω to $\mathcal{S}$ is equal to the pullback of $v \cdot (dy \wedge \omega)$. (To show this, just contract with any three vectors tangent to $\mathcal{S}$.) The integral of ω on $\mathcal{S}$ can thus be expressed as

$$\int_{\mathcal{S}} \omega = \int_{\mathcal{S}} v \cdot (dy \wedge \omega). \tag{A7}$$

This is useful when the 4-form $dy \wedge \omega$ has properties that allow efficient computation of the integral.

A2 Hodge dual operator

To begin with an intuitive definition, the Hodge dual $*$ of a decomposable p-form is simply the orthogonal decomposable form with the same squared norm, up to a sign. The sign of the squared norm of the dual is opposite in four dimensional spacetime, because the dual of a spacelike form is timelike, and vice versa. This definition is extended linearly to linear combinations of such forms, and it defines the dual of a form up to a sign. A more precise definition fixes all of these signs, up to one overall sign that depends on the choice of an orientation. Keeping track of the signs can be tedious, but for many purposes they need not be determined. The dual of the dual $*^2$ brings one back to the same form, up to a sign that depends on the dimension of the space, the rank of the form, and the signature of the metric. In two- or four-dimensional Lorentzian spacetime, $*^2 = \pm 1$, with the $+$ sign for odd rank forms and the $-$ sign for even rank forms, while for Euclidean signature the signs are opposite to these.

An explicit definition of the dual in an n-dimensional space can be given in terms of the metric-compatible volume element $\epsilon_{a_1\cdots a_n}$, which is the unique, up to a sign, totally antisymmetric tensor normalized by $\epsilon_{a_1\cdots a_n}\epsilon^{a_1\cdots a_n} = \pm n!$, with the $+$ sign for Euclidean and the $-$ sign for Lorentzian signature. (The indices are raised by inverse metrics as usual.) The choice between the two volume elements is a choice of orientation. The dual of a p-form with respect to this orientation is defined by

$$*\beta_{b_1\cdots b_{n-p}} = \frac{1}{p!}\beta^{a_1\cdots a_p}\epsilon_{a_1\cdots a_p b_1\cdots b_{n-p}}. \tag{A8}$$

For any pair of p-forms α and β, one has the useful relation

$$\alpha \wedge *\beta = \langle \alpha, \beta\rangle_g * 1, \tag{A9}$$

where

$$\langle \alpha, \beta\rangle = \frac{1}{p!}\alpha_{m_1\cdots m_p}\beta^{m_1\cdots m_p}, \tag{A10}$$

and $*1$ is the volume element (with a choice of orientation). In fact, the dual is defined implicitly by the relation (A9).

A2.1 *Diagonal metrics*

If the line element is written in diagonal form, it is particularly easy to find the action of the dual. For example, consider the Schwarzschild line element

$$ds^2 = -A\,dt^2 + A^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\varphi^2, \tag{A11}$$

with $A = 1 - 2M/r$ (in units with $G = c = 1$). We can read off that the four 1-forms $A^{1/2}\,dt$, $A^{-1/2}\,dr$, $rd\theta$, and $r\sin\theta\, d\varphi$ are orthonormal, the first being timelike and the others spacelike. The dual $*(d\theta \wedge d\varphi)$ is therefore proportional to $dt \wedge dr$. To determine the coefficient function, we can simply scale all the forms so they have unit norm. Thus

$$\begin{aligned} *(d\theta \wedge d\varphi) &= \frac{1}{r^2\sin\theta} *((rd\theta)\wedge(r\sin\theta\, d\varphi)) \\ &= \pm\frac{1}{r^2\sin\theta}(A^{1/2}dt)\wedge(A^{-1/2}dr) \\ &= \pm\frac{1}{r^2\sin\theta}\,dt\wedge dr. \end{aligned} \tag{A12}$$

Since $** = -1$ on spacetime 2-forms, it follows that also $*(dt \wedge dr) = \mp r^2\sin\theta\, d\theta \wedge d\varphi$. As explained above, the overall sign depends upon the orientation. According to equation (A9), we have $d\theta \wedge d\varphi \wedge *(d\theta\wedge d\varphi) = \langle d\theta\wedge d\varphi, d\theta\wedge d\varphi\rangle * 1$, and since the metric on the angular subspace is positive definite this is a positive number times the volume element $*1$. Hence, the sign in equation (A12) is $+$ for the orientation of $d\theta\wedge d\varphi\wedge dt\wedge dr$ and $-$ for the opposite orientation.

A2.2 *Orthogonal subspaces*

Suppose the metric space V of dimension n is the direct sum of two orthogonal subspaces, $V = A \oplus B$, of dimensions n_A and n_B, and let the orientations be chosen so that the volume elements are related by $\epsilon = \epsilon^A \wedge \epsilon^B$. Then the dual of a wedge product $\alpha \wedge \beta$ of an A-p-form with a B-q-form is given by

$$*(\alpha\wedge\beta) = (-1)^{q(n_A - p)} \star\alpha \wedge \star\beta, \tag{A13}$$

where the symbol $\star$ denotes the Hodge dual on the subspaces A or B, defined with respect to ϵ^A and ϵ^B. We will find this very useful for simple 2-forms in stationary, axisymmetric spacetimes, in which case $p = q = 1$ and $n_A = 2$, so the sign is $-$.

A2.3 *Dual of a null 2-form*

A null 2-form has the composition $\alpha \wedge n$, with n a null 1-form and α a spacelike 1-form orthogonal to n. This is orthogonal to itself, and has zero norm, so the intuitive definition we began with does not specify the dual. However, we can decompose the space as in the previous subsection, with α in A and n in B, so that $*(\alpha\wedge n) = -\star\alpha \wedge \star n$. Now the dual of a null 1-form n in a two-dimensional space satisfies $n \wedge \star n = 0$, so $\star n \propto n$. Also $\star^2 n = n$, so $\star n = \pm n$. One of the two null directions has the plus sign and the other has the minus sign, but which is which depends on the orientation of ϵ^B. We thus have

$$*(\alpha\wedge n) = \pm\star\alpha\wedge n, \tag{A14}$$

where $\pm$ corresponds to $\star n = \mp n$. Note that the 1-form $\star\alpha$ is not unique, because any multiple of n can be added to it without changing the wedge product in equation (A14). Thus the particular $2+2$ decomposition of the space plays no role: $\star\alpha$ can be defined as any 1-form with the same norm as α and orthogonal to both α and n.

An example to be used in the text involves the retarded time coordinate

$$u = t - r^* \tag{A15}$$

on the Schwarzschild spacetime, where r^* is the radial 'tortoise coordinate' defined by $dr^* = A^{-1}dr$. In terms of u, the Schwarzschild line element takes the Eddington–Finkelstein form

$$ds^2 = -A\,du^2 - 2\,du dr + r^2 d\theta^2 + r^2\sin^2\theta\, d\varphi^2. \tag{A16}$$

A surface of constant u is an outgoing lightlike, spherical surface in spacetime, so du is a null 1-form. The dual of null 2-forms involving du is given by

$$*(d\theta\wedge du) = \pm\sin\theta\, d\varphi\wedge du, \tag{A17}$$

$$*(d\varphi\wedge du) = \mp(\sin\theta)^{-1}\, d\theta\wedge du. \tag{A18}$$

The relative sign of these two duals is fixed by the fact that $** = -1$ on 2-forms. To fix the overall sign, we adopt the orientation of $d\theta\wedge d\varphi\wedge dt\wedge dr$ for ϵ, $d\theta\wedge d\varphi$ for ϵ^A, and $dt\wedge dr = du\wedge dr$ for ϵ^B. It is then simple to see that $\star d\theta = \sin\theta\, d\varphi$. To find the sign of $\star du = \pm du$, we compute $(\star du)_a = g^{bc}(\epsilon^B)_{ca}(du)_b = g^{ur}(\epsilon^B)_{ra} = (-1)(-du)_a$. Hence, $\star du = du$, so the *upper* signs in equations (A17) and (A18) apply. Had we used instead the advanced time coordinate, a similar calculation would have yielded $\star dv = -dv$, since $g^{vr} = 1$.

A2.4 *Dual of a null 3-form*

The electric current density is a 3-form, and we shall be interested in the case in which this 3-form is null. A null 3-form has the composition $\alpha\wedge\beta\wedge n$, where n is null and orthogonal to both α and β. The dual of this is a 1-form that is orthogonal to all three of the forms in this triple wedge product, hence is a multiple of n. We can understand this, and the coefficient, using the method of orthogonal subspaces described in Section A2.2, with the $2+2$ decomposition into the subspace spanned by $\alpha\wedge\beta$ and the orthogonal subspace. Then we have $*(\alpha\wedge\beta\wedge n) = \star(\alpha\wedge\beta)\wedge\star n = |\alpha\wedge\beta|\star n$, so the proportionality factor is just the norm of the 2-form $\alpha\wedge\beta$:

$$*(\alpha\wedge\beta\wedge n) = \pm|\alpha\wedge\beta| n. \tag{A19}$$

For example, $d\theta\wedge d\varphi\wedge du$ is null, and

$$*(d\theta\wedge d\varphi\wedge du) = -(r^2\sin\theta)^{-1}\,du. \tag{A20}$$

A3 Electromagnetism and differential forms

Maxwell's equations (3) and (4) take an elegant form in the language of differential forms,

$$dF = 0, \tag{A21}$$

$$d * F = J, \tag{A22}$$

where F is the electromagnetic field F_{ab} and J is the current 3-form, related to the current vector j^a by $J_{abc} = j^m \epsilon_{mabc}$. Current conservation $dJ = 0$ is implied by $dd = 0$. The charge that flows (in spacetime) through a patch of oriented 3-surface Σ is the integral $\int_\Sigma J$. If Σ is spacelike with future orientation, this is the total charge in a spatial three-volume, whereas if Σ is timelike, it is the net charge that flows in the orientation direction across a spatial 2-surface over a lapse of time.[33]

The integral of the 2-form F on a spacelike 2-surface is the magnetic flux through that surface. The choice of orientation for the surface integral corresponds in 3+1 terms to the choice of sign for the normal to the surface when defining the flux of the magnetic field pseudo-vector.[34] This integral vanishes if the 2-surface is the (closed) boundary of a three-ball and $dF = 0$ holds everywhere in the interior. The magnetic flux through two homologous surfaces bounded by the same loop must therefore be the same, so the flux through a loop is well defined.[35]

A surface layer between two spatial regions can support a discontinuity in the field F by carrying a surface charge and/or current density. The *jump conditions* restricting such discontinuities are naturally formulated in terms of the three-dimensional world volume $\mathcal{S}$ of the surface, which allows for arbitrary motion of the surface in time. These conditions are that the pullback to $\mathcal{S}$ of the jump of F vanishes, and the pullback to $\mathcal{S}$ of the jump of $*F$ is equal to the current 2-form K on $\mathcal{S}$,

$$[F]_{\mathcal{S}} = 0, \qquad [*F]_{\mathcal{S}} = K. \tag{A23}$$

The jump is defined as the discontinuous change when crossing $\mathcal{S}$ in a given arbitrary 'jump direction. The 2-form K is defined so that when integrated on a patch of two-dimensional surface contained in $\mathcal{S}$ it yields the same result as J integrated on an infinitesimal thickening of that patch transverse to $\mathcal{S}$. The orientation ϵ_3 of the thickened patch should satisfy $v \cdot \epsilon_3 = \epsilon_2$ (up to positive rescalings), where ϵ_2 is the orientation of the patch and v is a vector pointing in the jump direction. The 2-form K may also be expressed in terms of a distributional current 3-form $J_{\rm surf}$, related to K via

$$J_{\rm surf} = \delta(s)\, ds \wedge K, \tag{A24}$$

where s is any function that is constant on $\mathcal{S}$ and increasing in the jump direction. The surface current $J_{\rm surf}$ does not depend on the jump direction.

The jump conditions (A23) are established by integrating Maxwell's equations (A21) and (A22) over the thickened patch and using Stokes' theorem, in the limit that the width of the thickening goes to zero. It is easily checked that for a surface at rest in an inertial frame in flat spacetime, these conditions agree with the familiar ones: the tangential electric field and normal magnetic field must be continuous, the jump in the normal electric field is the surface charge density, and the jump in the tangential magnetic field is the cross product of the surface current density with the unit normal to the surface in the direction the jump is defined.

[33] Given a spacetime orientation ϵ, a direction of flow across a three-surface Σ corresponds to an orientation $v \cdot \epsilon$ on Σ, where v is any vector transverse to Σ sharing the flow direction. The integral $\int_\Sigma J$ with respect to this orientation gives the current flowing across Σ in the sense of v.

[34] The surface orientation ϵ_2 is related to a vector a defining the 'outward' direction for the flux by $\epsilon_2 = a \cdot (u \cdot \epsilon)$ (up to positive rescalings), where u is a future timelike vector and ϵ is the spacetime orientation.

[35] An eternal black hole provides an example with non-trivial homology. If the black hole carries a magnetic monopole charge, then the fluxes through two surfaces spanning a loop will not be the same if the two surfaces together enclose the horizon.

APPENDIX B: POYNTING FLUX EXAMPLES

The fact that stationary field configurations can carry energy away from a source in FFE is counter to intuition from the vacuum case, where this role is normally reserved for time-dependent fields sourced by accelerated charges. In order to have finite, non-vanishing net flux from a central source, the Poynting vector (or at least its angular integral) must fall off as $1/r^2$. This indicates that E and B should fall off as $1/r$, which in vacuum occurs only for time-dependent, radiative behaviour. In the stationary, vacuum case, the E and B fields fall off as $1/r^2$ and $1/r^3$ (or $1/r^2$, if we allow monopoles), respectively. The Poynting flux is thus at best $1/r^5$ and so there is no net flux through a large sphere. (By Poynting's theorem, there is then no net flux through any closed surface surrounding the source.)

The situation is different, however, if charge current extends from the source out into the surroundings. A helpful example is an electric circuit with a battery and a resistor. In a steady state, power flows from the battery into the resistor, and the energy is carried by a Poynting flux largely in vacuum between battery and resistor (e.g. Galili & Goihbarg 2005). A force-free plasma is, in effect, a distributed circuit, in which a similar effect can take place. In particular, considering for example a rotating conductor as a localized energy source, if charge–current extends to infinity, then $1/r$ behaviour for the fields is possible in the stationary case, provided the charge and current fall off as $1/r^2$. This entails an unphysical infinite total charge on some (most) conical wedges of space, even though the total charge may be zero. However, in reality, the force-free magnetosphere extends only a finite distance, and quantities that are finite in the infinite-r limit (such as the net flux) should adequately represent the physics of a real configuration that extends to large but finite r.

In the remainder of this appendix, we consider three simple, quantitative examples that help to elucidate the role of the current in allowing for energy transport by Poynting flux in stationary fields. The examples are a plane symmetric vacuum solution, a coaxial cable, and a cylindrical plasma-filled waveguide. The last two examples were used by Punsly (2008) to illustrate features and provide intuition about the physics of MHD magnetospheres, and we adapt them here to the force-free setting.

B1 Planar symmetry in vacuum

A simple vacuum solution with planar symmetry is given by

$$F^{\rm plane} = f(u)\, dx \wedge du, \tag{B1}$$

where (t, x, y, z) are Minkowski coordinates, $u = t - z$, and f is an arbitrary function. (To check that Maxwell's equations are satisfied, note that $dF^{\rm plane} = 0$ follows immediately from $du \wedge du = 0$, and $F^{\rm plane}$ is a null 2-form so (cf. Appendix A2.3) $*F^{\rm plane} = -f(u) dy \wedge du$, which is similarly closed, $d * F^{\rm plane} = 0$.) Notice the similarity to the non-vacuum force-free solution (30). If the function $f(u)$ is sinusoidal, $f \sim \sin(\omega u)$, equation (B1) would typically be described as a plane wave polarized in the x-direction, but any function $f(u)$ gives a solution. The energy flux (Poynting vector) is proportional to $f(u)^2$ and persists in the stationary case $f =$ const, for which equation (B1) represents static crossed electric and magnetic fields filling all of space.

This stationary case is evidently an ‘energy-transporting field’, but it has no physical source. Nevertheless, a solution with global planar symmetry reveals a possible *local* behaviour of electromagnetic fields. Vacuum electrodynamics does not allow this local behaviour to be extended globally with a localized source. On the other hand, the force-free case does allow such a global extension. The correspondence can be made precise by noting that equation (B1) arises in a planar limit of equation (30), where z and x, y are identified with the normal and tangential directions (respectively) to the sphere about a point. The charge-current vanishes in this limit.

B2 Coaxial cable

A coaxial cable consists of a pair of concentric, cylindrical conductors, and supports transverse electromagnetic (TEM) modes whose behaviour is closely analogous to the planar case (B1). The relevant solution is

$$F^{\mathrm{coax}} = \frac{f(u)}{\rho}\, d\rho \wedge du, \tag{B2}$$

where ρ is the cylindrical radius of the x, y plane. The demonstration that it satisfies the vacuum Maxwell equations is essentially the same as for the planar case (B1). This field tensor corresponds to a radial electric field and a circumferential magnetic field, both of which are transverse to the propagation direction. Assuming that there is no radial magnetic field in the conductor, the boundary condition at the vacuum/conductor interface is that the pullback of F^{coax} to the world volume of the conducting walls vanishes (see discussion of boundary conditions in Section A3). The world volume of a cylinder contains no radial vector, while F^{coax} has a $d\rho$ factor, so this is satisfied. Thus F^{coax} is indeed a TEM mode, which propagates at the speed of light, and is terminated at the cylindrical conductors on which it induces charge and current. If there were no central conductor, the field would be singular on the axis (at $\rho = 0$).

As in the planar case, oscillatory solutions $f(u) \sim \sin u$ are usually considered, viewed as transmission modes in a coaxial cable. However, also as in the planar case, the local energy flux (Poynting vector) is proportional to $f(u)^2$, and persists for any $f(u)$. The stationary solution is just static crossed electric and magnetic fields, sourced by an infinite line charge and current in the conductors.

While energy transport is not physically realizable in the strictly planar configuration (B1), it *is* realizable in the coaxial case, even for static fields. Imagine embedding a finite-length coaxial waveguide in a longitudinal magnetic field, and attaching a conducting disc to one end and a resistor connecting the inner and outer cylinders at the other. If the conducting disc is set spinning (while the walls are fixed), it becomes a ‘Faraday disc’ electric generator, driving current in the z-direction along the inner cylinder and in the opposite direction along the outer cylinder. Far from either end, and after initial transients, field takes the form of a uniform magnetic field in the z-direction, $B_0 \rho\, d\rho \wedge d\varphi$, plus a transverse part,

$$F^{\mathrm{coax\&res}} = \frac{1}{2\pi\rho}\, d\rho \wedge (\lambda\, dt + I\, dz). \tag{B3}$$

The constant λ is the linear charge density, which determines the strength of the radial electric field, and the constant I is the current along the z-direction in the central conductor, which determines the azimuthal magnetic field strength. The linear charge density is fixed by the voltage drop V between the walls (which is in turn fixed by the disc rotation rate and magnetic field strength B_0), while the current is given by Ohm's law $V = IR$ in terms of the resistance R of the resistor. The static Poynting vector points from the disc to the resistor, and we may regard the Poynting flux as delivering the energy from the agent spinning the wheel to the resistor on the other end.

The solution (B2) with $f(u)$ constant arises when we take the special case of equation (B3) with $I = -\lambda$. This case is selected by some unremarkable particular value for the resistance, but it can also be selected by a sort of ‘no outer boundary’ condition. Suppose the waveguide is infinitely long, and that the Faraday disc starts turning at time $t = 0$. Then at any time the fields should remain zero beyond some distance from the wheel, and we may model this by $F^{\mathrm{propagating}} = \theta(vt - z) F^{\mathrm{coax\&res}}$, where θ is the Heaviside step function and v is a constant speed. The Maxwell equations then imply $d\theta \wedge F^{\mathrm{coax\&res}} = 0$ and $d\theta \wedge {*F^{\mathrm{coax\&res}}} = 0$, which in turn imply $v = 1$ and $I = -\lambda$. (Note that this implies that the current is null.) That is, the wavefront propagates at the speed of light, and behind it we are left with the solution (B2) in the static case.

B3 Plasma-filled waveguide

Now suppose we take out the central cylinder in the coaxial waveguide and fill the cylinder with force-free plasma. Instead of the current being carried on the central cylinder, it is distributed throughout the plasma. The field must satisfy the perfect conductor boundary condition on the outer cylinder and on the Faraday disc rotating with angular velocity Ω, and we suppose it has a uniform magnetic field of magnitude B_0 in the z-direction. In a stationary, axisymmetric configuration, the total field must then have the form

$$F^{\mathrm{waveguide}} = B_0 \rho\, d\rho \wedge (d\varphi - \Omega dt + d\psi_2(\rho, z)) \tag{B4}$$

as explained in Appendix D. The function ψ_2 can be determined by the requirement that the field satisfies the two force-free conditions. The first one is $d\rho \wedge d * F = 0$, which implies $\psi_2 = f(\rho) z$, and the second one (or the stream equation) then implies $f(\rho) = \pm\Omega$. The result (after rejecting a solution that has a line current singularity on the axis) is $d\psi_2 = \pm\Omega\, dz$; hence,

$$F^{\mathrm{waveguide}} = B_0 \rho\, d\rho \wedge (d\varphi - \Omega d(t \mp z)). \tag{B5}$$

One thus has a uniform magnetic field superposed with a Poynting flux in either the positive or negative z-direction. This is a precise cylindrical analogue of the Michel monopole solution (41) for a rotating, conducting sphere generating a spherical outgoing flux. As in that solution, the current associated with equation (B5) is null, i.e. the charge density and three-current vector have equal magnitude. The three-current is uniform and in the z-direction. The force-free condition is satisfied by virtue of a balance between the radial inward Lorentz force acting on the three-current and the repulsive radial electric force acting on the charge density. Note that, in contrast to the coaxial vacuum waveguide (B3), the current and charge density in the force-free plasma-filled waveguide are not independently specifiable.

APPENDIX C: KERR METRIC

In this appendix, we present a number of useful formulae related to the Kerr spacetime. For a more detailed treatment, see (e.g.) Poisson (2004). The Kerr metric for a black hole of mass M and angular momentum aM is given in BL coordinates by

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar\sin^2\theta}{\Sigma} dt d\varphi + \frac{A}{\Sigma}\sin^2\theta\, d\varphi^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \quad \text{(C1)}$$

$$= -\frac{\Sigma\Delta}{A}dt^2 + \frac{A}{\Sigma}\sin^2\theta\,(d\varphi - \Omega_Z dt)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2, \quad \text{(C2)}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2Mr + a^2 \quad \text{(C3)}$$

$$A = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta, \quad \Omega_Z = 2Mar/A. \quad \text{(C4)}$$

The inner/outer horizons $r_\pm$ are the roots of $\Delta = (r - r_+)(r - r_-)$, $r_\pm = M \pm \sqrt{M^2 - a^2}$. The Killing field $\partial_t + \Omega_H\partial_\varphi$ generates the horizon, where $\Omega_H = a/(r_+^2 + a^2)$ is called the horizon angular velocity. The relevant metric determinants are given by

$$\sqrt{-g^T} = \sqrt{\Delta}\sin\theta, \qquad \sqrt{g^P} = \Sigma/\sqrt{\Delta}, \quad \text{(C5)}$$

$$\sqrt{-g} = \sqrt{-g^T g^P} = \Sigma\sin\theta, \quad \text{(C6)}$$

where 'T' and 'P' refer to toroidal ($t\varphi$) and poloidal ($r\theta$), respectively. The inverse metric components are

$$g^{tt} = -A/(\Sigma\Delta),\ g^{t\varphi} = -2Mar/(\Sigma\Delta),$$
$$g^{\varphi\varphi} = (\Delta - a^2\sin^2\theta)/(\Sigma\Delta\sin^2\theta),\ g^{rr} = \Delta/\Sigma,\ g^{\theta\theta} = 1/\Sigma. \quad \text{(C7)}$$

The BL coordinates are singular on the future and past event horizons. The ingoing Kerr coordinates v and $\tilde{\varphi}$ are regular on the future horizon (but not the past horizon), and are related to t and φ by

$$dt = dv - [(r^2 + a^2)/\Delta]dr, \quad \text{(C8)}$$

$$d\varphi = d\tilde{\varphi} - (a/\Delta)dr. \quad \text{(C9)}$$

(In Schwarzschild spacetime $a = 0$, v is the ingoing Eddington–Finklekstein coordinate.) The Kerr metric becomes

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dv^2 + 2dvdr - 2a\sin^2\theta dr d\tilde{\varphi} - \frac{4Mar\sin^2\theta}{\Sigma}dvd\tilde{\varphi} + \frac{A}{\Sigma}\sin^2\theta d\tilde{\varphi}^2 + \Sigma d\theta^2 \quad \text{(C10)}$$

$$= -dv^2 + 2dr(dv - a\sin^2\theta d\tilde{\varphi}) + (r^2 + a^2)\sin^2\theta d\tilde{\varphi}^2 + \Sigma d\theta^2 + \frac{2Mr}{\Sigma}(dv - a\sin^2\theta d\tilde{\varphi})^2. \quad \text{(C11)}$$

It follows immediately by inspection of equation (C11) that the 1-form $dv - a\sin^2\theta d\tilde{\varphi}$ is equal to $(\partial_r)_a$, is null, and is orthogonal to dv, $d\tilde{\varphi}$, and $d\theta$. We note that $(\partial_r)^a$ is proportional to the ingoing principal congruence. The inverse metric components are

$$g^{vv} = (a^2\sin^2\theta)/\Sigma,\ g^{vr} = (r^2 + a^2)/\Sigma,\ g^{v\tilde{\varphi}} = a/\Sigma,$$
$$g^{rr} = \Delta/\Sigma,\ g^{r\tilde{\varphi}} = a/\Sigma,\ g^{\theta\theta} = 1/\Sigma,\ g^{\tilde{\varphi}\tilde{\varphi}} = 1/(\Sigma\sin^2\theta). \quad \text{(C12)}$$

Alternatively, one may use outgoing Kerr coordinates u and $\bar{\varphi}$, defined by

$$dt = du + [(r^2 + a^2)/\Delta]dr, \quad \text{(C13)}$$

$$d\varphi = d\bar{\varphi} + (a/\Delta)dr. \quad \text{(C14)}$$

(In Schwarzschild spacetime $a = 0$, u is the outgoing Eddington–Finklekstein coordinate.) These coordinates are regular on the past horizon (but not the future horizon). They are useful for describing outgoing radiation processes, such as the Poynting flux solution discussed in the text. Analogous formulae for the metric may be obtained by exploiting the time reversal symmetry $t \to -t$ and $\varphi \to -\varphi$ of the Kerr metric. That is, one sends $v \to -u$ and $\tilde{\varphi} \to -\bar{\varphi}$ in equations (C10) and (C11), which yields

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) du^2 - 2dudr + 2a\sin^2\theta dr d\bar{\varphi} - \frac{4Mar\sin^2\theta}{\Sigma}dud\bar{\varphi} + \frac{A}{\Sigma}\sin^2\theta d\bar{\varphi}^2 + \Sigma d\theta^2 \quad \text{(C15)}$$

$$= -du^2 - 2dr(du - a\sin^2\theta d\bar{\varphi}) + (r^2 + a^2)\sin^2\theta d\bar{\varphi}^2 + \Sigma d\theta^2 + \frac{2Mr}{\Sigma}(du - a\sin^2\theta d\bar{\varphi})^2. \quad \text{(C16)}$$

As in the ingoing case, we see that the 1-form $du - a\sin^2\theta d\bar{\varphi}$ is equal to $-(\partial_r)_a$, is null, and is orthogonal to du, $d\bar{\varphi}$, and $d\theta$. The vector $(\partial_r)^a$ is proportional to the outgoing principal congruence. The inverse metric components are

$$g^{uu} = (a^2\sin^2\theta)/\Sigma,\ g^{ur} = -(r^2 + a^2)/\Sigma,\ g^{u\bar{\varphi}} = a/\Sigma,$$
$$g^{rr} = \Delta/\Sigma,\ g^{r\bar{\varphi}} = -a/\Sigma,\ g^{\theta\theta} = 1/\Sigma,\ g^{\bar{\varphi}\bar{\varphi}} = 1/(\Sigma\sin^2\theta). \quad \text{(C17)}$$

APPENDIX D: EULER POTENTIALS WITH SYMMETRY

When a degenerate electromagnetic field has a symmetry, the corresponding Euler potentials do not in general have the same symmetry, but the form of their dependence on the ignorable coordinates is very constrained. Uchida (1997b) solved the problem of finding their form in the presence of one symmetry or two commuting symmetries. In this appendix, we follow his treatment, making use of differential forms to streamline the analysis. In the first and second subsections, we consider the case of one and two symmetries, respectively, and in the final subsection, we apply the results to the case of stationary axisymmetry.

D1 One symmetry

Suppose the vector field X generates a symmetry of the field, $\mathcal{L}_X F = 0$. Since the force-free equations involve the metric, X should presumably also generate a symmetry of the metric, i.e. it should be a Killing vector. However, the arguments in this section rely only on the symmetry properties of F and the metric-independent subset of Maxwell's equations, Faraday's law, $dF = 0$.

Cartan's 'magic formula' [$\mathcal{L}_X = (X\cdot)d + d(X\cdot)$] and $dF = 0$ imply that this symmetry condition is equivalent to $d(X\cdot F) = 0$, which implies (modulo homological obstructions) that $X\cdot F$ is exact, i.e.

$$X\cdot F = df \quad \text{(D1)}$$

for some function f. A degenerate field can be expressed in terms of Euler potentials as $F = d\phi_1 \wedge d\phi_2$, in terms of which equation (D1) becomes

$$(X \cdot d\phi_1)\, d\phi_2 - (X \cdot d\phi_2)\, d\phi_1 = df. \tag{D2}$$

Since the differential of f can be written as a linear combination of $d\phi_1$ and $d\phi_2$, evidently $f = f(\phi_1, \phi_2)$. If $X \cdot F = 0$, then both potentials are invariant under the symmetry, but in general that is not the case.

Now recall that the Euler potentials are not unique: we may choose any other pair $(\tilde{\phi}_1, \tilde{\phi}_2)$ such that $d\tilde{\phi}_1 \wedge d\tilde{\phi}_2 = d\phi_1 \wedge d\phi_2$, which amounts to the requirement that the map $(\phi_1, \phi_2) \to (\tilde{\phi}_1, \tilde{\phi}_2)$ have unit Jacobian determinant. We may exploit this freedom to choose potentials that are adapted to the symmetry. In fact, if $df \neq 0$ we may choose $\tilde{\phi}_1 = -f$ to be one of a new pair of potentials. The unit Jacobian requirement then imposes a single, first-order partial differential equation on the other potential $\tilde{\phi}_2(\phi_1, \phi_2)$, which can be satisfied by integration with respect to ϕ_2, provided $\partial f/\partial\phi_1 \neq 0$, or by integration with respect to ϕ_1, provided $\partial f/\partial\phi_2 \neq 0$. The new pair will satisfy the analogue of equation (D2), with the same function f since that was defined in equation (D1) without reference to the potentials. In terms of this new pair, equation (D2) becomes

$$(X \cdot d\tilde{\phi}_1)\, d\tilde{\phi}_2 - (X \cdot d\tilde{\phi}_2)\, d\tilde{\phi}_1 = -d\tilde{\phi}_1, \tag{D3}$$

from which we read off the symmetry properties

$$X \cdot d\tilde{\phi}_1 = 0, \quad X \cdot d\tilde{\phi}_2 = 1. \tag{D4}$$

One of the potentials can thus be taken to be invariant, while the other has a constant, unit derivative along the symmetry flow.

D2 Two commuting symmetries

Suppose now that there are two commuting symmetry vectors, X and Y, such that $[X, Y] = \mathcal{L}_X Y = -\mathcal{L}_Y X = 0$, and $\mathcal{L}_X F = \mathcal{L}_Y F = 0$. Then, as in (D1), we also have

$$Y \cdot F = dg \tag{D5}$$

for some g. To assess the relation between df and dg, we compute their wedge product:

$$df \wedge dg = (X \cdot F) \wedge (Y \cdot F) = (Y \cdot X \cdot F) F. \tag{D6}$$

The scalar $Y \cdot X \cdot F$ ($=F_{ab}X^a Y^b$) must be constant:

$$d(Y \cdot X \cdot F) = \mathcal{L}_Y(X \cdot F) - Y \cdot d(X \cdot F) = 0. \tag{D7}$$

(The first term can be expanded using the Leibniz rule for the Lie derivative, and both the resulting terms vanish, while the second term vanishes since $d(X \cdot F) = \mathcal{L}_X F$.) Hence, there are two cases to consider: $Y \cdot X \cdot F = 0$, which Uchida called Case I, and $Y \cdot X \cdot F \neq 0$, which he called Case II. Case II will not be relevant when the two Killing fields are time translation and rotations around an axis since, as explained below, no such field configuration is regular on the axis, even if it is not force-free everywhere.

If both $X \cdot F$ and $Y \cdot F$ vanish, then both potentials are simply invariant under both symmetries. Suppose now that $X \cdot F \neq 0$. Considering first only the vector field X, we may then conclude, as in the one-symmetry case, that one of the Euler potentials may be chosen to be $\phi_1 = -f$. In Case I, we have $df \wedge dg = 0$, from which it follows that $g = g(\phi_1)$. The symmetry condition (D5) for Y then reads

$$Y \cdot F = (Y \cdot d\phi_1)\, d\phi_2 - (Y \cdot d\phi_2)\, d\phi_1 = g'(\phi_1) d\phi_1. \tag{D8}$$

It follows from equations (D4) and (D8), without the tildes, that the potentials have the symmetry properties

$$X \cdot d\phi_1 = 0, \quad X \cdot d\phi_2 = 1, \tag{D9}$$

$$Y \cdot d\phi_1 = 0, \quad Y \cdot d\phi_2 = \kappa(\phi_1), \tag{D10}$$

where $\kappa(\phi_1) = -g'(\phi_1)$.

The function $\kappa(\phi_1)$ has an interesting geometric interpretation. The potentials are both invariant with respect to the flow of the vector field

$$Z = Y - \kappa(\phi_1) X; \tag{D11}$$

hence, Z generates a symmetry of the field and is tangent to the field surfaces. If X and Y are spacetime Killing vectors, then Z is also only if $\kappa(\phi_1)$ is constant, but it is always a Killing vector of the induced metric on the field sheets because ϕ_1 is constant on each field sheet. That is, Z is a field sheet Killing vector.

In Case II, both $X \cdot F$ and $Y \cdot F$ must be non-vanishing, and according to equations (D6) and (D7), we may choose the Euler potentials to be $\phi_1 = -f$ and $\phi_2 = g/\lambda$, where $\lambda = Y \cdot X \cdot F$. Then equations (D2), without the tildes, and (D8) imply the symmetry conditions

$$X \cdot d\phi_1 = 0, \quad X \cdot d\phi_2 = 1, \tag{D12}$$

$$Y \cdot d\phi_1 = \lambda, \quad Y \cdot d\phi_2 = 0. \tag{D13}$$

In this case, no linear combination of X and Y is tangent to the field surfaces.

D3 Stationary axisymmetry

In stationary axisymmetry, there are two commuting Killing fields, ∂_t and ∂_φ, where t and φ are Killing coordinates in some coordinate system. (For example, they could be the usual BL coordinates for the Kerr metric, but they could also be, say, the ingoing Kerr coordinate v and azimuthal angle $\tilde{\varphi}$, respectively. These differ by the addition of functions of the coordinate r, so yield different choices for the potentials below.) Case II does not occur for such fields, since ∂_φ vanishes on the symmetry axis, so the constant $\partial_t \cdot \partial_\varphi \cdot F$ always vanishes. Thus we consider only Case I.

If $\partial_t \cdot F$ and $\partial_\varphi \cdot F$ both vanish, then both the potentials are independent of t and φ; hence, the field tensor is purely poloidal ($F \sim dr \wedge d\theta$).

Now let $X = \partial_\varphi$ and $Y = \partial_t$, in the notation of the previous subsection. If $\partial_\varphi \cdot F \neq 0$, we have from equations (D9) and (D10) that for stationary, axisymmetric fields, the Euler potentials may always be taken to have the form

$$\phi_1 = \psi(r, \theta), \quad \phi_2 = \psi_2(r, \theta) + \varphi - \Omega_{\mathrm{F}}(\psi) t \tag{D14}$$

for some function $\Omega_{\mathrm{F}}(\psi)$. We have replaced the notation κ by $-\Omega_{\mathrm{F}}$ since, as explained in the text, Ω_{F} corresponds to the 'angular velocity of the field lines'.

An exceptional case not mentioned by Uchida occurs if instead $\partial_\varphi \cdot F = 0$, i.e. if there is no poloidal magnetic field. Then we must reverse the roles of X and Y before invoking the results of the previous subsection, and the Euler potentials may always be taken to have the form

$$\phi_1 = \psi(r, \theta), \quad \phi_2 = \psi_2(r, \theta) + t. \tag{D15}$$

This can be viewed as a singular limit of equation (D14) in which Ω_{F} and ψ_2 go to infinity while ψ goes to zero, with the products held finite.

APPENDIX E: CONSERVED NOETHER CURRENT ASSOCIATED WITH A SYMMETRY

Let L be the Lagrangian 4-form of some field theory, i.e. it depends on various dynamical fields and perhaps on some background fields. Suppose the vector field ξ^a generates a symmetry of the dynamics, in the sense that when the dynamical fields are varied by their Lie derivative with respect to ξ^a, the net induced variation of L is simply the Lie derivative of L itself as a 4-form, $\mathcal{L}_\xi L = d(\xi \cdot L)$. Since this is a total derivative, the variation of the action $\int L$ will be at most a boundary term. Typically, this will be the case if the background fields in L have zero Lie derivative with respect to ξ. In this case, there is a Noether current 3-form $\mathcal{J}_\xi$ that is closed (conserved) when the dynamical equations of motion are satisfied. To see how this comes about, and how J_ξ is defined, we can just make the two variations in question.

Let Φ stand for all the dynamical fields, and let E be their equations of motion form. Then the two variations are

$$\delta_{\mathrm{dynamical}} L = E\,\mathcal{L}_\xi \Phi + d\theta(\mathcal{L}_\xi \Phi) \tag{E1}$$

$$\delta_{\mathrm{total}} L = \mathcal{L}_\xi L = d(\xi \cdot L). \tag{E2}$$

The 3-form θ depends linearly on the variation $\mathcal{L}_\xi \Phi$, and also on the fields. It is called the symplectic potential, and its integral over a spacelike (Cauchy) surface is the field-theoretic analogue of $p_i\, dq^i$ in mechanics. Setting these two variations equal, we have

$$d(\theta(\mathcal{L}_\xi \Phi) - \xi \cdot L) = -E\,\mathcal{L}_\xi \Phi. \tag{E3}$$

When the equations of motion are satisfied, $E = 0$, the Noether current $\mathcal{J}_\xi$ is closed, where

$$\mathcal{J}_\xi = \theta(\mathcal{L}_\xi \Phi) - \xi \cdot L. \tag{E4}$$

For instance, when ξ is the time-translation vector ∂_t, the current $\mathcal{J}_\xi$ is the field theory analogue of $p\dot{q} - L$, the canonical Hamiltonian.

If the only background field is the spacetime metric, then we get a conserved current for every Killing vector, and perhaps more conserved currents, if L does not depend on all aspects of the metric. For example, in vacuum or force-free electromagnetism, L depends only on the conformal structure, so also conformal Killing vectors produce conserved currents. Note that for a field configuration such that the total variation $\mathcal{L}_\xi L = d(\xi \cdot L)$ vanishes, the second term in the Noether current (E4) is automatically conserved by itself, without appeal to field equations. In this case, the first term also is conserved by itself, when the equations of motion hold.

E1 Noether currents for electromagnetic field

In the usual Lagrangian formulation of electrodynamics, the dynamical field is the vector potential A, the field strength is $F = dA$, and the Lagrangian 4-form is $-\frac{1}{2} F \wedge *F$, plus any interaction term. The Lagrangian is invariant (possibly only up to addition of an exact form, i.e. a total derivative) under gauge transformations of the potential, $A \to A + d\lambda$, where λ is any scalar function λ.

The electromagnetic Noether current associated with a vector field ξ can be constructed as above, yielding

$$\mathcal{J}_\xi = -\mathcal{L}_\xi A \wedge *F + \tfrac{1}{2}\xi \cdot (F \wedge *F). \tag{E5}$$

If A is treated as an ordinary 1-form, substituting

$$\mathcal{L}_\xi A = \xi \cdot F + d(\xi \cdot A) \tag{E6}$$

into equation (E5), the result is not gauge invariant, although it is still a correct contribution to a conserved current. The terms that violate gauge invariance consist of one that vanishes by the equations of motion, and an exact form that is automatically conserved by itself. One could drop those terms to arrive at a gauge-invariant Noether current. A more insightful way to arrive at the same current is to note that the response of A to the diffeomorphism generated by ξ is defined only up to a gauge transformation. We can define a gauge-invariant response by omitting the gauge transformation term $d(\xi \cdot A)$ from equation (E6), yielding a 'gauge-invariant Lie derivative',

$$\mathcal{L}'_\xi A = \xi \cdot F. \tag{E7}$$

The reasoning leading to the conserved Noether current (E4) can be applied using this variation, which leads directly to the Noether current

$$\mathcal{J}_\xi = -(\xi \cdot F) \wedge *F + \tfrac{1}{2}\xi \cdot (F \wedge *F). \tag{E8}$$

Note that for configurations that share the Killing symmetry, i.e. such that $\mathcal{L}_\xi F = 0$, the second term in equation (E8) is conserved independently of field equations, so the first term is conserved by itself when the field equations hold.

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