Relativistic tidal properties of neutron stars

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We study the various linear responses of neutron stars to external relativistic tidal fields. We focus on three different tidal responses, associated to three different tidal coefficients: (i) a gravito-electric-type coefficient $G\mu_\ell = [\text{length}]^{2\ell+1}$ measuring the ℓth-order mass multipolar moment $GM_{a_1 \ldots a_\ell}$ induced in a star by an external ℓth-order gravito-electric tidal field $G_{a_1 \ldots a_\ell}$; (ii) a gravito-magnetic-type coefficient $G\sigma_\ell = [\text{length}]^{2\ell+1}$ measuring the ℓth spin multipole moment $GS_{a_1 \ldots a_\ell}$ induced in a star by an external ℓth-order gravito-magnetic tidal field $H_{a_1 \ldots a_\ell}$; and (iii) a dimensionless "shape" Love number h_ℓ measuring the distortion of the shape of the surface of a star by an external ℓth-order gravito-electric tidal field. All the dimensionless tidal coefficients $G\mu_\ell/R^{2\ell+1}$, $G\sigma_\ell/R^{2\ell+1}$, and h_ℓ (where R is the radius of the star) are found to have a strong sensitivity to the value of the star's "compactness" $c \equiv GM/(c_0^2 R)$ (where we indicate by c_0 the speed of light). In particular, $G\mu_\ell/R^{2\ell+1} \sim k_\ell$ is found to strongly decrease, as c increases, down to a zero value as c is formally extended to the "black hole (BH) limit" $c^{\text{BH}} = 1/2$. The shape Love number h_ℓ is also found to significantly decrease as c increases, though it does *not* vanish in the formal limit $c \to c^{\text{BH}}$, but is rather found to agree with the recently determined shape Love numbers of black holes. The formal vanishing of μ_ℓ and σ_ℓ as $c \to c^{\text{BH}}$ is a consequence of the no-hair properties of black holes. This vanishing suggests, but in no way proves, that the effective action describing the gravitational interactions of black holes may not need to be augmented by nonminimal worldline couplings.

I. MOTIVATION AND INTRODUCTION

Coalescing binary neutron stars are one of the most important (and most secure) targets of the currently operating network of ground-based detectors of gravitational waves. A key scientific goal of the detection of the gravitational-wave signal emitted by coalescing binary neutron stars is to acquire some knowledge on the equation of state (EOS) of neutron-star matter. Recent breakthroughs in numerical relativity have given an example of the sensitivity of the gravitational-wave signal to the EOS of the neutron stars [1–4]. However, this sensitivity is qualitatively striking only during and after the merger of the two neutron stars, i.e. for gravitational-wave (GW) frequencies $f_{\text{GW}} \gtrsim 1000$ Hz, which are outside the most sensitive band of interferometric detectors. It is therefore important to study to what extent the gravitational-wave signal emitted within the most sensitive band of interferometric detectors (around $f_{\text{GW}} \sim 150$ Hz) is *quantitatively* sensitive to the EOS of neutron stars. In such a regime, the two neutron stars are relatively far apart, and the problem can be subdivided into three separate issues, namely:

  (i) to study the response of each neutron star to the tidal field generated by its companion;
  (ii) to incorporate the corresponding tidal effects within a theoretical framework able to describe the gravitational-wave signal emitted by inspiralling compact binaries; and

  (iii) to assess the measurability of the tidal effects within the signal seen by interferometric detectors.

A first attack on these three issues has been recently undertaken by Flanagan and Hinderer [5,6]. [See also [4] for an attempt at addressing the third issue.] Our aim in this work, and in subsequent ones, is to improve the treatment of Refs. [5,6] on several accounts. The present work will focus on the first issue (i) above, namely, the study of the tidal response of a neutron star. Our treatment will complete the results of [6] in several directions. First, we shall study not only the usually considered "electric-type," "tidal," "quadrupolar" Love number $G\mu_2 = \frac{2}{3}k_2R^5$, but also several of the other tidal coefficients of a self-gravitating body. This includes not only the higher multipolar analogues $G\mu_\ell \propto k_\ell R^{2\ell+1}$ of μ_2, but their "magnetic-type" analogues $G\sigma_\ell$ (first introduced in [7]), as well as their (electric) "shape-type" kin h_ℓ. Second, we shall study in detail the strong sensitivity of these tidal coefficients to the *compactness* parameter[1] $c \equiv GM/c_0^2 R$ of the neutron star. Note, indeed, that the published version of Ref. [6] was marred by errors which invalidate the conclusions drawn there that k_2 has only a mild dependence on the compactness c [see, e.g., Eq. (27) or Fig. 2 there]. [These errors were later corrected in an erratum,

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[1]To avoid confusion with the compactness, we sometimes denote the velocity of light as c_0.

which, however, did not correct Eq. (27), nor Fig. 2.] We shall interpret below the strong sensitivity of μ_ℓ and σ_ℓ to c, and contrast the vanishing of μ_ℓ and σ_ℓ in the formal "black hole limit" $c \to 1/2$, to the nonvanishing of the "shape" Love numbers h_ℓ in the same limit. In order to approach the black hole limit (which is, however, disconnected from the perfect-fluid star models), we shall particularly focus on the incompressible models which can reach the maximum compactness of fluid models, namely $c_{\max} = 4/9$.

In subsequent works, we shall show how to incorporate the knowledge acquired here on the various tidal responses of neutron stars into the effective one body framework. Indeed, recent investigations [8,9] have shown that the effective one body formalism is the most accurate theoretical way of describing the motion and radiation of inspiralling compact binaries.

This paper is organized as follows: Section II is an introduction to the various possible tidal responses of a neutron star. Section III discusses the relevant equations to deal with stationary perturbations of neutron stars that are then used in Secs. IV and V to compute the electric-type (μ_ℓ) and magnetic-type (σ_ℓ) tidal coefficients. Section VI is devoted to the computation of the shape Love numbers h_ℓ. Sections VII, VIII, and IX provide explicit numerical results related to μ_ℓ, σ_ℓ, and h_ℓ, respectively. The concluding section, Sec. X, summarizes our main results.

II. THE VARIOUS TIDAL RESPONSES OF A NEUTRON STAR

Let us first recall that the motion and radiation of a system of well separated, strongly self-gravitating ("compact"), bodies can be theoretically investigated by a "matching" approach which consists in splitting the problem into two subproblems:

  (i) the outer problem where one solves field equations in which the bodies are "skeletonized" by worldlines endowed with some global characteristics (such as mass, spin, or higher-multipole moments), and

  (ii) the inner problem where one obtains the near-worldline behavior of the outer solution from a study of the influence of the other bodies on the structure of the fields in an inner world tube around each body.

This matching approach has been used: to obtain the dynamics of binary black holes at low post-Newtonian orders [10–12], to prove that the tidal deformation of compact bodies will start to introduce in the outer problem a dependence on the internal structure of the constituent bodies (measured by a "relativistic generalization of the second Love number" k) only at the fifth post-Newtonian (5 PN) level [13], and to derive the dynamics of compact bodies in alternative theories of gravitation [14–16]. Finite-size corrections to the leading skeletonized dynamics can

be taken into account by adding nonminimal worldline couplings to the effective action [17,18].

Let us start by considering the "inner problem" for a neutron star, i.e., the influence of the other bodies in the considered gravitationally interacting system.[2] As explained, e.g., in Ref. [13], the matching method uses a multichart approach which combines the information contained in several expansions. One uses both a global weak-field expansion $g_{\mu\nu}(x) = \eta_{\mu\nu} + h^{(1)}_{\mu\nu}(x) + h^{(2)}_{\mu\nu}(x) + \ldots$ for the outer problem, and several local expansions of the type

$$G^A_{\alpha\beta}(X^\gamma_A) = G^{(0)}_{\alpha\beta}(X^\gamma_A) + H^{(1)}_{\alpha\beta}(X^\gamma_A) + \ldots \quad (1)$$

for each inner problem. Here, $G^{(0)}_{\alpha\beta}$ denotes the metric generated by an isolated neutron star, as seen in a local inner coordinate system X^α_A, which is nonlinearly related to the global ("barycentric") coordinate system x^μ by an expansion of the form

$$x^\mu = z^\mu_A(X^0_A) + e^\mu_a(X^0_A)X^a_A + \tfrac{1}{2}f^\mu_{ab}(X^0_A)X^a_A X^b_A + \ldots \quad (2)$$

Here, the suffix $A = 1, 2, \ldots, N$ labels the considered member of the N-body system, while $H^{(1)}_{\alpha\beta}(X^\gamma_A)$ denotes the metric perturbation, seen in the local A frame, due to the combined influence of the various companions $B \neq A$ of A. In the leading approximation $H^{(1)}_{\alpha\beta}$ is a sum of separate contributions due to each $B \neq A$: each contribution then contains both the far-away field generated by the B worldline, its deformation as it propagates on the "background" metric $G^{(0)}_{\alpha\beta}$ generated by A, and the tidally-induced effect of the deformation of A by the effect of B.

Before tackling the technical problem of computing $H^{(1)}_{\alpha\beta}$, let us recall the general structure of tidal expansions in general relativity [7,12,19]. We will use here the notation and results of the general multichart approach to the general relativistic dynamics of N self-gravitating, deformable bodies developed by Damour, Soffel, and Xu (DSX) [7,19–21].

Using the DSX notation (with $T \equiv X^0_A/c$),

$$G^A_{00}(X) = -\exp(-2W^A/c^2), \quad (3)$$

$$G^A_{0a}(X) = -\frac{4}{c^3}W^A_a, \quad (4)$$

$$E^A_a(X) = \partial_a W^A + \frac{4}{c^2}\partial_T W^A_a, \quad (5)$$

$$B^A_a(X) = \epsilon_{abc}\partial_b(-4W^A_c), \quad (6)$$

one defines, in the local frame of each body A, two sets of "gravito-electric" and "gravito-magnetic" relativistic ti-

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[2]In the following, we shall have in mind a binary system made either of two neutron stars or of a neutron star and a black hole.

dal moments, G_L^A and H_L^A, respectively, as[3]

$$G_L^A(T) \equiv [\partial_{\langle L-1} \bar{E}_{a_\ell\rangle}^A(T, \mathbf{X})]_{X^a \to 0}, \qquad (7)$$

$$H_L^A(T) \equiv [\partial_{\langle L-1} \bar{B}_{a_\ell\rangle}^A(T, \mathbf{X})]_{X^a \to 0}, \qquad (8)$$

where $\bar{E}_a^A$ and $\bar{B}_a^A$ denote the externally-generated parts of the local gravito-electric and gravito-magnetic fields E_a^A and B_a^A. In the presently considered approximation where $G_{\alpha\beta}^{A(0)}$ is stationary, and where it is enough to consider the linearized, multipole expanded, perturbation $H_{\alpha\beta}^{(1)}$ in Eq. (1), the externally-generated parts $\bar{E}_a^A$ and $\bar{B}_a^A$ are well defined and capture the terms in E_a^A and B_a^A that asymptotically grow as $R^{\ell-1}$ as $R \equiv |\mathbf{X}| \to \infty$. The (seemingly contradictory) formal limit $X^a \to 0$ indicated in Eqs. (7) and (8) refers to the matching performed in the outer problem (where, roughly speaking, the outer limit $X_{\text{outer}}^a \to 0$ can still refer to a world tube which is large, in internal units, compared to the radius of body A).

Besides the externally-generated "tidal moments" (7) and (8), one also defines the internally-generated "multipole moments" of body A, $M_L^A(T)$ (mass moments) and $S_L^A(T)$ (spin moments) as the symmetric-trace-free tensors that parametrize the body-generated terms in the metric coefficients W^A, W_a^A that asymptotically decrease (in the A-body zone) as $R^{-(\ell+1)}$ as $R \equiv |\mathbf{X}| \to \infty$. The normalization of these quantities is defined by Eqs. (6.9) of [19] (and agrees with the usual one in post-Newtonian theory).

In the stationary case (which is relevant to our present "adiabatic" approach to tidal effects), this normalization is such that the "internally-generated" post-Newtonian metric potentials $W^{+A}(X)$, $W_a^{+A}(X)$ read

$$W^{+A}(X) = G \sum_{\ell \geq 0} \frac{(-)^\ell}{\ell!} \partial_L \left(\frac{M_L^A}{R}\right), \qquad (9)$$

$$W_a^{+A}(X) = -G \sum_{\ell \geq 1} \frac{\ell}{\ell+1} \frac{(-)^\ell}{\ell!} \epsilon_{abc} \partial_{bL-1} \left(\frac{S_{cL-1}^A}{R}\right)$$
$$- \frac{1}{4} \partial_a(\Lambda^A - \lambda), \qquad (10)$$

where $\Lambda^A - \lambda$ is a gauge transformation (which would drop out if we had considered the gravito-magnetic field B_a^{+A}).

In the first post-Newtonian approximation considered by DSX, the separation of the local-frame potential $W_\alpha^A(X)$ into an "externally-generated" part $\bar{W}_\alpha^A$ and an internally-generated one W_α^{+A}, is well defined (thanks to the structure of Einstein's equations). In the case we are considering here of a linearly perturbed, quasistationary, fully relativistic neutron star, the asymptotically growing character (as $R \to \infty$) of the externally-generated potentials allows one

to uniquely define the tidal moments (7) and (8). On the other hand, the asymptotic decrease $\propto R^{-(\ell+1)}$ of the internally-generated multipolar potentials (9) and (10) introduces an ambiguity in their definition. For an attempt to uniquely define the gravito-electric quadrupole moment M_{ab}^A induced on a black hole by an external tidal moment G_{ab}^A see [22]. Here, instead of relying on such a conventional (harmonic-coordinates related) definition of the induced multipole moments, we shall follow the spirit of Sec. 5 of [13] in defining M_L^A, S_L^A as parametrizing the (uniquely defined) pieces in the local-frame metric $G_{\alpha\beta}^A(X)$ which violate the "effacing principle," in that they directly depend on the body A being a neutron star, rather than a black hole. Reference [13] explicitly treated the dominant even-parity case, and introduced [see Eq. (11) there] a "dimensionless constant k" ($\equiv a_2$ as defined below) as a relativistic generalization of the second Love number. This minimal definition (which will be made fully precise below) is rather natural, and coincides with the definition adopted in [5,6].

With this notation in hand, we can define the two "tidal-polarizability" coefficients μ_ℓ and σ_ℓ introduced in Eqs. (6.19) of [7]. These coefficients relate the (electric or magnetic) tidal induced[4] multipole moments to the corresponding external tidal moments, i.e.,

$$M_L^A = \mu_\ell^A G_L^A, \qquad (11)$$

$$S_L^A = \sigma_\ell^A H_L^A. \qquad (12)$$

The electric-type (or "even-parity") tidal coefficient μ_ℓ generalizes the (k_ℓ-type) Newtonian "Love number." For the leading quadrupolar tide, μ_2, as defined by Eq. (11), agrees with the quantity denoted λ in [5,6]. The magnetic-type (or "odd-parity") quadrupolar tidal coefficient σ_2 is proportional to the quantity γ which has been considered in the investigations of Favata [23] which were, however, limited to the first post-Newtonian approximation. Here, we shall consider the case of strongly self-gravitating bodies (neutron stars), and study the dependence of both μ_ℓ^A and σ_ℓ^A on the compactness $c_A \equiv (GM/c^2R)_A$ of the considered neutron star. Let us also note that, in terms of finite-size corrections to the leading point-particle effective action $S_{\text{pointmass}} = -\sum_A \int M_A ds_A$, the two tidal effects parametrized by μ_ℓ and σ_ℓ correspond to nonminimal worldline couplings, respectively, proportional to

$$\mu_\ell^A \int ds_A(G_L^A)^2 \quad \text{and} \quad \sigma_\ell^A \int ds_A(H_L^A)^2. \qquad (13)$$

The leading quadrupolar corrections (13) can be reproduced (using the link between G_{ab}^A and $\bar{u}_A^\mu \bar{u}_A^\nu \bar{R}_{\mu a\nu b}^A$, and

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[3]As in DSX, L denotes a multi-index $a_1 a_2 \dots a_\ell$ and $\langle a_1 \dots a_\ell\rangle$ a symmetric-trace-free projection.

[4]Here, we consider a nonrotating star which is spherically symmetric (with vanishing multipole moments) when it is isolated, so that M_L^A and S_L^A represent the multipole moments induced by the influence of the external tidal fields G_L^A and H_L^A.

H_{ab}^A and $\epsilon_b{}^{cd}\bar{u}_A^\mu \bar{R}_{\mu acd}^A$, see Sec. 3.D of [12,19]) as the following nonminimal couplings involving the Weyl tensor:

$$\mu_2 \int ds \mathcal{E}_{\alpha\beta}\mathcal{E}^{\alpha\beta} \quad \text{and} \quad \sigma_2 \int ds \mathcal{B}_{\alpha\beta}\mathcal{B}^{\alpha\beta}, \qquad (14)$$

where $u^\mu = dz^\mu/ds$, and we have introduced the tensors $\mathcal{E}_{\alpha\beta} \equiv u^\mu u^\nu C_{\mu\alpha\nu\beta}$ and $\mathcal{B}_{\alpha\beta} \equiv u^\mu u^\nu C^*_{\mu\alpha\nu\beta}$, with $C^*_{\mu\nu\alpha\beta} \equiv \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}C_{\rho\sigma\alpha\beta}$ being the dual of the Weyl tensor. In $D = 3 + 1$ dimensions, and in absence of parity-violating couplings, the two terms (14) are the only possible isotropic couplings. In higher dimensions, there are three nonminimal isotropic couplings quadratic in the Weyl tensor as indicated in Eq. (90) of [18]. Note that we are using here the freedom of locally redefining the dynamical variables to eliminate terms proportional to the (zeroth-order) equations of motion, such as terms involving the Ricci tensor; see, e.g., the discussion of finite-size effects in tensor-scalar gravity in Appendix A of Ref. [17].

Let us finally note that there are other "tidal coefficients" which might be interesting to discuss. First, though the linear relations (11) and (12) are the most general ones that can exist in the (parity-preserving) case of a nonspinning neutron star, the tidal properties of a spinning neutron star will involve other tidal coefficients, proportional to the spin, and associated to a mixing between electric and magnetic effects. Such electric-magnetic mixing terms would correspond, say in the leading quadrupolar case, to nonminimal worldline couplings quadratic in $C_{\mu\nu\alpha\beta}$ and linear in the spin tensor $S_{\mu\nu}^A$.

There exist also other tidal coefficients which do not have a direct dynamical meaning, but which generalize the "first type" of Love numbers introduced in the theory of Newtonian tides. Indeed, it is physically meaningful to define, for any ℓ, a shape Love number measuring the proportionality between the external tidal influence and the deformation of the geometry of the surface of the considered (neutron) star. More precisely, limiting ourselves to the electric-type tides, one can define a dimensionless number h_ℓ by writing, as one does in Newtonian theory,

$$g(\delta R)_\ell = h_\ell U_\ell^{\text{disturb}}(R), \qquad (15)$$

or, equivalently,

$$\left(\frac{\delta R}{R}\right)_\ell = h_\ell \frac{U_\ell^{\text{disturb}}(R)}{gR}, \qquad (16)$$

where $(\delta R/R)_\ell \propto P_\ell(\cos\theta)$ represents the fractional deformation of the (areal) radius R of the neutron star (measured in a geometrically invariant way, by relating it to the inner geometry of the deformed surface), where $U_\ell^{\text{disturb}}(R) \propto R^\ell P_\ell(\cos\theta)$ represents the usual, external, Newtonian tidal potential deforming the star, formally evaluated at the radius of the star (as if one were in flat space), and where

$g \equiv GM/R^2$ represents the usual Newtonian surface gravity of the neutron star. This h_ℓ, shape Love number has been recently considered in the theory of the gravitational polarizability of black holes [24] and it will be interesting to compare and contrast the values of the h_ℓ for black holes to the values of h_ℓ for neutron stars, especially in the limit where the compactness gets large. See Sec. VI below which will give the exact definition of the quantity $(\delta R/R)_\ell$.

III. STATIONARY PERTURBATIONS OF A NEUTRON STAR

The unperturbed structure of an isolated (nonrotating) neutron star is described by a metric of the form

$$G_{\alpha\beta}^{A(0)}dX^\alpha dX^\beta = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2 d\Omega^2. \quad (17)$$

Here, and in the following, for notational simplicity we shall denote the local (spherical) coordinates of the A-body frame simply as (t, r, θ, φ) (with $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\varphi^2$), instead of the upper case letters (T, R, Θ, Φ) that would more closely follow the DSX notation recalled above. Introducing as usual the radial dependent mass parameter $m(r)$ by[5]

$$e^{\lambda(r)} \equiv \left(1 - \frac{2m(r)}{r}\right)^{-1}, \qquad (18)$$

and assuming a perfect-fluid energy-momentum tensor

$$T_{\mu\nu} = (e + p)u_\mu u_\nu + p g_{\mu\nu}, \qquad (19)$$

the spherically symmetric metric coefficients $\nu(r)$, $m(r)$, and the pressure $p(r)$ satisfy the Tolman-Oppenheimer-Volkoff (TOV) equations of stellar equilibrium

$$\frac{dm}{dr} = 4\pi r^2 e, \qquad (20)$$

$$\frac{dp}{dr} = -(e + p)\frac{m + 4\pi r^3 p}{r^2 - 2mr}, \qquad (21)$$

$$\frac{d\nu}{dr} = \frac{2(m + 4\pi r^3 p)}{r^2 - 2mr}. \qquad (22)$$

These equations are integrated from the center outward once that a barotropic EOS relating p to e is provided. We shall consider several types of barotropic EOS, namely, two different types of "polytropic EOS" (an e polytrope, with $p = \kappa e^\gamma$, and a μ polytrope," with $p = k\mu^\gamma$ and $e = \mu + p/(\gamma - 1)$, where $\mu = nm_b$ is the baryonic rest-mass density), and two different tabulated ("realistic") EOS (the FPS one [25] and the SLy [26] one). In view of the current large uncertainty in the correct description of dense nuclear matter, we are not claiming that our selection of realistic EOS is physically preferred (see, e.g., Ref. [27] and refer-

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[5]Henceforth, we shall often set $G = c = 1$.

ences therein for a thorough comparison among models from various EOS). We have chosen them because they have been used in recent numerical relativity simulations of binary neutron-star coalescence [28,29]. As for the polytropic EOS, they have also been often used in numerical relativity simulations (especially the μ-polytrope one), and their dependence on the adiabatic index[6] is a convenient way of varying the "stiffness" of the EOS (the limit $\gamma \to \infty$ representing the stiffest possible EOS, namely, incompressible matter with $e = $ const and an infinite speed of sound).

Because of the spherical symmetry of the background, the metric perturbation

$$G^A_{\alpha\beta}(X) = G^{A(0)}_{\alpha\beta}(X) + H_{\alpha\beta}(X), \qquad (23)$$

here considered at the linearized level, can be analyzed in (tensor) spherical harmonics. The metric is expanded in even-parity and odd-parity tensor harmonics as

$$H_{\alpha\beta} = H^{(e)}_{\alpha\beta} + H^{(o)}_{\alpha\beta}. \qquad (24)$$

In the Regge-Wheeler gauge, and following standard definitions for the expansion coefficients and the sign conventions of [30,31], one has

$$H^{(e)}_{\alpha\beta}dX^\alpha dX^\beta = -[e^\nu H^{\ell m}_0 dt^2 + 2H^{\ell m}_1 dt dr + H^{\ell m}_2 e^\lambda dr^2$$
$$+ r^2 K^{\ell m} d\Omega^2]Y_{\ell m}, \qquad (25)$$

while the nonvanishing components of $H^{(o)}_{\alpha\beta}$ are $H^{(o)}_{tA} = h_0 \epsilon_A{}^B \nabla_B Y_{\ell m}$ and $H^{(o)}_{rA} = h_1 \epsilon_A{}^B \nabla_B Y_{\ell m}$ where $(A, B) = (\theta, \varphi)$ and where $\epsilon_A{}^B$ is the mixed form of the volume form on the sphere S^2_r.

Our aim is then to solve the *coupled system* of the perturbed Einstein's equations, together with the perturbed hydrodynamical equations $\nabla^\alpha \delta T_{\alpha\beta}[e, p] = 0$, so as to describe a star deformed by an external tidal field. We shall only consider *stationary* perturbations ("adiabatic tides").

A. Even-parity, stationary barotropic perturbations

Even-parity, stationary perturbations of a barotropic star simplify in that (i) the metric perturbations reduce to two functions $H = H_0 = H_2$, and K (with $H_1 = 0$); (ii) the fluid perturbations are described by the logarithmic enthalpy function h, such that $\delta h = \delta p/(e + p)$; and (iii) the latter logarithmic enthalpy function is simply related (in absence of entropy perturbation) to the metric function H by

$$\delta h = -\tfrac{1}{2}H. \qquad (26)$$

It was then showed by Lindblom, Mendell, and Ipser [31] how to convert the system of first-order radial differential

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equations relating H', K', H, and K to a single second-order radial differential equation for the metric variable H (such that $H_{00} = -e^\nu H Y_{\ell m}$) of the form

$$H'' + C_1 H' + C_0 H = 0. \qquad (27)$$

[As usual, we shall generally drop the multipolar index ℓ on the various metric perturbations. The presence of a factor $Y_{\ell m}(\theta, \varphi)$, or $P_\ell(\cos\theta)$, in (or to be added to) the considered metric perturbation is also often left implicit.] Taking the stationary limit ($\omega \to 0$) of the results given in Appendix A of [31] (together with the barotropic relation $\tilde{\delta}U = 0$) one gets

$$C_1 = \frac{2}{r} + \frac{1}{2}(\nu' - \lambda') = \frac{2}{r} + e^\lambda\left[\frac{2m}{r^2} + 4\pi r(p - e)\right], \qquad (28)$$

$$\begin{aligned}
C_0 &= e^\lambda\left[-\frac{\ell(\ell + 1)}{r^2} + 4\pi(e + p)\frac{de}{dp} + 4\pi(e + p)\right] \\
&\quad + \nu'' + (\nu')^2 + \frac{1}{2r}(2 - r\nu')(3\nu' + \lambda') \\
&= e^\lambda\left[-\frac{\ell(\ell + 1)}{r^2} + 4\pi(e + p)\frac{de}{dp} + 4\pi(5e + 9p)\right] \\
&\quad - (\nu')^2, \qquad (29)
\end{aligned}$$

where we have used the background (TOV) equations to rewrite C_1 and C_0. As a check, we have also derived from scratch Eq. (27) by starting from the "gauge-invariant" formalism of Ref. [32]. Equation (27) generalizes to an arbitrary value of the multipolar order ℓ Eq. (15) of Ref. [6], which concerned the leading quadrupolar even-parity tide.

For completeness, let us note that the other metric variable, K, can be expressed as a linear combination of H and H', namely

$$K = \alpha_1 H' + \alpha_2 H, \qquad (30)$$

where the explicit expressions of the coefficients α_1 and α_2 can also be deduced by taking the stationary limit of the results given in Appendix A of [31].

B. Odd-parity, stationary perturbations

It was shown by Thorne and Campolattaro [33] that odd-parity perturbations of a nonrotating perfect-fluid star consists only of metric fluctuations, and do not affect the star's energy density and pressure. One might naively think that this means that an odd-parity tidal field will induce no (gauge-invariant) spin multipole moment in a (nonrotating) star. This conclusion is, however, incorrect because the "gravitational potential well" generated by the stress-energy tensor of the star does affect the "radial propagation" of the external odd-parity tidal fields and necessarily adds an asymptotically decreasing "induced" tidal response to the "incoming" tidal field. To describe this

phenomenon, it is convenient to describe the odd-parity perturbation by means of the (static limit of the) "master equation" derived by Cunningham, Price, and Moncrief [34] (see also Ref. [35]). In the stationary limit, and in terms of the ordinary radial variable r (rather than the "tortoise" coordinate r_*) this equation reads

$$\psi'' + \frac{e^\lambda}{r^2}[2m + 4\pi r^3(p - e)]\psi'$$
$$- e^\lambda\left[\frac{\ell(\ell + 1)}{r^2} - \frac{6m}{r^3} + 4\pi(e - p)\right]\psi = 0. \quad (31)$$

In terms of the variables (h_0, h_1) entering the odd-parity perturbations, the odd-parity master function ψ can be taken to be either $e^{(\nu-\lambda)/2}h_1/r$, or the combination $r\partial_t h_1 - r^3\partial_r(h_0/r^2)$ [see, e.g., [36] for more details]. As h_1 vanishes in the stationary limit, we can define ψ as being

$$\psi = r^3\partial_r\left(\frac{h_0}{r^2}\right) = rh_0' - 2h_0. \quad (32)$$

IV. COMPUTATION OF THE ELECTRIC-TYPE TIDAL COEFFICIENT μ_ℓ

The electric-type tidal response coefficient μ_ℓ, defined by Eq. (11) above, can be obtained by going through three steps: (i) numerically solving the even-parity master equation (27) within the neutron star; (ii) analytically solving the same master equation (27) in the exterior of the star; and (iii) matching the interior and exterior solutions across the star surface, taking into account the definition (11) to normalize the ratio between the "growing" and "decreasing" parts of $H(r)$, namely $H^{\text{growing}} \sim r^\ell$ versus $H^{\text{decreasing}} \sim \mu_\ell r^{-(\ell+1)}$.

A. The internal problem

The internal value of the metric function $H(r)$ is obtained by numerically integrating Eq. (27), together with the TOV equations (20)–(22), from the center (or, rather some very small cutoff radius $r_0 = 10^{-6}$) outwards, starting with some central values of m, p, ν, H, and H'. For H, one takes as starting values at the cutoff radius $H(r_0) = r_0^\ell$ and $H'(r_0) = \ell r_0^{\ell-1}$. The latter boundary conditions follow from the analysis of Eq. (27) around the regular-singular point $r = 0$, which shows that $H(r) \simeq \bar{h}r^\ell$ (where $\bar{h}$ is an arbitrary constant) is the most general regular solution around $r = 0$. As Eq. (27) is homogeneous in H, the scaling constant $\bar{h}$ is irrelevant and will drop out when we shall match the logarithmic derivative

$$y^{\text{int}}(r) \equiv \frac{rH'}{H}, \quad (33)$$

across the star surface. This is why it is enough to use $\bar{h} = 1$ as the initial boundary conditions for H.

The main output of this internal integration procedure is to compute (for each value of ℓ) the value of the internal logarithmic derivative (33) at the star's surface, say $r = R$

$$y_\ell \equiv y_\ell^{\text{int}}(R). \quad (34)$$

B. The external problem

As noticed long ago by Regge and Wheeler [37] and Zerilli [38], the exterior form of the stationary, even-parity master equation (27) $[e = p = 0, m(r) = M]$ can be recast as an associated Legendre equation (with $\ell = \ell$ and $m = 2$). More precisely, in terms of the independent variable $x \equiv r/M - 1$, the exterior form of (27) reads

$$(x^2 - 1)H'' + 2xH' - \left(\ell(\ell + 1) + \frac{4}{x^2 - 1}\right)H = 0, \quad (35)$$

where the prime stands now for d/dx. Its general solution can be written as

$$H = a_P\hat{P}_{\ell 2}(x) + a_Q\hat{Q}_{\ell 2}(x), \quad (36)$$

where the hat indicates that the associated Legendre functions of first, $P_{\ell 2}$, and second,[7] $Q_{\ell 2}$, kind have been normalized so that $\hat{Q}_{\ell 2} \simeq 1/x^{\ell+1} \simeq (M/r)^{\ell+1}$ and $\hat{P}_{\ell 2} \simeq x^\ell \simeq (r/M)^\ell$ when $x \to \infty$ or $r \to \infty$; a_Q and a_P are integration constants to be determined by matching to the internal solution. Defining $a_\ell \equiv a_Q/a_P$, the exterior logarithmic derivative $y^{\text{ext}} \equiv rH'/H$ reads

$$y_\ell^{\text{ext}}(x) = (1 + x)\frac{\hat{P}_{\ell 2}'(x) + a_\ell\hat{Q}_{\ell 2}'(x)}{\hat{P}_{\ell 2}(x) + a_\ell\hat{Q}_{\ell 2}(x)}. \quad (37)$$

C. Matching at the star's surface and computation of the "electric" tidal Love number

As Eq. (27) is second order in the radial derivative of H, one expects that H and H' will be continuous at the star's surface. Actually, the issue of regularity at the star surface is somewhat subtle because some of the thermodynamic variables (such as pressure) do not admit regular Taylor expansions in $r - R$ as $r \to R$. For instance, while the logarithmic enthalpy $h(p) = \int_0^p dp/(e + p)$ vanishes smoothly $[h(r) \propto r - R]$ across the surface, one finds that (for any polytrope) $p(r) \propto (r - R)^{\gamma/(\gamma-1)}$ and that the term involving the inverse of the squared sound velocity $c_s^2 = dp/de$ in Eq. (27) is singular (when $\gamma > 2$), namely

$$(e + p)\frac{de}{dp} \propto (r - R)^{((2-\gamma)/(\gamma-1))}. \quad (38)$$

Despite this mildly singular behavior of the coefficient C_0 of (27) and despite the fact that the exact location of the tidally-deformed star surface is slightly displaced from the background value $r = R$, one checks that it is correct

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[7]Note that contrary to the usual mathematical definition of $Q_{\ell m}(x)$, which is tuned to the real interval $-1 < x < +1$, we need to work with $Q_{\ell m}(x)$ in the interval $x > 1$. This means replacing $\log(\frac{1+x}{1-x})$ with $\log(\frac{x+1}{x-1})$.

(when $\gamma < \infty$) to impose the continuity of H and H' at $r = R$. [Note that we consider here the case of a finite adiabatic index γ. The incompressible limit $\gamma \to \infty$ leads to a master equation which is singular at the surface, and which must be considered with care. See below our discussion of the incompressible limit.] This continuity then imposes the continuity of the logarithmic derivative rH'/H. This leads to the condition $y^{\text{ext}}(R) = y^{\text{int}}(R) = y_\ell$, which determines the value of the ratio $a_\ell = a_Q/a_P$ in terms of the compactness $c \equiv M/R$ of the star

$$a_\ell = -\frac{\hat{P}'_{\ell 2}(x) - cy_\ell \hat{P}_{\ell 2}(x)}{\hat{Q}'_{\ell 2}(x) - cy_\ell \hat{Q}_{\ell 2}(x)} \bigg|_{x=1/c-1}. \quad (39)$$

On the other hand, the ratio $a_\ell \equiv a_Q/a_P$ can be related to the tidal coefficient μ_ℓ by comparing (modulo an overall factor -2),

$$-(\delta H_{00}e^{-\nu})^{\text{growing}} = H^{\text{growing}}(r) = a_P \hat{P}_{\ell 2}(x)Y_{\ell m}$$

$$\simeq a_P \left(\frac{r}{M}\right)^\ell Y_{\ell m}, \quad (40)$$

$$-(\delta H_{00}e^{-\nu})^{\text{decreasing}} = H^{\text{decreasing}}(r) = a_Q \hat{Q}_{\ell 2}(x)Y_{\ell m}$$

$$\simeq a_Q \left(\frac{r}{M}\right)^{-(\ell+1)} Y_{\ell m}, \quad (41)$$

respectively to

$$\bar{W} = \frac{1}{\ell!}\hat{X}^L G_L^A = \frac{1}{\ell!}r^\ell \hat{n}^L G_L^A, \quad (42)$$

$$W^+ = G\frac{(-)^\ell}{\ell!}\partial_L\left(\frac{M_L^A}{r}\right), \quad (43)$$

[see, e.g., Eq. (4.15a) of Ref. [7]] where $n^a \equiv X^a/r$ is a radial unit vector. Using the fact that

$$\partial_L r^{-1} = (-)^\ell (2\ell - 1)!!\hat{n}^L r^{-(\ell+1)}, \quad (44)$$

and $M_L = \mu_\ell G_L$, and remembering that $G_L \hat{n}^L \propto Y_{\ell m}(\theta, \varphi)$, we see that

$$(2\ell - 1)!!G\mu_\ell = \frac{a_Q}{a_P}\left(\frac{GM}{c_0^2}\right)^{2\ell+1} = a_\ell\left(\frac{GM}{c_0^2}\right)^{2\ell+1}. \quad (45)$$

Note that $G\mu_\ell$ has the dimensions of $[\text{length}]^{2\ell+1}$. There are then two natural ways of expressing $G\mu_\ell$ in terms of a dimensionless quantity. Either by scaling it by the $(2\ell + 1)$th power of GM/c_0^2, which leads to

$$(2\ell - 1)!!\frac{G\mu_\ell}{(GM/c_0^2)^{2\ell+1}} = a_\ell, \quad (46)$$

or by scaling it by the $(2\ell + 1)$th power of the star radius R, which gives

$$(2\ell - 1)!!\frac{G\mu_\ell}{R^{2\ell+1}} \equiv 2k_\ell = a_\ell c^{2\ell+1}. \quad (47)$$

Alternatively, we can write

$$G\mu_\ell = \frac{a_\ell}{(2\ell - 1)!!}\left(\frac{GM}{c_0^2}\right)^{2\ell+1} = \frac{2k_\ell}{(2\ell - 1)!!}R^{2\ell+1}. \quad (48)$$

The scaling of $G\mu_\ell$ by means of $R^{2\ell+1}$ is the traditional "Newtonian" way of proceeding, and leads to the introduction of the dimensionless "second tidal Love number" k_ℓ [conventionally normalized as in Eq. (47) above].

One can finally write k_ℓ as

$$k_\ell = \frac{1}{2}c^{2\ell+1}a_\ell$$

$$= -\frac{1}{2}c^{2\ell+1}\frac{\hat{P}'_{\ell 2}(x) - cy_\ell \hat{P}_{\ell 2}(x)}{\hat{Q}'_{\ell 2}(x) - cy_\ell \hat{Q}_{\ell 2}(x)}\bigg|_{x=1/c-1}. \quad (49)$$

The dimensionless Love number k_ℓ has the advantage of having a weaker sensitivity on the compactness $c \equiv GM/(c_0^2 R)$ (especially as the compactness formally tends to zero, i.e., in the Newtonian limit). Note, however, that the dimensionless quantity which will most directly enter the gravitational-wave phase of inspiralling binary neutron stars (NS) is $G\mu_\ell/(GM/c_0^2)^{2\ell+1} \sim a_\ell \sim c^{-(2\ell+1)}k_\ell$.

The evaluation of the result (49) for k_ℓ yields the following explicit expressions for $2 \le \ell \le 4$ (with, for simplicity, $y \equiv y_\ell$):

$$k_2 = \frac{8}{5}(1 - 2c)^2 c^5 [2c(y - 1) - y + 2]\Big\{2c(4(y + 1)c^4 + (6y - 4)c^3 + (26 - 22y)c^2 + 3(5y - 8)c - 3y + 6)$$

$$- 3(1 - 2c)^2(2c(y - 1) - y + 2)\log\left(\frac{1}{1 - 2c}\right)\Big\}^{-1}, \quad (50)$$

$$k_3 = \frac{8}{7}(1 - 2c)^2 c^7 [2(y - 1)c^2 - 3(y - 2)c + y - 3]\Big\{2c[4(y + 1)c^5 + 2(9y - 2)c^4 - 20(7y - 9)c^3 + 5(37y - 72)c^2$$

$$- 45(2y - 5)c + 15(y - 3)] - 15(1 - 2c)^2(2(y - 1)c^2 - 3(y - 2)c + y - 3)\log\left(\frac{1}{1 - 2c}\right)\Big\}^{-1}, \quad (51)$$

$$k_4 = \frac{32}{147}(1-2c)^2 c^9 [12(y-1)c^3 - 34(y-2)c^2 + 28(y-3)c - 7(y-4)] \Big\{ 2c[8(y+1)c^6 + (68y-8)c^5$$

$$+ (1284 - 996y)c^4 + 40(55y - 116)c^3 + (5360 - 1910y)c^2 + 105(7y-24)c - 105(y-4)]$$

$$- 15(1-2c)^2 [12(y-1)c^3 - 34(y-2)c^2 + 28(y-3)c - 7(y-4)] \log\Big(\frac{1}{1-2c}\Big) \Big\}^{-1}. \tag{52}$$

Equation (50) above agrees with the corrected version of Eq. (23) of [6]. Note that independently of the values of y (as long as it does not introduce a pole singularity, which will be the case), the results (50)–(52) [and, more generally, the result (49)] contain an overall factor $(1-2c)^2$ which formally tends (quadratically) to zero when the compactness $c = GM/R$ "tends" toward the compactness of a black hole (BH), namely $c^{\mathrm{BH}} = GM/(2GM) = 1/2$. [The singular logarithm $\log[1/(1-2c)]$ in the denominator is also easily checked to be always multiplied by $(1-2c)^2$ and thereby not to affect the $\propto (1-2c)^2$ formal vanishing of k_ℓ as $c \to 1/2$.] This property can be easily understood as a consequence of the "no-hair" properties of black holes. Indeed, among the two solutions of the exterior tidal perturbation equation (27), the no-hair property means that the solution which is "rooted" within the horizon, i.e., the "asymptotically decreasing" solution $Q_{\ell 2}(x)$ is *singular* at the horizon, i.e., when $x = R/M - 1 \to 1$. More precisely, this singular behavior is

$$Q_{\ell 2}(x) \sim (x^2 - 1)^{2/2} \frac{d^2 Q_\ell(x)}{dx^2}$$

$$\sim (x^2 - 1) \frac{d^2}{dx^2}\Big[\log\Big(\frac{x+1}{x-1}\Big) P_\ell(x)\Big]$$

$$\sim (x^2 - 1)(x-1)^{-2} \sim (x-1)^{-1}, \tag{53}$$

so that the most singular term in the denominator of a_ℓ or k_ℓ is $\hat{Q}'_{\ell 2}(x) \sim (x-1)^{-2} \sim (R-2M)^{-2} \sim (1-2c)^{-2}$ which is at the origin of the presence of a factor $(1-2c)^2$ in a_ℓ and k_ℓ. One might naively think that this behavior proves that the "correct" value of the k_ℓ tidal Love numbers of a black hole is simply $k_\ell^{\mathrm{BH}} = 0$. However, we do not think that this conclusion is warranted. Indeed, as we explained above, the definition used here (and in [5,6,13]) of the Love numbers of a (neutron) star consists in selecting, within the gravitational field of a tidally distorted star, the terms which violate the effacing principle (in the sense of Ref. [13]), i.e., the internal-structure-dependent terms which differentiate the tidal response of a (compact) star, from that of a black hole. From this point of view, the vanishing of k_ℓ as $c \to c^{\mathrm{BH}}$ is mainly a consistency check on this formal definition. The question of computing the correct value of k_ℓ for a black hole is a technically much harder issue which involves investigating in detail the many divergent diagrams that enter the computation of interacting point masses at the 5-loop (or 5 PN) level.

Indeed, the issue at stake is the following. When describing the motion of two black holes (as seen in the "outer

problem") by a skeletonized action of the form $S = S_{\mathrm{pointmass}} + S_{\mathrm{nonminimal}}$, the presence of nonminimal world-line couplings $S_{\mathrm{nonminimal}}$ of the type (13) and (14) can only be detected if one treats (when using perturbative expansions in powers of G) the general relativistic nonlinear self-interactions entailed by $S_{\mathrm{pointmass}} = -\sum_A \int M_A ds_A$ at the order of approximation corresponding to $S_{\mathrm{nonminimal}}$. For a black hole (of "radius" $R_A = 2GM_A/c_0^2$), the leading nonminimal coupling parameter scales as $\mu_2^A \sim k_2^A R_A^5/G \sim k_2^A G^4 M_A^5$, so that (using $\mathcal{E}_{\alpha\beta}^A \sim R_{\alpha\beta\gamma\delta}^A \propto GM_B$) the leading nonminimal interaction $\mu_2^A \int ds_A \mathcal{E}_{\alpha\beta}^A \mathcal{E}_A^{\alpha\beta}$ is proportional to $k_2^A G^6 M_A^5 M_B^2$. The presence of an overall factor G^6 [which is the same factor G^6 that appeared in Eq. (19) in Sec. 5 of [13]] signals that such an effect is G^5 smaller than the leading (Newtonian) interaction ($\propto GM_A M_B$) between two point masses, so that it corresponds to the 5 PN level. In the diagrammatic language of (post-Minkowskian or post-Newtonian) perturbation theory (as used, e.g., in [39]), this corresponds to the 5-loop level. Let us recall that the computation of the interaction of two black holes at the 3-loop level was a technically complex enterprise that necessitated the careful consideration of many divergent diagrams, and the use of the efficient method of dimensional regularization [40,41]. At the 3-loop level, the result of the computation was (essentially) *finite*, though the use of harmonic coordinates in one of the computations [41] introduced some gauge-dependent infinities. As argued long ago [13], and confirmed by an effective action approach [18], one expects to see real, gauge-independent infinities arising at 5-loop (5 PN), i.e., at the level where the effacing principle breaks down, and where, as explained above, a parameter ($\sim k_2$) linked to the internal structure of the considered compact body starts to enter the dynamics. Until a careful analysis of the 5 PN nonlinear self-interactions is performed, one cannot conclude from the above result ($k_2^{\mathrm{NS}} \to 0$ as $c \to c^{\mathrm{BH}}$) that the effective action describing the dynamics of interacting black holes is described by the pure point-mass action $-\sum_A \int m_A ds_A$ without the need of additional nonminimal couplings of the type of Eq. (14).

We have phrased here the problem within standard (post-Minkowskian or post-Newtonian) perturbation theory, because this is the clearest framework within which the issue of higher order nonlinear gravitational interactions of point masses is technically well defined (when using, say, dimensional regularization to define the perturbative interaction of point masses in general relativity

[40,41]). Note that, in the extreme mass ratio limit ($M_A \ll M_B$), where one might use black hole perturbation theory, the interaction associated to the leading nonminimal coupling parameter μ_2^A of M_A is proportional to M_A^5 (see above). This is well beyond the currently studied "gravitational self-force" effects, which are proportional to M_A^2, and correspond to a "1-loop" effect within a black hole background.

V. COMPUTATION OF THE MAGNETIC-TYPE TIDAL COEFFICIENT σ_ℓ

The magnetic-type tidal response coefficient σ_ℓ, defined by Eq. (12) above, can be obtained by following three steps, which are similar to those followed for the electric-type coefficient μ_ℓ.

A. The internal problem

The internal value of the odd-parity master function ψ is obtained by numerically integrating Eq. (31), together with the TOV equations. The boundary conditions are now obtained from the behavior $\psi \propto r^{\ell+1}$ of the general regular solution at the origin. Again, the main output of the internal integration procedure is to compute (for each value of ℓ) the value of the internal logarithmic derivative of ψ, at the star surface, say

$$y_\ell^{\text{odd}} \equiv y_\ell^{\text{int}}(R) \equiv \left[\frac{r \psi'_{\text{int}}}{\psi_{\text{int}}}\right]_{r=R}. \tag{54}$$

B. The external problem

As noticed long ago by Regge and Wheeler [37], the stationary odd-parity perturbations can be analytically solved in the exterior region. Similar to the even-parity case, there exist two types of exterior solutions: a growing type solution, say $\psi_P(\hat{r})$, with $\hat{r} \equiv r/M$, and a decreasing type one, say $\psi_Q(\hat{r})$. We normalize them so that $\psi_P(\hat{r}) \simeq \hat{r}^{\ell+1}$, and $\psi_Q(\hat{r}) \simeq \hat{r}^{-\ell}$ as $\hat{r} \to \infty$. The general analytical forms of ψ_P and ψ_Q, for any ℓ, can be obtained from Ref. [37]. In the case of the leading quadrupolar odd-parity perturbation, $\ell = 2$, the growing analytical exterior solution of (31) is the very simple polynomial

$$\psi_P^{\ell=2}(\hat{r}) = \hat{r}^3, \tag{55}$$

while the decreasing one can be expressed in terms of an hypergeometric function $F(a, b; c; z)$ as

$$\psi_Q^{\ell=2}(\hat{r}) = -\frac{1}{4}\hat{r}^3 \partial_{\hat{r}}\left[\hat{r}^{-4}F\left(1, 4; 6; \frac{2}{\hat{r}}\right)\right]. \tag{56}$$

The normalization of $\psi_Q(\hat{r})$ is such that $\psi_Q(\hat{r}) \simeq \hat{r}^{-2}$ as $r \to \infty$. Note also that, for the special values $a = 1$, $b = 4$, $c = 6$, the hypergeometric function is actually expressible in terms of elementary functions. The result has the form

$$\psi_Q^{\ell=2}(\hat{r}) = A_3 \hat{r}^3 \log\left(\frac{\hat{r} - 2}{\hat{r}}\right) + A_2 \hat{r}^2 + A_1 \hat{r} + A_0$$
$$+ A_{-1}\hat{r}^{-1}. \tag{57}$$

Going back to an arbitrary ℓ, the general exterior solution of Eq. (31) can be written, analogously to the even parity Eq. (36), as

$$\psi^{\text{ext}} = b_P \psi_P(\hat{r}) + b_Q \psi_Q(\hat{r}). \tag{58}$$

This result allows one to compute the logarithmic derivative $y^{\text{odd}} = r\psi'/\psi$ of ψ in the exterior domain, namely

$$y_{\text{odd}}^{\text{ext}}(\hat{r}) = \hat{r}\frac{\psi'_P(\hat{r}) + b_\ell \psi'_Q(\hat{r})}{\psi_P(\hat{r}) + b_\ell \psi_Q(\hat{r})}, \tag{59}$$

where $b_\ell \equiv b_Q/b_P$.

C. Matching at the star surface and computation of the "magnetic" tidal Love number

We again impose the continuity of ψ, ψ', and therefore $y^{\text{odd}} = r\psi'/\psi$, at the star's surface. Similar to the even-parity case, this determines the value of the ratio $b_\ell = b_Q/b_P$ in terms of the compactness of the star:

$$b_\ell = -\frac{\psi'_P(\hat{r}) - cy_{\text{odd}}\psi_P(\hat{r})}{\psi'_Q(\hat{r}) - cy_{\text{odd}}\psi_Q(\hat{r})}\bigg|_{\hat{r}=1/c}. \tag{60}$$

Again, we see at work the effect of the no-hair property in that the term $\psi'_Q(\hat{r})$ in the denominator of (60) will become [from (57)] singular as $(\hat{r} - 2)^{-1}$ when $\hat{r} \to 2$. This implies that b_ℓ will vanish proportionally to the first power of $1 - 2c$ in the formal limit where the star's compactness $c \to c^{\text{BH}} = 1/2$.

The dimensionless quantity b_ℓ, Eq. (60), is the odd-parity analog of the even-parity quantity a_ℓ, Eq. (39). In the even-parity case, a_ℓ was, essentially, the tidal response coefficient $G\mu_\ell$ scaled by $(GM/c_0^2)^{2\ell+1}$. In the present odd-parity case, the tidal response coefficient $G\sigma_\ell$ has again the dimension $[\text{length}]^{2\ell+1}$, and b_ℓ (for a general ℓ) is essentially $b_\ell \sim G\sigma_\ell(GM/c_0^2)^{-(2\ell+1)}$. Before working out the exact numerical coefficient in this proportionality, we can note that the odd-parity analog of k_ℓ (i.e., essentially $k_\ell \sim G\mu_\ell/R^{2\ell+1} \sim c^{2\ell+1}a_\ell$) will be obtained by scaling $G\sigma_\ell$ by the $(2\ell + 1)$th power of the star radius R, and will therefore involve the new dimensionless combination

$$j_\ell \equiv c^{2\ell+1}b_\ell = -c^{2\ell+1}\frac{\psi'_P(\hat{r}) - cy_{\text{odd}}\psi_P(\hat{r})}{\psi'_Q(\hat{r}) - cy_{\text{odd}}\psi_Q(\hat{r})}\bigg|_{\hat{r}=1/c}. \tag{61}$$

One expects that the new odd-parity dimensionless combination j_ℓ (61) will, like k_ℓ, depend less strongly on the value of the compactness c than b_ℓ itself.

Let us now derive the precise link between the odd-parity tidal response coefficient $G\sigma_\ell$, defined by

Eq. (12), and the dimensionless quantities b_ℓ, Eq. (60), or $b_\ell c^{2\ell+1}$, Eq. (61). To relate them, we start by noting that the Regge-Wheeler metric function h_0 entering the odd-parity master quantity ψ, Eq. (32), parametrizes the time $\times$ angle off diagonal component of the metric perturbation

$$H_{0A} \propto h_0(r)\epsilon_A{}^B \nabla_B Y_{\ell m}(\theta, \varphi), \tag{62}$$

where $A, B = 2, 3 = \theta, \varphi$ are indices on the background coordinate sphere S_r^2 of radius r. The metric on S_r^2 is $\gamma_{AB}dx^A dx^B = r^2 d\Omega^2$, while $\epsilon_A{}^B \equiv \gamma^{BC}\epsilon_{AC}$ denotes the mixed form of the volume form $\frac{1}{2}\epsilon_{AB}dx^A \wedge dx^B = r^2\sin\theta d\theta \wedge d\varphi$ on S_r^2. Let us now consider the gravito-magnetic field B_a, as defined by DSX. Modulo an irrelevant numerical factor, it is the 3-dimensional curl of the time-space off diagonal metric component: $B_a \propto \epsilon_{abc}\partial_b H_{0c}$. Let us focus on the "radial component" of the gravito-magnetic field B_a, i.e., the pseudoscalar

$$\mathbf{n} \cdot \mathbf{B} = n^a B_a \propto n^a \epsilon_{abc}\partial_b H_{0c} \propto \epsilon^{AB}\nabla_A H_{0B}. \tag{63}$$

Using then Eq. (62), one finds that

$$\mathbf{n} \cdot \mathbf{B} \propto -h_0(r)\epsilon^{AB}\epsilon_A{}^C \nabla_B \nabla_C Y_{\ell m} = \ell(\ell+1)\frac{h_0(r)}{r^2}Y_{\ell m}, \tag{64}$$

where one used $\epsilon^{AB}\epsilon_A{}^C = \gamma^{BC}$ and the fact that $\gamma^{AB}\nabla_A \nabla_B Y_{\ell m} = -\ell(\ell+1)r^{-2}Y_{\ell m}$, where the factor r^{-2} comes from the fact that γ_{AB} is the metric on a sphere of radius r, rather than a unit sphere. (The various proportionality signs refer to irrelevant, coordinate-independent, numerical factors.} Finally, we have the link

$$\psi = r^3 \partial_r\left(\frac{h_0}{r^2}\right) \propto r^3 \partial_r(\mathbf{n} \cdot \mathbf{B}). \tag{65}$$

Focusing on the two crucial (growing or decreasing) asymptotic terms in the odd-parity metric, we can now compare the definition of b_ℓ, namely

$$\psi \propto \left\{r^{\ell+1} + b_\ell\left(\frac{GM}{c_0^2}\right)^{2\ell+1}r^{-\ell}\right\}Y_{\ell m}(\theta, \varphi), \tag{66}$$

to the stationary limit of the general gravito-magnetic fields in a local A frame [see Eqs. (2.19) and (4.16) of Ref. [7]],

$$B_a = \bar{B}_a + B_a^+$$

$$= \sum_\ell \frac{1}{\ell!}X^L H_{aL} + \sum_\ell 4G\frac{(-)^\ell}{\ell!}\frac{\ell}{\ell+1}\partial_{aL}\left(\frac{S_L}{r}\right). \tag{67}$$

Inserting now $S_L = \sigma_\ell H_L$, contracting B_a with n^a, and recalling that one has $n^a \partial_a = \partial_r$ and $\partial_L(r^{-1}) = (-)^\ell \times (2\ell-1)!!r^{-(\ell+1)}$, one finds

$$\mathbf{n} \cdot \mathbf{B} = \sum_\ell \frac{1}{(\ell-1)!}r^{\ell-1}H_L n^L + \sum_\ell 4G\sigma_\ell \frac{(-)^\ell}{\ell!}\frac{\ell}{\ell+1}$$
$$\times \partial_r \partial_L\left(\frac{H_L}{r}\right)$$
$$= \sum_\ell \frac{1}{(\ell-1)!}\left\{r^{\ell-1} - 4G\sigma_\ell(2\ell-1)!!\frac{1}{r^{\ell+2}}\right\}n_L H_L, \tag{68}$$

so that

$$r^3 \partial_r(\mathbf{n} \cdot \mathbf{B}) = \sum_\ell \frac{1}{(\ell-2)!}$$
$$\times \left\{r^{\ell+1} + 4G\sigma_\ell(2\ell-1)!!\frac{\ell+2}{\ell-1}\frac{1}{r^\ell}\right\}n_L H_L. \tag{69}$$

The comparison with Eq. (66) finally yields

$$\frac{4(\ell+2)}{\ell-1}(2\ell-1)!!G\sigma_\ell = b_\ell\left(\frac{GM}{c_0^2}\right)^{2\ell+1} = b_\ell c^{2\ell+1}R^{2\ell+1} \tag{70}$$

or

$$G\sigma_\ell = \frac{\ell-1}{4(\ell+2)}\frac{b_\ell}{(2\ell-1)!!}\left(\frac{GM}{c_0^2}\right)^{2\ell+1}$$
$$= \frac{\ell-1}{4(\ell+2)}\frac{j_\ell}{(2\ell-1)!!}R^{2\ell+1}, \tag{71}$$

where we used the notation $j_\ell \equiv c^{2\ell+1}b_\ell$ of Eq. (61). As announced, we see that, modulo a numerical coefficient, the odd-parity analog of the R-scaled Love number k_ℓ is the combination $c^{2\ell+1}b_\ell$, Eq. (61). In the odd-parity, quadrupolar case ($\ell = 2$) the above link reads

$$G\sigma_2 = \frac{1}{48}j_2 R^5 = \frac{1}{48}b_2\left(\frac{GM}{c_0^2}\right)^5, \tag{72}$$

while the explicit expression of $j_2 = c^5 b_2$ reads

$$j_2 = \frac{96c^5(2c-1)(y-3)}{5(2c(12(y+1)c^4 + 2(y-3)c^3 + 2(y-3)c^2 + 3(y-3)c - 3y + 9) + 3(2c-1)(y-3)\log(1-2c))}. \tag{73}$$

VI. COMPUTATION OF THE SHAPE LOVE NUMBER h_ℓ

We have indicated above (following a recent study of the gravitational polarizability of black holes [24]) how one can generalize to a relativistic context the shape tidal constant h_ℓ (the "first Love number") introduced in Newtonian theory. Let us first point out that, though in general there is no direct connection between h_ℓ and k_ℓ, there is a simple relation between them in the case where the deformed object is a ball of (barotropic) perfect fluid, treated in Newtonian gravity. Indeed, in the Newtonian theory of tidal deformation (see [42,43]), we have the result that an external "disturbing" tidal potential

$$U^{\text{disturb}} = \sum_\ell c_\ell r^\ell P_\ell(\cos\theta) \qquad (74)$$

deforms the constant-pressure (and constant-density) level surfaces $[p = p(a)]$ of a fluid star into

$$r(a) = a\left(1 + \sum_\ell f_\ell(a)P_\ell(\cos\theta)\right). \qquad (75)$$

Here $f_\ell(a) = (\delta r/r)_\ell$ satisfies the Clairaut equation [in which $\bar\rho(a)$ indicates the mean density within $0 \leq r \leq a$]

$$a^2 f_\ell'' + \frac{6\rho(a)}{\bar\rho(a)}(af_\ell' + f_\ell) - \ell(\ell + 1)f_\ell = 0, \qquad (76)$$

and is related to the disturbing tidal coefficient c_ℓ via

$$f_\ell(A) = \frac{(2\ell + 1)c_\ell A^{\ell+1}}{G(\ell + \eta_\ell)M}, \qquad (77)$$

where A is the surface value of a (i.e., $A \simeq R$, the undisturbed radius) and where $\eta_\ell = [af_\ell'(a)/f_\ell(a)]_{a=A}$ denotes the surface logarithmic derivative of $f_\ell(a)$. The latter quantity is related to the "second" Love number k_ℓ via

$$k_\ell = \frac{\ell + 1 - \eta_\ell}{2(\ell + \eta_\ell)}. \qquad (78)$$

On the other hand, using the definition of h_ℓ, i.e.,

$$\left(\frac{\delta R}{R}\right)_\ell = f_\ell(A) = h_\ell \frac{c_\ell R^\ell}{GM/R}, \qquad (79)$$

we find

$$h_\ell = \frac{2\ell + 1}{\ell + \eta_\ell}. \qquad (80)$$

By eliminating η_ℓ between Eqs. (78) and (80), we finally get a simple relation between h_ℓ and k_ℓ, namely,

$$h_\ell = 1 + 2k_\ell. \qquad (81)$$

For instance, a Newtonian $\gamma = 2$ polytrope has a density profile $\rho(r) \propto \sin x/x$, where $x \equiv \pi r/R$, from which one deduces, using either the Clairaut equation, or the Newtonian limit of the ($\ell = 2$) even-parity master equation (27), namely, [6]

$$H'' + \frac{2}{r}H' + \left(4\pi G\rho\frac{d\rho}{dp} - \frac{6}{r^2}\right)H = 0, \qquad (82)$$

that $H \propto x^{-1/2}J_{5/2}(x)$. [Note that this result is misprinted as $x^{+1/2}J_{5/2}(x)$ in [6]]. This leads to [6]

$$k_2^N(\gamma = 2) = -\frac{1}{2} + \frac{15}{2\pi^2} \simeq 0.259\,91, \qquad (83)$$

and therefore

$$h_2^N = \frac{15}{\pi^2} \simeq 1.519\,82. \qquad (84)$$

When generalizing the definition of h_ℓ to the relativistic context it seems that we lose the existence of a functional relation between h_ℓ and k_ℓ. Let us explicate the meaning and implementation of the relativistic version of h_ℓ, Eq. (16). First, we define $\delta R/R$ as the fractional deformation of a sphere, embedded in an auxiliary 3-dimensional Euclidean manifold, such that the inner geometry of this deformed, embedded sphere is equal to the inner geometry (induced by the ambient curved spacetime) of the real, tidally-deformed neutron-star surface (considered at fixed coordinate time). In general, one would need to consider the Gaussian curvatures of both surfaces to express their identity (as done in the black hole case [24]). Here, things are simpler because we are using the Regge-Wheeler gauge. In that gauge, it is easily seen that the inner metric of the surface $r = r(\theta, \varphi)$ of an (even-parity) tidally-deformed star is

$$ds^2 = (r(\theta, \varphi))^2(1 - K)d\Omega^2$$
$$= R_0^2\left[1 + \left(2\frac{\delta r}{r} - K\right)\right]d\Omega^2, \qquad (85)$$

where $r(\theta, \varphi) = R_0(1 + \delta r/r)$ is the radial coordinate location of the star's surface. [Here we absorbed the $Y_{\ell m}(\theta, \varphi)$ factors in K and $\delta r/r$.] Because this inner metric is *conformal* to the usual sphere S^2 of the unit radius $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$, it is easily checked that it would be the inner geometry of a flat-embedded sphere (linearly) deformed by $\delta R/R$, with

$$\frac{\delta R}{R} = \frac{\delta r}{r} - \frac{1}{2}K. \qquad (86)$$

We can further compute the value of the coordinate deformation $\delta r/r$ by using the fact that the logarithmic enthalpy $h = \int_0^p dp/(e + p)$ must vanish on the star surface. Since $h(r) = h_0(r) + \delta h(r)$, where $h_0(r)$ is the enthalpy of the background undeformed star, and where $\delta h = -H/2$, we then find

$$\delta r = \frac{1}{2}\left(\frac{H}{h'}\right)_{r=R} = \frac{1}{2}\left(\frac{e + p}{p'}H\right)_{r=R}, \qquad (87)$$

where the prime denotes d/dr. Finally, we have the "flat-equivalent shape deformation":

$$\frac{\delta R}{R} = \frac{1}{2}\left(\frac{e+p}{rp'}H - K\right)_{r=R}. \tag{88}$$

As we said above, the metric variable K can be expressed as a linear combination [31] of H and H', namely, $K = \alpha_1 H' + \alpha_2 H$. The coefficients α_1 and α_2 evaluated on the unperturbed surface $r = R$ of the star read

$$\alpha_1 = \frac{2cR}{(\ell-1)(\ell+2)}, \tag{89}$$

$$\alpha_2 = \frac{1}{(\ell-1)(\ell+2)}\left\{\ell(\ell+1) + \frac{4c^2}{1-2c} - 2(1-2c)\right\}. \tag{90}$$

In addition, using the TOV equations, we also have on the star surface

$$\frac{rp'}{e+p} = -\frac{c}{1-2c}. \tag{91}$$

By replacing (on the surface) also $RH' = yH(R)$, we finally obtain

$$\frac{\delta R}{R} = -\frac{1}{2}H(R)\left\{\frac{1-2c}{c} + \frac{2cy}{(\ell-1)(\ell+2)} + \alpha_2\right\}. \tag{92}$$

At this stage, we have obtained an expression of the form $\delta R/R = H(R)f(c, y)$. To proceed further and compute h_ℓ, it remains to obtain the value of U_ℓ^{disturb}. Following Ref. [24], and the spirit of the Newtonian definition of h_ℓ, we define $-2U_\ell^{\text{disturb}}(R)$ as being the analytic continuation at radius $r = R$ of the leading asymptotically growing piece in H, i.e., the part of $H^{\text{growing}} = a_P\hat{P}_{\ell2}$ which grows as r^ℓ. In other words, we define it by the Newtonian-looking formula

$$U_\ell^{\text{disturb}}(R) = -\frac{1}{2}a_P\left(\frac{R}{M}\right)^\ell. \tag{93}$$

We can then compute $U_\ell^{\text{disturb}}(R)$ in terms of the full value of H on the surface $H(R) = a_P\hat{P}_{\ell2}(x) + a_Q\hat{Q}_{\ell2}(x)$ (with, we recall, $x = R/M - 1 = 1/c - 1$) by separating two "correcting factors" out of $H(R)$, namely,

$$-\frac{1}{2}H(R) = [c^\ell\hat{P}_{\ell2}(x)]\left[1 + a_\ell\frac{\hat{Q}_{\ell2}(x)}{\hat{P}_{\ell2}(x)}\right]U_\ell^{\text{disturb}}(R). \tag{94}$$

Putting all the pieces together, and inserting our general result (39) for $a_\ell = a_Q/a_P$, we obtain the following final result for h_ℓ:

$$h_\ell = c^{\ell+1}\hat{P}_{\ell2}(x)\left\{\frac{1-2c}{c} + \frac{1}{(\ell-1)(\ell+2)}\right.$$
$$\times \left[2cy_\ell + \ell(\ell+1) + \frac{4c^2}{1-2c} - 2(1-2c)\right]\right\}$$
$$\times \left(1 - \frac{\partial_x\log\hat{P}_{\ell2}(x) - cy_\ell}{\partial_x\log\hat{Q}_{\ell2}(x) - cy_\ell}\right)\Bigg|_{x=1/c-1}. \tag{95}$$

VII. RESULTS FOR THE EVEN-PARITY TIDAL COEFFICIENT μ_ℓ

Having explained how to compute the various tidal constants μ_ℓ (or k_ℓ), σ_ℓ (or j_ℓ), and h_ℓ, let us discuss the dependence of these quantities on the compactness $c = GM/(c_0^2R)$, for various kinds of EOS. As we already mentioned, we shall consider a sample of EOS.

A. Polytropic equations of state

First, we consider two kinds of relativistic polytropes: the "energy polytrope," or e polytrope, such that $p = \kappa e^\gamma$, where e is the total energy density, and the "rest-mass polytrope," or μ polytrope, with $p = \kappa\mu^\gamma$ and $e = \mu + p/(\gamma - 1)$, where $\mu = nm_b$ is the baryon rest-mass density. For these polytropes, we shall focus on the adiabatic index $\gamma = 2$, which is known to give a rather good representation of the overall characteristics of neutron stars. We shall also briefly explore what happens when γ takes values larger (or smaller) than 2. In particular, we shall discuss, in the next subsection, the limit $\gamma \to \infty$, which leads to an *incompressible* model, with uniform energy density e. Let us note that the compactness of the $\gamma = 2$ e polytrope models ranges between 0 (for a formally vanishingly small central pressure) and 0.265 for the maximum mass model, while the compactness of the μ polytrope ranges between 0 and 0.2145. Note that the limit of a vanishing compactness $c = GM/(c_0^2R) \to 0$ formally corresponds to the Newtonian limit. Augmenting γ allows one to reach higher compactnesses, and thereby to better explore the effects of general relativistic strong-field gravity. In particular, the incompressible limit, $\gamma \to \infty$, yields a range of compactnesses which extends up to $c_{\max} = 4/9 = 0.4444\ldots$, quite close to the "black hole compactness" $c^{\text{BH}} = 1/2 = 0.5$. We note that a theorem guarantees that $4/9$ is the highest possible compactness of a general relativistic perfect-fluid ball (see, e.g., [44,45]).

Figure 1 exhibits the dependence of the dimensionless, even-parity, Love number k_ℓ on the compactness c of $\gamma = 2$ polytropes for three values of the multipole order: $\ell = 2$, 3,and 4. The results for the μ polytrope (solid lines) are compared with those for the e polytrope (dashed line). The limiting values of k_ℓ for $c \to 0$ (which numerically means $c \simeq 10^{-4}$) do agree, as they should, with the known Newtonian results [46,47]. In particular, $k_2^N(\gamma = 2) = 0.259\,91$ (as mentioned above) $k_3^N(\gamma = 2) = 0.106\,45$, and $k_4^N(\gamma = 2) = 0.060\,24$.

The most striking structure of Fig. 1 is the very strong decrease of k_ℓ with increasing compactness. For typical neutron-star compactness, say $c \sim 0.15$, the general relativistic value of k_ℓ is about 4 times smaller than its Newtonian estimate k_ℓ^N. This might have an important negative impact on the measurability of neutron-star characteristics through gravitational-wave observations. Leaving this issue to a future investigation [48], let us
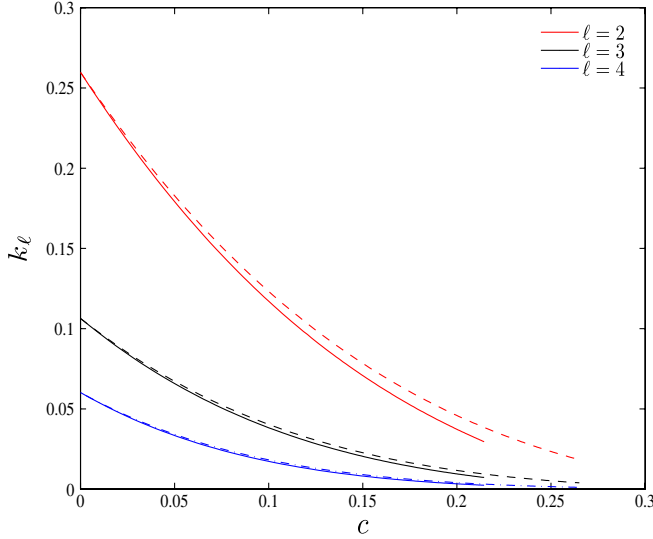
FIG. 1 (color online). Polytropic EOS: gravito-electric Love numbers k_ℓ (or apsidal constants) for $\ell = 2$, 3, 4 versus compactness $c = M/R$. We use two different polytropic EOS's, either of the rest-mass type ($p = \kappa \mu^\gamma$; solid lines) or of the energy type ($p = \kappa e^\gamma$; dashed lines). For both EOS's we use $\gamma = 2$. Note that the maximum compactness allowed by the e polytrope is larger than that for the μ polytrope.

focus here on a deeper understanding of the c sensitivity of k_ℓ.

The main origin of the strong decrease of k_ℓ, when c increases, is the universal presence of an overall factor $(1 - 2c)^2$ in k_ℓ. As discussed above, the presence of this term is linked to the no-hair property of black holes, i.e., the fact that the star-rooted contribution $\propto a_Q \hat{Q}_{\ell 2}$ in the metric variable H around a tidally-deformed star becomes singular, in the black hole limit $R \to 2GM/c_0^2$. In addition, there are other c-dependent effects that tend to decrease the value of k_ℓ. This is illustrated in Fig. 2 which plots the ratios $\hat{k}_\ell \equiv k_\ell/[k_\ell^{\rm N}(1 - 2c)^2]$ for the $\gamma = 2$, μ polytropic EOS. In the case of k_2, Fig. 2 shows that the "normalized" Love number $\hat{k}_2$ is, to a good approximation, a linearly decreasing function of c, $\hat{k}_2(c) \simeq 1 - \beta c$, with a slope $\beta \sim$ 3. In other words, the c dependence of k_2 (for $\gamma = 2$) is approximately describable as

$$k_2(c) \simeq k_2^{\rm N}(1 - 2c)^2(1 - \beta c) \qquad (\gamma = 2), \qquad (96)$$

with $\beta \simeq 3$. To get a more accurate representation, one must include more terms in the c expansion of the normalized k_2, or more generally k_ℓ, say

$$k_\ell = k_\ell^{\rm N}(1 - 2c)^2 \sum_{n=0}^{4} a_n^\ell c^n. \qquad (97)$$

Such a nonlinear fit yields an extremely accurate representation of the c dependence of $k_\ell(c)$. The performance of such fits is illustrated in Fig. 2 (dashed lines) and the best fit values of the coefficients a_n^ℓ, $0 \le n \le 4$ (fitted over the full
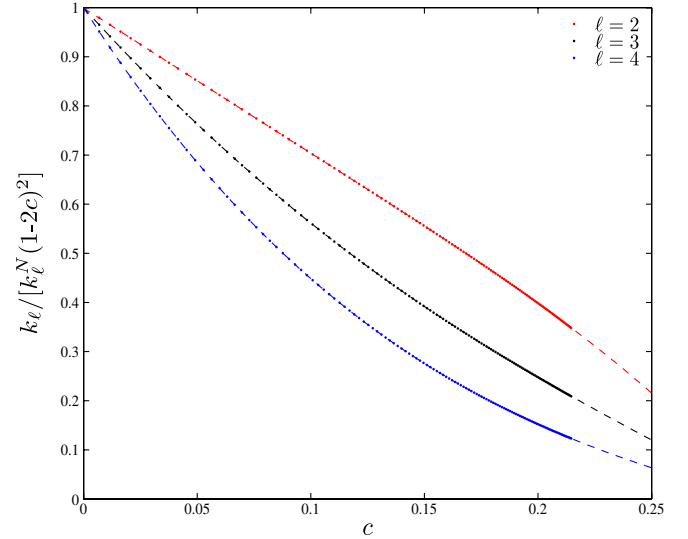

FIG. 2 (color online). Normalized Love numbers versus c for a $\gamma = 2$ μ polytrope (points) and performance of the fitting with the template given by Eq. (97) (dashed lines). The coefficients of the fit for each ℓ are listed in Table I.

range $0 < c < c_{\rm max}$) are listed in Table I. Note that, if we were to trust these fits beyond the range $0 < c < c_{\rm max}(\gamma)$ where $k_\ell(c)$ is defined, they would predict that $k_\ell(c)$ vanishes for a value c_*^ℓ slightly smaller than $c^{\rm BH} = 1/2$, and would become negative before vanishing again (now quadratically) at $c = c^{\rm BH}$. The critical value c_*^ℓ is around $1/3$, and approximately independent of ℓ.

B. Incompressible equation of state

To further explore what happens for large compactnesses, we have studied in detail the limit $\gamma \to \infty$, i.e., the incompressible EOS, $e = {\rm const}$. Let us recall that in this case the TOV equations can be solved analytically giving

$$p = e \frac{\sqrt{1 - 2c} - \sqrt{1 - 2cr^2}}{\sqrt{1 - 2cr^2} - 3\sqrt{1 - 2c}}. \qquad (98)$$

Here r denotes the dimensionless ratio $r^{\rm phys}/R$, so that $0 \le r \le 1$. Note that the central values ($r = 0$) of the pressure are $p_c = e(\sqrt{1 - 2c} - 1)/(1 - 3\sqrt{1 - 2c})$, so that $p_c \to \infty$ when $c \to 4/9$, which shows that $c_{\rm max} = 4/9$ is the

TABLE I. Fitting coefficients for k_ℓ as defined in Eq. (97) for a $\gamma = 2$ μ polytrope, up to $\ell = 4$.

ℓ	2	3	4
a_0^ℓ	0.9991	0.9997	0.9998
a_1^ℓ	−2.9287	−5.0933	−7.1938
a_2^ℓ	−1.1373	7.2008	18.9509
a_3^ℓ	14.0013	1.0826	−21.8488
a_4^ℓ	−50.9711	−18.7750	4.9031

maximum compactness reachable by an incompressible star.

Let us now discuss the computation of k_ℓ for an incompressible star. The limit $e = \text{const}$ creates a technical problem in the use of the master equation (27). Indeed, the coefficient C_0 of H, Eq. (29), contains a contribution $\propto (e + p)de/dp$ which formally vanishes in the incompressible limit $e = \text{const}$. However, as $e(r)$ is an inverted step function which vanishes outside the star, $e(r) = e_0(1 - \theta(r - 1))$, this term actually contributes a term $\propto \delta(r - 1)$ which must crucially be taken into account in the computation of k_ℓ. We can then proceed as follows. First, one numerically integrates the incompressible limit of Eq. (27) in the open interval $0 < r < 1$ representing the interior of the star. The output of this integration is the value of the logarithmic derivative of H at $r = 1^-$, say $y_{\rm in} \equiv y(R^-)$. Second, one corrects this value into the value $y_{\rm out} \equiv y(R^+)$ just outside the star. To compute the correction let us evaluate the "strength" of the delta-function singularity in the only singular piece of C_0, namely, $C_0^{\rm sing} = 4\pi G e^\lambda(e + p)de/dp$. Using the TOV equations giving the radial derivative of p, we can write (reverting to general units, with $r = r^{\rm phys}$)

$$C_0^{\rm sing} = -\frac{4\pi Gr^2}{m(r) + 4\pi Gr^3 p}\frac{de}{dr}. \tag{99}$$

In the incompressible limit $de/dr = -e_0\delta(r - R)$, $m(r) = (4\pi G/3)e_0R^3$, and $p(R) = 0$, so that

$$C_0^{\rm sing} = +\frac{3}{R}\delta(r - R). \tag{100}$$

Then the effect of the singular term in Eq. (27), or, more clearly, in the corresponding Riccati equation for $y(r) = rH'/H$,

$$ry' + y(y - 1) + rC_1y + r^2C_0 = 0, \tag{101}$$

is easily found to introduce a step function singularity in $y(r)$ with strength $y^{\rm sing}(r) = -3\theta(r - R)$. This shows that the correct value $y_{\rm out} = y(R^+)$ to be used in evaluating k_ℓ is (independent of the value of ℓ)

$$y_\ell^{\rm out} = y_\ell^{\rm in} - 3. \tag{102}$$

As a check on this result, we can consider the Newtonian limit of Eq. (27). In this limit, the exterior solutions $\hat{P}_{\ell2}(x)$ and $\hat{Q}_{\ell2}(x)$ reduce to $(r/M)^\ell$ and $(M/r)^{\ell+1}$, respectively, so that one has

$$k_\ell^N = \frac{1}{2}\frac{\ell - y}{\ell + 1 + y}, \tag{103}$$

which generalizes the $\ell = 2$ result of [6] to an arbitrary ℓ. Then, the incompressible limit, in the interior, of the Newtonian limit of Eq. (27) reads

$$H'' + \frac{2}{r}H' - \frac{\ell(\ell + 1)}{r^2}H = 0, \tag{104}$$

which coincides with the exterior, Newtonian equation for H, with general solution $H = a_P(r/M)^\ell + a_Q(M/r)^{\ell+1}$. Regularity at the origin selects the a_P term, so that $y_\ell^{\rm in}(r) = \ell = y_\ell^{\rm in}(R)$. Then, Eq. (102) determines

$$y_\ell^{\rm outN} = \ell - 3, \tag{105}$$

so that

$$k_\ell^{N({\rm incomp})} = \frac{3}{4(\ell - 1)}. \tag{106}$$

This result agrees with the known result for a limiting $\gamma = \infty$ ($n = 0$) polytrope [46].

Figure 3 shows our numerical results for the k_ℓ Love numbers of a *relativistic* incompressible star, as a function of c, in the range $0 \le c \le c_{\rm max} = 4/9$. We exhibit the three first multipolar order $\ell = 2$, 3, and 4. Note that the $c \to 0$ values of $k_\ell(c)$ agree with the Newtonian limit, Eq. (106). The phenomenon of the "quenching" of k_ℓ as c increases is even more striking, in this incompressible case, than in the $\gamma = 2$ case considered in Fig. 1 above. Note, in particular, the very small values reached by k_ℓ for the maximum compactness $c = 4/9$. To understand better the quenching of k_ℓ by strong-field effects, we have analytically studied the incompressible model in the limit of maximum compactness $c \to 4/9$. This limit is singular (because $p_c \to \infty$), but is amenable to a full analytical treatment of k_ℓ. This analytical study is thereby a useful *strong-field* analog of the analytical study of the $\gamma = 2$ model in the *weak-field* (Newtonian) limit. Let us sketch how one can analytically solve the $\gamma = \infty$, $c \to 4/9$ model. First, by introducing the variable
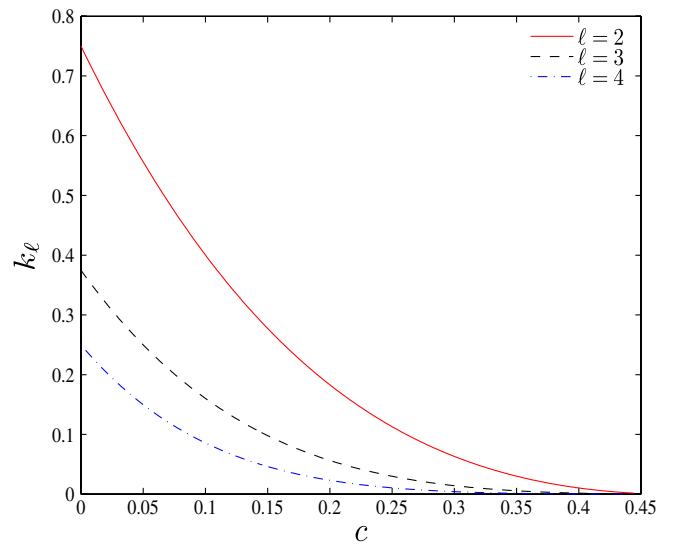

FIG. 3 (color online).   Incompressible EOS: gravito-electric Love numbers k_ℓ (or apsidal constants) for $\ell = 2$, 3, 4 versus compactness $c = M/R$.

$$x = \sqrt{1 - \frac{8}{9}r^2}, \qquad (107)$$

Equation (27) becomes

$$(x^2 - 1)^2 \frac{d^2 H}{dx^2} + (4x^3 + x^2 - 4x - 1)\frac{dH}{dx}$$
$$+ (2x^2 + x - \ell(\ell + 1) - 1)H = 0. \quad (108)$$

We found two exact, analytical solutions of this equation, which are both of the form $(1 - x)^\alpha (1 + x)^\beta$ with some rational exponents α, β. More precisely, either

$$\alpha = \frac{\ell - 1}{2}, \qquad \beta = -\frac{\ell + 1}{2}, \qquad (109)$$

or

$$\alpha = -\frac{\ell + 2}{2}, \qquad \beta = \frac{\ell}{2}. \qquad (110)$$

Regularity at the origin[8] selects the first solution, namely,

$$H(x) = (1 - x)^{(\ell-1)/2}(1 + x)^{-(\ell+1)/2}. \qquad (111)$$

As a result, the *interior* value of the logarithmic derivative y_{int} of H has the simple form

$$y(x) = \frac{\ell}{x} - 1. \qquad (112)$$

Adding the effect of the δ function at the surface ($r = 1$, i.e., $x = 1/3$), Eq. (102) finally leads to

$$y_\ell^{\text{out}} = y_{\text{int}}(1/3) - 3 = 3\ell - 4. \qquad (113)$$

Inserting this result in our general result (49) for k_ℓ gives an analytical expression for $k_\ell^{(\text{incomp})}(c_{\max})$. For instance, one finds the following value for $\ell = 2$

$$k_2^{(\text{incomp})}(c_{\max}) = \frac{4096}{10\,935(308 - 81\log 3)} \simeq 0.001\,710\,3. \qquad (114)$$

Note the striking quenching of k_2, by nearly 3 orders of magnitude, from the $c \to 0$ value $k_2^N = 0.75$ to this result for $c_{\max} = 4/9$.

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[8]Note, however, that because of the singular behavior of $p(r)$ at the origin, the "regular" solution $H(r)$ is less regular than usual: $H(r) \propto r^{\ell-1}$ instead of r^ℓ.

C. Realistic equations of state

Before discussing our results for the other tidal coefficients (b_ℓ, σ_ℓ, h_ℓ), let us end this section devoted to the k_ℓ (and μ_ℓ) Love numbers by briefly considering the k_ℓ, for the dominant quadrupolar order $\ell = 2$, predicted by the two realistic (tabulated) EOS's, FPS and SLy.[9] We recall that we have chosen them here because they have been used in some recent numerical relativity simulations of coalescing neutron-star binaries [28,29]. The maximum compactness of the two realistic EOS's that we retained are $c_{\max}^{\text{FPS}} = 0.2856$ and $c_{\max}^{\text{SLy}} = 0.303$.

The corresponding results for k_ℓ are shown in the left panel of Fig. 4 for $2 \leq \ell \leq 4$. The right panel focuses only on the results for the $\ell = 2$ case, that are plotted together with several illustrative μ polytropes, namely, $\gamma = 1.8$, 2, and 2.3. The $\gamma = 1.8$ polytrope illustrates the reason why the realistic EOS's lead to a decrease of k_ℓ as $c \to 0$. This is due to the fact that the "local" adiabatic index

$$\Gamma = \left(1 + \frac{p}{e}\right)\frac{d\log p}{d\log e} = \frac{d\log p}{d\log \mu} \qquad (115)$$

of these EOS's varies with the density (or pressure). As shown, e.g., in the bottom panel of Fig. 1 of Ref. [27] for ~nuclear densities it stays in the range $2 \lesssim \Gamma \lesssim 2.3$, while for lower densities (around neutron drip) it drops to low values $\Gamma \lesssim 1$, before rising again towards $\Gamma \sim 4/3$ for low densities. Let us recall in this respect that Newtonian polytropes have a finite radius only for $\gamma > 1.2$ ($n < 5$). When $\gamma \to 1.2$, a Newtonian polytrope has finite mass, but its radius R tends to infinity. As k_2 uses a scaling of μ_2 by a power of R, this causes $k_2^N = k_2(c = 0)$ to tend to zero as $\gamma \to 1.2$ (see [46]). Anyway, the decrease of $k_2^{\text{realistic}}(c)$ as $c \to 0$, linked to the small value of Γ for low (central) densities and pressures is a mathematical property which is physically irrelevant for our main concern, namely, the tidal properties of neutron stars. Indeed, neutron stars have a minimal mass determined by setting the mean value of Γ equal to the critical value ~4/3 for radial stability against collapse [49]. Moreover, we are mainly interested in neutron-star masses ~$1.4M_\odot$. Such neutron stars are expected to have radii varying at most in the range 10 km $\lesssim R \lesssim$ 15 km, corresponding to compactnesses $0.13 \lesssim c \lesssim 0.2$. To be on the safe side, we shall consider the interval $0.12 \leq c \leq 0.22$, which is indicated by vertical lines in the right panel of Fig. 4. Focusing our attention on this interval, we can draw the following conclusions from the inspection of the right panel of Fig. 4:

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[9]Since these EOS's are given through tables, to use them in a numerical context it is necessary to interpolate between the tabulated values. As in [27], we use simple linear interpolation (instead of third-order Hermite or spline ones) to avoid the introduction of spurious oscillations in the speed of sound. See Ref. [27] for further details.
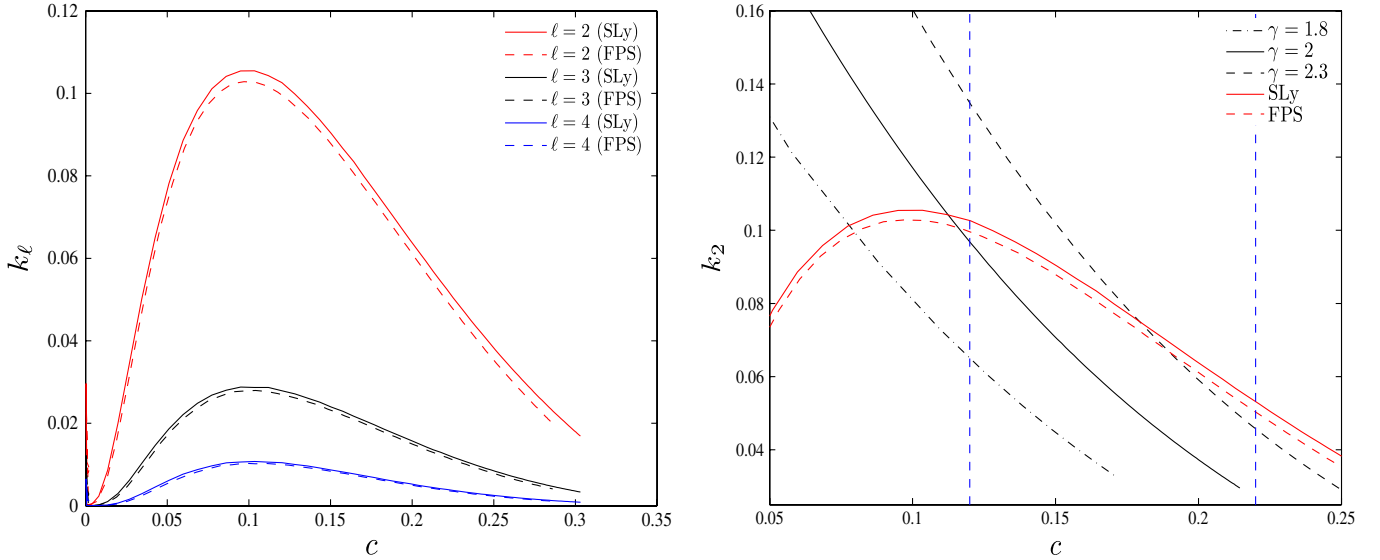
FIG. 4 (color online). The gravito-electric Love numbers k_ℓ (or apsidal constants) for $\ell = 2$, 3, 4 versus compactness $c = M/R$ for the two tabulated realistic equations of state FPS and SLy (left panel). Right panel: comparison between k_2 from various relativistic μ polytropes (with different γ) and the FPS and SLy realistic EOS's.

(1) The μ polytropes $\gamma = 2$ and $\gamma = 2.3$ approximately bracket the $k_2(c)$ sequence predicted by the two realistic EOS's.

(2) Actually, the two realistic EOS's that we retained here, FPS and SLy, lead to rather close predictions for $k_2(c)$.

(3) In the range $0.12 \leq c \leq 0.22$ the two, realistic $k_2(c)$ can be approximately represented by the following linear fit

$$k_2^{(\text{FPS;SLy})} \simeq A - Bc, \tag{116}$$

with $A \simeq 0.165$ and $B \simeq 0.515$.

VIII. RESULTS FOR THE ODD-PARITY TIDAL COEFFICIENTS σ_ℓ

As explained above, the odd-parity tidal coefficients σ_ℓ are obtained by solving the master equation (31). Let us start by noticing that the formal Newtonian limit of this master equation is simply

$$r^2 \psi'' = \ell(\ell + 1)\psi. \tag{117}$$

Indeed, all the matter-dependent contributions to Eq. (31) are, fractionally, of order $Gm(r)/(c_0^2 r) \sim 4\pi G r^2 e/c_0^2$ or $4\pi G r^2 p/c_0^2$, and vanish in the Newtonian limit $c \to 0$. In this limit, Eq. (117) does not contain any effect of the star, and, in particular, is the same in the interior or in the exterior of the star. This shows that the origin-regular solution of Eq. (117) is, everywhere, of the form

$$\psi^N(r) \propto r^{\ell+1} \tag{118}$$

and does not contain any decreasing, Q-type contribution $b_Q \psi_Q^N \propto r^{-\ell}$. This proves that the odd-parity Love number

$b_\ell = b_Q/b_P$ *vanishes* in the Newtonian limit. More precisely, as the first post-Newtonian corrections to Eq. (117) are fractionally of order $c = GM/(c_0^3 R)$, we can then easily see that the R-normalized odd-parity Love number $j_\ell = c^{2\ell+1} b_\ell$ will be of order c as $c \to 0$ (in agreement with the results of [23] concerning the $\ell = 2$ case).

As we have pointed out above that b_ℓ also contains a factor $1 - 2c$, we conclude that j_ℓ vanishes *both* when $c \to 0$ and $c \to 1/2$, and should qualitatively be of the type

$$j_\ell = c^{2\ell+1} b_\ell \simeq B_\ell c(1 - 2c). \tag{119}$$

This approximate result suggests that a 1 PN-accurate calculation of the solution of the master equation (31) should give us access to the coefficient B_ℓ, and thereby, in view of Eq. (119), to a global understanding of the c dependence of j_ℓ. In particular, one expects from (119) that $|j_\ell| = |c^{2\ell+1} b_\ell|$ will attain a maximum value somewhere around $c = 1/4$, with the value

$$|j_\ell|_{\max} \sim \frac{|B_\ell|}{8}. \tag{120}$$

Therefore, a 1 PN computation of the coefficient B gives an indication of the maximum strength of the odd-parity Love number. We have analytically computed the coefficient B_ℓ [defined by $j_\ell = B_\ell c + \mathcal{O}(c^2)$] by solving Eq. (31) by perturbation theory $\psi = \psi_0 + \psi_1$. Here $\psi_0 = r^{\ell+1}$ is the solution of the $c \to 0$ limit, Eq. (117), of Eq. (31), and ψ_1 is the first-order effect of the matter terms in Eq. (31). Actually, we found it convenient to get $\psi = \psi_0 + \psi_1$ by using Lagrange's method of variation of constants: $\psi(r) = c_1(r) r^{\ell+1} + c_2(r) r^{-\ell}$, with $c_1(r) \to 1$ and $c_2(r) \to 0$ as $r \to 0$. Skipping technical details, we found that the logarithmic derivative

$$y = \frac{r\psi'}{\psi} \simeq (\ell+1)\left[1 - \frac{2\ell+1}{\ell+1}\frac{c_2(r)}{c_1(r)}r^{-(2\ell+1)}\right] \quad (121)$$

takes the following value at the star surface:

$$y(R) = (\ell+1)\left\{1 + \frac{\ell+2}{\ell+1}\frac{1}{R^{2\ell+1}}\int_0^R dr\, r^{2\ell} \right.$$
$$\left. \times \left[2(\ell-2)\frac{m}{r} + 4\pi r^2(e-p)\right]\right\}. \quad (122)$$

The integral term in this result represents the 1 PN correction due to the presence of matter. On the other hand, the small-c limit of the general result (61) yields for $\ell = 2$ (chosen for simplicity)

$$j_2 \simeq -\frac{y-3}{y+2} \simeq -\frac{y-3}{5}, \quad (123)$$

where we used the fact $y = 3 + \mathcal{O}(c)$. Combining this result with Eq. (122) above yields

$$j_2 = c^5 b_2 \simeq -\frac{4}{5R^5}\int_0^R dr\, 4\pi G(e-p)r^6. \quad (124)$$

One can analytically compute this integral in the case of a Newtonian polytrope with $\gamma = 2$. Recalling that in this case we have $e - p \simeq \rho c_0^2 \simeq \rho_c c_0^2 \sin x / x$, with $x = \pi r/R$, the result is

$$j_2 = B_2 c + \mathcal{O}(c^2) \quad (\gamma = 2) \quad (125)$$

with

$$B_2 = -\frac{4}{5}\left(1 - \frac{20}{\pi^2} + \frac{120}{\pi^4}\right) \simeq -0.164\,395, \quad (126)$$

in agreement with [23] (modulo normalization issues that we did not check).

In view of the reasoning above, we then expect that $j_2 \simeq B_2 c(1-2c)$ will be *negative*, will vanish at $c = 0$ and (formally) at $c = 1/2$, and will reach a minimum value $(j_2)_{\gamma=2}^{\min} \simeq B_2/8 = -0.020\,55$ around $c = 1/4$. For completeness, let us also mention that the case of an incompressible neutron star leads to

$$B_2^{(\text{incomp})} = -\frac{12}{35} = -0.342\,857 \quad (127)$$

and $(j_2)_{\text{incomp}}^{\min} \simeq -0.042\,857$. We have qualitatively and semiquantitatively confirmed these results on the c dependence of the odd-parity Love number by numerically integrating Eq. (31). We display in Fig. 5 the resulting odd-parity quadrupolar Love number j_2, versus c, for both a $\gamma = 2$ μ polytrope (solid line) and an incompressible EOS (dashed line). We have numerically checked that the slope at the origin of the c axis is indeed B_2 as analytically determined above. In both cases (though it is more evident in the incompressible case, where higher values of c are allowed) j_2 has a negative minimum before rising again towards zero. The numerically determined minimum of j_2 is $\min(j_2) \simeq -0.0216$ (reached around $c \simeq 0.21$) for $\gamma = 2$
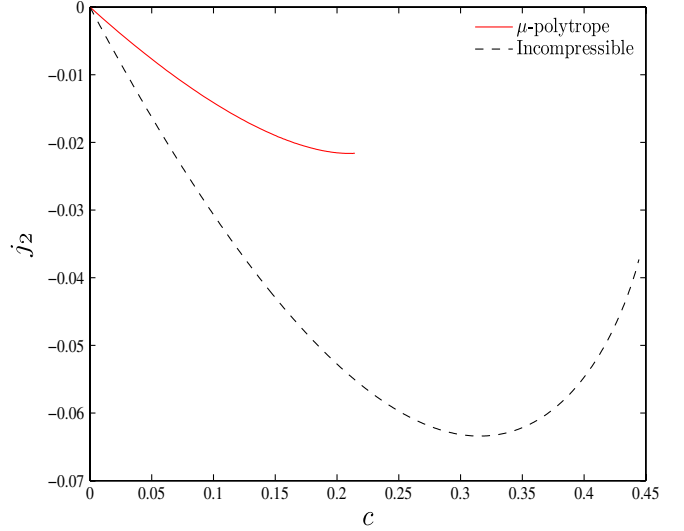

FIG. 5 (color online). The j_2, odd-parity, Love number for the $\gamma = 2$, μ polytrope and for the incompressible EOS.

and $\min(j_2) \simeq -0.0634$ (reached around $c \simeq 0.315$) for the incompressible EOS.

From the conceptual point of view, these results on the odd-parity Love number [and, via Eq. (72), on the corresponding magneticlike tidal coefficient σ_ℓ] are interesting counterparts of the even-parity results discussed above. They have points in common (their vanishing in the formal limit $c \to 1/2$), and they also strongly differ in other aspects: σ_ℓ vanishes when $c \to 0$, while μ_ℓ has a well-known Newtonian limit, and σ_ℓ is proportional to the first power of $1 - 2c$, while $\mu_\ell \propto (1 - 2c)^2$. Moreover, σ_ℓ, as naturally defined, is negative, while μ_ℓ is positive. As, with our DSX-like normalization, the interaction energies associated to both types of couplings are proportional (modulo positive numerical constants) to $-M_L G_L = -\mu_\ell G_L^2$ and $-S_L H_L = -\sigma_\ell H_L^2$, respectively, this sign difference can be interpreted as being linked to the well-known, Lorentz-signature related, fact that current-current interactions have always the opposite sign to charge-charge, or mass-mass, interactions. Concerning the formal vanishing of σ_ℓ as $c \to c^{\text{BH}} = 1/2$, the same remarks we made above for the even-parity case apply here. This fact is essentially, given the no-hair properties of black holes, a consistency check on the definition of σ_ℓ as measuring a violation of the effacing principle. As said above, although it suggests that the correct value of σ_ℓ for black holes might be zero, it is far from proving such a statement which has a meaning only within a more complex nonlinear context.

From the practical point of view, an interesting output of the investigation of σ_2 is that its numerical value happens to be quite small. Indeed, for a $\gamma = 2$ μ polytrope we have

$$\frac{|G\sigma_2|^{\max}}{R^5} = \frac{1}{48}|j_2|^{\max} \simeq 4 \times 10^{-4}. \quad (128)$$
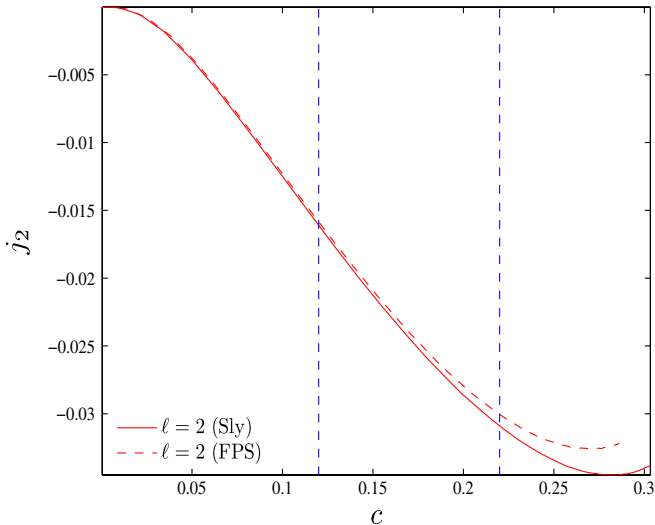
We shall discuss in another work the precise dynamical

FIG. 6 (color online). The j_2, odd-parity, Love number for FPS and SLy EOS's.

FIG. 7 (color online). Shape Love numbers h_ℓ versus c for the two $\gamma = 2$ polytropic EOS's: the μ polytrope (solid lines) and the e polytrope (dashed lines).

meaning of this small number (e.g., for the dynamics of binary neutron stars), but the appearance of such a small number clearly means that it will be an enormous challenge to measure it via gravitational-wave observations.

For completeness, we conclude this section by showing, in Fig. 6, the behavior of j_2 also for the two different realistic EOS's, FPS and SLy, that we have introduced above. Similarly to the case of k_2 [see Eq. (116) above], in the range $0.12 \lesssim c \lesssim 0.22$ (i.e., between the two dashed vertical lines) the two realistic $j_2(c)$ can be approximately represented by the following linear fit:

$$j_2^{(\text{FPS;SLy})} \simeq A - Bc, \tag{129}$$

with $A \simeq 1 \times 10^{-4}$ and $B \simeq 0.1411$.

IX. RESULTS FOR THE SHAPE LOVE NUMBERS h_ℓ

Equation (95) gave the final expression for the (even-parity) shape Love number h_ℓ. This expression contains several terms that are singular as $c \to c^{\text{BH}} = 1/2$: (i) the long curly bracket contains an explicit term $\propto 1/(1 - 2c)$; (ii) the logarithmic derivative of $\hat{Q}_{\ell 2}$ behaves, when $x \to 1$ (given that $\hat{Q}_{\ell 2}(x) \sim (x - 1)^{-1}$), as $\partial_x \log Q_{\ell 2}(x) \simeq -(x - 1)^{-1}$; and (iii) the logarithmic derivative of $\hat{P}_{\ell 2}$ behaves, when $x \to 1$ [given that $\hat{P}_{\ell 2}(x) \simeq (x - 1)^{+1}$, see below], as $\partial_x \log \hat{P}_{\ell 2}(x) \simeq +(x - 1)^{-1}$. The latter behaviors mean that, in the limit $R/(2M) \to 1$, the last factor in Eq. (95) tends to $1 + 1 = 2$. As for the $(1 - 2c)^{-1}$ singularity in the curly bracket, it is compensated by the linear vanishing of $\hat{P}_{\ell 2}(x)$ as $x \to 1$, indeed

$$P_{\ell 2}(x) \sim (x^2 - 1) \frac{d^2 P_\ell(x)}{dx^2} \sim x - 1 = \frac{1}{c} - 2. \tag{130}$$

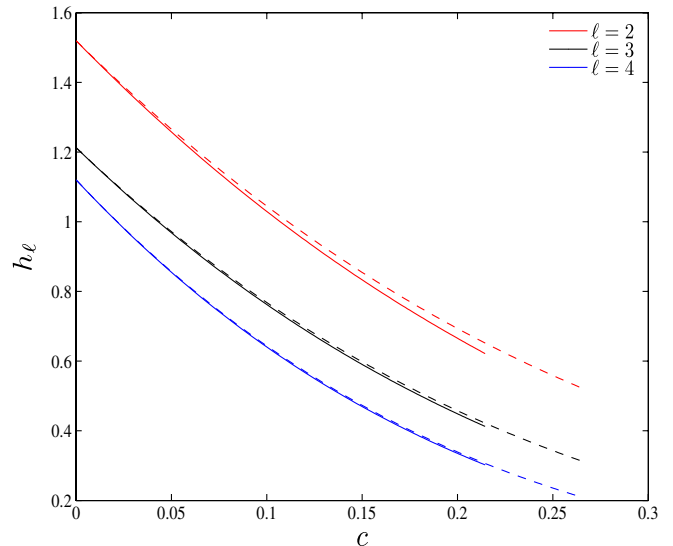Finally, the formal black hole limit, $c \to c^{\text{BH}} = 1/2$ of the

shape Love number $h_\ell(c)$ is finite and nonzero. Actually, we found that this limit agrees with the results of a recent direct investigation of the "gravitational polarizability" of a black hole [24]. The general result for h_ℓ^{BH} can be found in the latter reference. Let us only mention here the values of the first two shape Love numbers:

$$\lim_{c \to 1/2} h_2(c) = h_2^{\text{BH}} = \frac{1}{4}, \tag{131}$$

$$\lim_{c \to 1/2} h_3(c) = h_3^{\text{BH}} = \frac{1}{20}. \tag{132}$$

Figure 7 shows the results of inserting the numerically determined value of $y_\ell(R)$ into the expression (95) of h_ℓ. We give the results for the first three multipolar orders, $\ell = 2, 3, 4$, and for the two $\gamma = 2$ polytropes (μ polytrope and e polytrope). We have also investigated the results for the incompressible EOS ($\gamma \to \infty$). They are shown in Fig. 8. This information is completed in Fig. 9, where we investigate the effect of changing the EOS on the c behavior of the leading, quadrupolar, shape Love number h_2. In all cases, we see that, somewhat similarly to the k_2 case, the strong self-gravity of a neutron star tends to "quench" the value of h_2. For instance, as we discussed above, the Newtonian limit of a $\gamma = 2$ polytrope yields $h_2^N(\gamma = 2) = 15/\pi^2 \simeq 1.52$. As we see in Fig. 9, this value is reduced below 1, i.e., by more than 33%, for typical neutron-star compactnesses. When exploring stronger self-gravity effects, notably for the incompressible model, one gets an even more drastic quenching of h_2, by an order of magnitude, from the Newtonian value $h_2^{N(\text{incomp})} = 2.5$ down to a value near the "black hole" value $h_2^{\text{BH}} = 1/4 = 0.25$. From the theoretical point of view, it is nice to see this continuity, as the compactness increases, between the
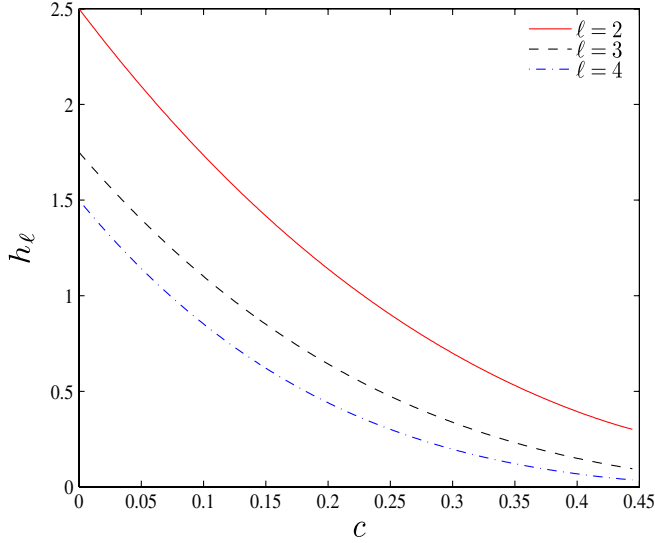
FIG. 8 (color online). Shape Love numbers h_ℓ versus c for the incompressible model.

neutron-star case and the black-hole case. Note that neither the no-hair property of black holes, nor the related effacing principle, are relevant to the present result. What is relevant is that the inner geometry of the horizon of a black hole is well defined and that a black hole is an elastic object, like a neutron star.

Similarly to what we did for k_ℓ discussed above, it is also convenient and useful to represent h_ℓ as a c expansion of the form

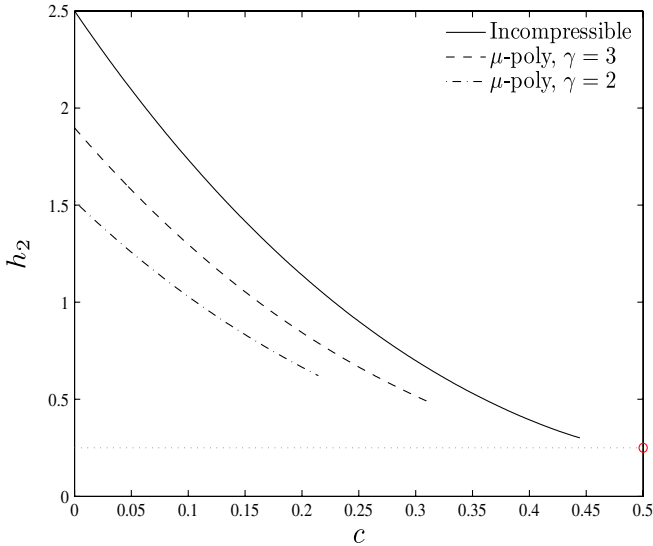$$h_\ell = h_\ell^N \sum_{n=0}^{4} b_n^\ell c^n, \qquad (133)$$

FIG. 9 (color online). Influence of the EOS on $h_2(c)$: comparison between the incompressible case and μ polytropes with two values of the polytropic index: $\gamma = 2$ and $\gamma = 3$. The red circle on the right of the plot indicates the formal $c \rightarrow 1/2$ result.

TABLE II. Fitting coefficients for h_ℓ as defined in Eq. (133) for a $\gamma = 2$ μ polytrope, up to $\ell = 4$.

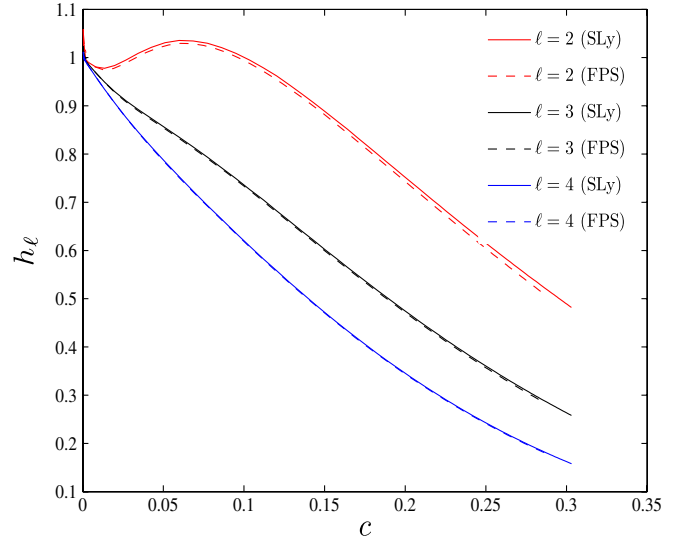| ℓ	2	3	4
b_0^ℓ	0.9999	0.9999	0.9999
b_1^ℓ	-3.6764	-4.3700	-5.2361
b_2^ℓ	4.5678	6.9775	10.4578
b_3^ℓ	-0.0192	-4.1964	-9.6026
b_4^ℓ	-5.8466	-1.028	3.0415

FIG. 10 (color online). Shape Love numbers h_ℓ versus c for the two tabulated realistic equations of state FPS and SLy.

where h_ℓ^N is the Newtonian value [obtained from k_ℓ^N through Eq. (81)] and the coefficients b_n^ℓ are obtained from a fit. As an example, Table II lists these coefficients for a $\gamma = 2$, μ polytrope up to $\ell = 4$ (i.e., they are obtained by fitting the solid lines in Fig. 7).

For completeness, we conclude this section by discussing the h_ℓ results for the two different realistic EOS's, FPS and SLy, that we have introduced above. In Fig. 10 we display the h_ℓ Love numbers (for $\ell = 2, 3, 4$) versus compactness c. The fact that $h_\ell \rightarrow 1$ when $k_\ell \rightarrow 0$ (because of the small value of the local adiabatic index Γ for low central densities and pressures) is understood via the Newtonian link (81).

X. CONCLUSIONS

We have studied the various tidal responses of neutron stars to external tidal fields. We have considered both electric-type (even-parity) and magnetic-type (odd-parity) external tidal fields. As indicated by Damour, Soffel, and Xu [7] some time ago, one can correspondingly introduce two types of linear response coefficients: an electric-type tidal coefficient $G\mu_\ell = [\text{length}]^{2\ell+1}$ measuring the ℓth mass multipole GM_L induced in a star by an external

ℓth-order (electric) tidal field G_L, and a magnetic-type tidal coefficient $G\sigma_\ell = [\text{length}]^{2\ell+1}$ measuring the ℓth spin multipole GS_L induced in a star by an external ℓth-order magnetic tidal field H_L. Dividing $G\mu_\ell$ and $G\sigma_\ell$ by the $(2\ell + 1)$th power of the star's radius R leads to dimensionless numbers of the type introduced by Love long ago in the Newtonian theory of tides. In addition, one can define a third[10] dimensionless Love number (for any ℓ), measuring the distortion of the *shape* of the surface of a star by external tidal fields.

We have studied, both analytically and numerically, these various tidal response coefficients, thereby generalizing a recent investigation of Flanagan and Hinderer. The main results of our study are

(1) A detailed study of the strong quenching of the electric-type tidal coefficients μ_ℓ (or its dimensionless version $k_\ell \sim G\mu_\ell/R^{2\ell+1}$) as the compactness $c \equiv GM/(c_0^2 R)$ of the neutron star increases. This quenching was studied both for polytropic EOS's (of two different types, see Fig. 1), for the incompressible EOS (where the quenching is particularly dramatic, see Fig. 3) and for two realistic (tabulated) EOS's (see Fig. 4).

(2) Part (though not all) of this quenching mechanism can be related to the no-hair property of black holes. The latter property ensures that some of the tidal response coefficients of neutron stars must vanish in the formal limit where $c \to c^{\text{BH}} = 1/2$. At face value, this suggests that the correct value of the μ_ℓ and σ_ℓ tidal coefficients of black holes is simply zero. We, however, argued that this conclusion is premature, until a 5 PN (5-loop) nonlinear analysis of the effective worldline action describing gravitationally interacting black holes is performed.

(3) We gave accurate nonlinear fitting formulas for the dependence of the tidal coefficients k_ℓ and h_ℓ of a $\gamma = 2$ μ polytrope on the compactness [see Eqs. (97) and (133)]. We also found that two realistic EOS's give rather close values both for the electric and magnetic tidal coefficients of neutron stars. In particular, this suggests a possible, approximately universal analytical representation of the leading, quadrupolar (electric) Love number for neutron stars of the expected compactnesses, $0.12 \lesssim c \lesssim 0.22$, namely,

$$k_2(c) \simeq 0.165 - 0.515c. \qquad (134)$$

Even if this simple linear fit reproduces with only a few percent accuracy, the c dependence of the known realistic EOS, it might suffice to deduce from future gravitational-wave observations an ac-

curate value of the neutron-star compactness. Indeed, the dimensionless parameter which is crucially entering the gravitational-wave observations is the dimensionless ratio

$$\hat{\mu}_2(c) \equiv \frac{G\mu_2}{(GM/c_0^2)^5} = \frac{2}{3}c^{-5}k_2(c). \qquad (135)$$

The strong c dependence of $\hat{\mu}_2(c)$ coming from the c^{-5} power implies that even an approximate fit such as (134) might allow one to deduce from the measurement of $\hat{\mu}_2(c)$ a rather accurate (say to better than 1%) estimate[11] of c.

(4) We surprisingly found that the magnetic-type Love numbers of neutron stars are negative, and quite small. We showed, by analytical arguments, that they can be approximately represented as $\propto Bc(1 - 2c)$ with a calculable coefficient B (that we computed in a few cases).

(5) Following a recent investigation of the gravitational polarizability of black holes [24], we studied the shape Love numbers h_ℓ of neutron stars. Again the quantity $h_\ell(c)$ is found to be drastically quenched when c increases. However, in that case $h_\ell(c)$ does *not* tend to zero as $c \to c^{\text{BH}} = 1/2$. Rather we found that $h_\ell^{\text{NS}}(c)$ tends to the nonzero black hole value h_ℓ^{BH} [24] as c formally tends to $c^{\text{BH}} = 1/2$.

In future work, we will come back to the other issues mentioned in the Introduction, namely:

(1) the incorporation of tidal effects within the effective one body formalism, starting from the additional term in the effective action

$$\Delta S = +\frac{1}{4}\mu_2 \int ds \mathcal{E}_{\alpha\beta}\mathcal{E}^{\alpha\beta}; \qquad (136)$$

(2) the study of the measurability of various tidal coefficients within the signal seen by interferometric detectors of gravitational waves.

After the submission of this work, a related paper by Binnington and Poisson [50] appeared on the archives. Reference [50] develops the theory of electric and magnetic Love numbers in a different gauge. Their results seem to be fully consistent with ours, but are less general: (i) their treatment is limited to e polytropes, (ii) they did not consider the shape Love numbers, and (iii) they do not discuss the effective action terms associated to tidal effects.

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[10]One could also introduce magnetic-like shape Love numbers by considering other aspects of the geometry around the star surface.

[11]We have in mind here, for simplicity, a black-hole-neutron-star system where only one unknown compactness enters.

[1] L. Baiotti, B. Giacomazzo, and L. Rezzolla, Phys. Rev. D **78**, 084033 (2008).

[2] L. Baiotti, B. Giacomazzo, and L. Rezzolla, Classical Quantum Gravity **26**, 114 005 (2009).

[3] K. Kiuchi, Y. Sekiguchi, M. Shibata, and K. Taniguchi, Phys. Rev. D **80**, 064037 (2009).

[4] J. S. Read *et al.*, Phys. Rev. D **79**, 124033 (2009).

[5] E. E. Flanagan and T. Hinderer, Phys. Rev. D **77**, 021502 (R) (2008).

[6] T. Hinderer, Astrophys. J. **677**, 1216 (2008).

[7] T. Damour, M. Soffel, and C. Xu, Phys. Rev. D **45**, 1017 (1992).

[8] T. Damour and A. Nagar, Phys. Rev. D **79**, 081503 (2009).

[9] A. Buonanno *et al.*, Phys. Rev. D **79**, 124028 (2009).

[10] P. D. D'Eath, Phys. Rev. D **11**, 1387 (1975).

[11] P. D. D'Eath, Phys. Rev. D **12**, 2183 (1975).

[12] K. S. Thorne and J. B. Hartle, Phys. Rev. D **31**, 1815 (1985).

[13] T. Damour, in *Gravitational Radiation*, edited by N. Deruelle and T. Piran (North-Holland, Amsterdam, 1983), p. 59.

[14] D. M. Eardley, Astrophys. J. **196**, L59 (1975).

[15] C. M. Will, *Theory And Experiment In Gravitational Physics* (Cambridge University Press, Cambridge, England, 1981).

[16] T. Damour and G. Esposito-Farese, Classical Quantum Gravity **9**, 2093 (1992).

[17] T. Damour and G. Esposito-Farese, Phys. Rev. D **58**, 042001 (1998).

[18] W. D. Goldberger and I. Z. Rothstein, Phys. Rev. D **73**, 104029 (2006).

[19] T. Damour, M. Soffel, and C. Xu, Phys. Rev. D **43**, 3273 (1991).

[20] T. Damour, M. Soffel, and C. Xu, Phys. Rev. D **47**, 3124 (1993).

[21] T. Damour, M. Soffel, and C. Xu, Phys. Rev. D **49**, 618 (1994).

[22] W. M. Suen, Phys. Rev. D **34**, 3633 (1986).

[23] M. Favata, Phys. Rev. D **73**, 104005 (2006).

[24] T. Damour and O. M. Lecian, Phys. Rev. D **80**, 044017 (2009).

[25] B. Friedman and B. R. Pandharipande, Nucl. Phys. **A361**, 502 (1981); C. P. Lorenz, D. G. Ravenhall, and C. J. Pethick, Phys. Rev. Lett. **70**, 379 (1993).

[26] F. Douchin and P. Haensel, Astron. Astrophys. **380**, 151 (2001).

[27] S. Bernuzzi and A. Nagar, Phys. Rev. D **78**, 024024 (2008).

[28] M. Shibata, K. Taniguchi, and K. Uryu, Phys. Rev. D **71**, 084021 (2005).

[29] M. Shibata, Phys. Rev. Lett. **94**, 201101 (2005).

[30] J. R. Ipser and R. H. Price, Phys. Rev. D **43**, 1768 (1991).

[31] L. Lindblom, G. Mendell, and J. R. Ipser, Phys. Rev. D **56**, 2118 (1997).

[32] C. Gundlach and J. M. Martin-Garcia, Phys. Rev. D **61**, 084024 (2000).

[33] K. Thorne and A. Campolattaro, Astrophys. J. **149**, 591 (1967).

[34] C. T. Cunningham, R. H. Price, and V. Moncrief, Astrophys. J. **224**, 643 (1978).

[35] Z. Andrade and R. H. Price, Phys. Rev. D **60**, 104037 (1999).

[36] A. Nagar and L. Rezzolla, Classical Quantum Gravity **22**, R167 (2005); **23**, 4297(E) (2006).

[37] T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063 (1957).

[38] F. J. Zerilli, Phys. Rev. D **2**, 2141 (1970).

[39] T. Damour and G. Esposito-Farese, Phys. Rev. D **53**, 5541 (1996).

[40] T. Damour, P. Jaranowski, and G. Schaefer, Phys. Lett. B **513**, 147 (2001).

[41] L. Blanchet, T. Damour, and G. Esposito-Farese, Phys. Rev. D **69**, 124007 (2004).

[42] Z. Kopal, *Dynamics of Close Binary Systems* (Reidel, Dordrecht, 1978).

[43] T. Mora and C. M. Will, Phys. Rev. D **69**, 104021 (2004); **71**, 129901(E) (2005).

[44] R. M. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984).

[45] N. Straumann, *General Relativity With Applications to Astrophysics* (Springer-Verlag, Berlin, 2004).

[46] R. A. Brooker and T. W. Olle, Mon. Not. R. Astron. Soc. **115**, 101 (1955).

[47] E. Berti, S. Iyer, and C. M. Will, Phys. Rev. D **77**, 024019 (2008).

[48] T. Damour, A. Nagar, and M. Trias (unpublished).

[49] S. L. Shapiro and S. A. Teukolsky, *Black Holes, White Dwarfs, and Neutron Stars: The Physics of Compact Objects* (Wiley, New York, 1983).

[50] T. Binnington and E. Poisson, Phys. Rev. D **80**, 084018 (2009).