On the ergoregion instability

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Rotating, ultra-compact stars in general relativity can have an ergo-region, in which all trajectories are dragged in the direction of the star's rotation. The existence of the ergoregion leads to a classical instability to emission of scalar, electromagnetic and gravitational radiation from the star. In this paper we calculate eigenfrequencies (including e-folding times) for stable and unstable modes of a scalar field on a background metric which has an ergoregion. Within a W.K.B.J. approximation for modes with angular dependence $\exp(im\phi)$, we find that unstable modes exist for all $|m| > m_0$ (m_0 depending upon the star), but that the e-folding time is asymptotically $\tau = \tau_0 \exp(2\beta m)$, where β is of order 1. Typically, τ_0 is several orders of magnitude longer than the age of the universe. However, the techniques evolved here should be applicable to other 'rotational dragging' instabilities in general relativity. Particularly useful should be the result that links the eigenfrequencies to resonances in the effective potentials governing photon motion in the metric; these potentials are rotationally 'split'.

1. INTRODUCTION

The rotation of a star in general relativity causes the trajectories of particles and photons to be dragged in the sense of the rotation. An observer far from the star would measure, for example, the period of a circular equatorial orbit near the star to be shorter in the prograde direction than in the retrograde direction. This effect, the dragging of inertial frames, can in principle be so strong that in some region all trajectories must rotate in the prograde direction: no trajectory, no matter how strongly accelerated, could rotate backwards around the star relative to a distance observer (see, for example, Misner, Thorne & Wheeler 1973). This region of space is called an ergoregion (e.r.). We have elsewhere given a preliminary discussion of the likelihood that e.rs will arise in important astrophysical situations (Schutz & Comins 1978). Here we examine in detail the instability found by Friedman (1975): a star without a horizon but with an e.r. is *unstable* to the emission of scalar, electromagnetic and gravitational waves, in the sense that any small perturbation will grow unboundedly large within the approximation of linear perturbation theory.

There are two reasons for our interest in this question. First, if the instability is a strong one, growing large within a few rotation periods of the star, then it will have

important consequences in those stars which develop e.rs even for brief periods. Second, and more important in view of the evident rarity of e.rs, is that the e.r. instability is just one example of a class of instabilities that Friedman & Schutz (1978) have called 'rotational dragging instabilities'. Another example of this class is the gravitational-radiation instability in rotating stars, discovered in the Maclaurin spheroids by Chandrasekhar (1970) and shown to exist in any rotating star, no matter how slowly rotating, by Friedman & Schutz (1978). We develop below an effective-potential approach to solving the wave equation which helps one understand the instability by making a close analogy with the more familiar problem of quantum-mechanical resonant scattering. This technique should prove applicable to all the rotational-dragging instabilities.
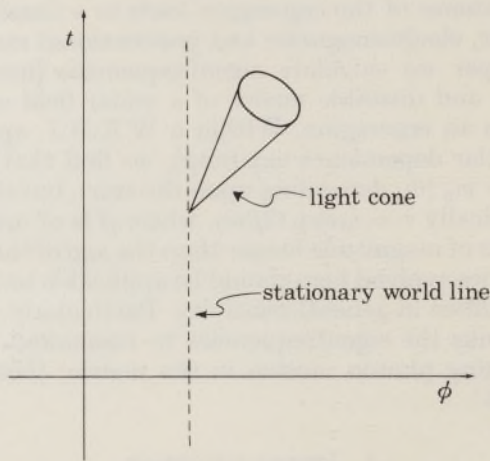

FIGURE 1. The tilting of light cones in a rotating star. The region where the stationary world lines are spacelike is called the ergoregion (e.r.). Since energy at infinity is measured by referring to timelike vectors parallel to the stationary world lines and is positive, the energy in the e.r. (where the time vector at infinity is seen as spacelike) is negative. Friedman has shown that this region is unstable to radiation of scalar, electromagnetic and gravitational radiation. Topologically the e.r. is a toroid. For more discussion of the structure of the e.r. see Schutz & Comins (1978).

The physical reason for the ergoregion instability is best understood by looking at the trajectories of photons and particles. Rotation causes the light cones in the vicinity of the star to 'tilt over' relative to those at infinity. In the e.r., the light cones have tilted so much that world lines which are stationary relative to infinity are actually outside these light cones (figure 1). It follows that all particles and photons in the e.r. must rotate in the sense of the star's rotation as seen by an observer at infinity. However, the observer at infinity measures energies by projecting four-momentum vectors onto his own time-vector, which may be thought of as parallel to the stationary world lines. As far as particles in the e.r. are concerned, this distant observer is using a spacelike – or tachyonic – energy measure, with the result that some e.r. particles are assigned a negative total energy by the distant

observer. What makes this significant is that the energy so defined is actually conserved along a freely moving particle's path. It is therefore possible to create, at zero net cost, a negative-energy photon and a positive-energy photon in the e.r. The negative-energy photon is trapped while the positive-energy one can be sent to infinity, removing energy from the star. This 'Penrose process', first suggested for black holes (Penrose & Floyd 1971), extracts rotational kinetic energy and angular momentum from the source of the metric. It has a direct analogue in the propagation of waves in the metric. For rotating black holes it implies the existence of 'super-radiant' wave modes, in which incident radiation is amplified by scattering off the hole. The amplification factor is, apparently, always finite (Press & Teukolsky 1975). For stars without horizons, however, the process actually leads to an instability, in which a small initial negative-energy perturbation radiates positive energy to infinity, thereby amplifying itself in order to conserve energy. (This does not lead to an instability in the case of a black hole because the negative energy can flow across the horizon faster than the positive energy flows to infinity, damping the wave. For a star, the negative energy is trapped inside the e.r.)

We shall limit our discussion to the massless scalar field because we have found that for a certain class of metrics having e.rs the scalar wave equation separates completely using spherical harmonics. This facilitates the mathematical treatment, while still permitting us to draw conclusions about the more physical cases of electromagnetism and gravitational radiation. We apply standard W.K.B.J. techniques in the limit of large axial eigenvalue m, and find a family of unstable modes whose e-folding time increases exponentially with m. The relativistic calculations below follow the notation of Misner *et al.* (1973).

2. SCALAR WAVE EQUATION IN A CLASS OF SPACETIMES WITH ERGOREGIONS

The metric of a stationary, axisymmetric, rotating star can be written in the form

$$ds^2 = -e^{2\Phi}\,dt^2 + e^{2\Lambda}\,dr^2 + r^2 e^{2\mu}\,d\theta^2 + r^2 e^{2\gamma}\sin^2\theta(d\phi - \omega\,dt)^2,$$

where Φ, Λ, μ, γ and ω are functions of r and θ. If the metric does not contain a horizon, none of the metric coefficients is singular. The boundary of the e.r. occurs when the asymptotically timelike killing vector ξ_t is null (lies along the boundary of the tilted light cone of figure 1). The boundary is given by

$$0 = \xi_t \cdot \xi_t = g_{00} = -e^{2\Phi} + e^{2\gamma}r^2\omega^2\sin^2\theta. \tag{1}$$

Solutions of (1) are topologically toroids, symmetric under reflexion through the equator.

In our analysis of the occurrence of e.rs in stars (Schutz & Comins 1978), we have found that realistic uniformly rotating stars with e.rs have a metric which is nearly spherical. This can be written in approximate form as

$$ds^2 = -e^{2\Phi(r)}\,dt^2 + e^{2\Lambda(r)}\,dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta(d\phi - \omega(r)\,dt)^2, \tag{2}$$

where the functions Φ, Λ and ω are computed numerically. This metric can describe a spacetime with an e.r.; it is not a solution of Einstein's equations, but when used as a background metric for the analysis of the scalar wave equation it should give reliable results. Its principal advantage is that it permits the scalar wave equation to separate completely, by using spherical harmonics. The scalar wave equation, $\nabla_{\mu}\nabla^{\mu}\psi = 0$, may be written:

$$\psi_{,\mu}^{,\mu} + \tfrac{1}{2}(\ln g)^{,\mu}\psi_{,\mu} = 0, \tag{3}$$

where $g \equiv |\det g_{\mu\nu}| = r^4\sin^2\theta\,e^{2(\Phi+\Lambda)}$. It is easy to verify that, since Φ and Λ are functions only of r, (3) separates with ψ in the form

$$\psi(t,r,\theta,\phi) = \chi_{lm}(r)\,\psi_{lm}(\theta,\phi)\,e^{i\sigma t}, \tag{4}$$

where $\psi_{lm}(\theta,\phi)$ are the spherical harmonics, and σ is the complex frequency of the wave. This gives the radial equation

$$\chi_{,rr} + [2/r + \Phi' - \Lambda']\chi_{,r} + [(\sigma+m\omega)^2\,e^{2(\Lambda-\Phi)} - l(l+1)\,e^{2\Lambda}/r^2]\chi = 0. \tag{5}$$

It will be convenient to write this equation in the form

$$\bar{\chi}_{,rr} + m^2 T(r,\Sigma)\bar{\chi} = 0, \tag{6}$$

where m is the angular eigenvalue and $\Sigma = \sigma/m$ is the negative of the pattern speed of the wave. (The pattern speed is the angular velocity at which surfaces of constant phase rotate.) The appropriate transformation is

$$\chi = \bar{\chi}\exp\left\{-\frac{1}{2}\int\left(\frac{2}{r}+\Phi'-\Lambda'\right)dr\right\}. \tag{7}$$

Substituting this into (5) yields an equation of the form (6), with

$$T(r,\Sigma) = (\Sigma+\omega)^2\,e^{2(\Lambda-\Phi)} - (l(l+1)/m^2r^2)\,e^{2\Lambda} - (\tfrac{1}{2}y^2+y')/(2m^2), \tag{8a}$$

where
$$y \equiv 2/r + \Phi' - \Lambda'. \tag{8b}$$

In the high-m limit, where Friedman (1975) has suggested that the instability sets in, the equation for $\bar{\chi}$ simplifies considerably. For this case we take $m > 0$, set $l = m$, and consider modes for which $\Sigma = 0(1)$†. Equations (6) and (8a) become, keeping dominant terms,

$$\bar{\chi}_{,rr} + m^2 e^{2(\Lambda-\Phi)}\{(\Sigma+\omega)^2 - e^{2\Phi}/r^2\}\bar{\chi} = 0. \tag{9}$$

3. Effective potentials for scalar-wave scattering

There is evident similarity between (9) and the Schrödinger equation. In the limit as ω goes to zero there is a direct analogy between Σ^2 and E, the energy of a quantum mechanical particle. In our case ($\omega \neq 0$), one might simply speak of a 'frequency

† The assumption $\Sigma = 0(1)$ means that we may have errors for modes with $|\Sigma| \lesssim \omega/m$. We shall discuss this in detail at the end of §7. The assumption $l = m$ picks the most unstable mode; modes with $l \gg m$ must be close to modes with $m = 0$ for some l, and these are never unstable. The assumption $m > 0$ is no restriction at all. From (5) it is clear that if σ is an eigenvalue for m with eigenfunction χ, then $-\sigma^*$ is an eigenvalue for $-m$ with eigenfunction χ^*. If $\chi\exp(i\sigma t)$ is outgoing unstable, so is $\chi^*\exp(-i\sigma^* t)$.

dependent' potential, meaning that Σ enters T in a complicated way. One can, however, make a good analogy with the Schrödinger equation by factorizing the potential T in (6) using (9):

$$T = e^{2(A-\Phi)}(\Sigma - V_+)(\Sigma - V_-), \tag{10}$$

$$V_\pm = -\omega \pm e^\Phi/r. \tag{11}$$

This gives us two rotationally split 'effective potentials', V_+ and V_-, typical examples of which are shown in figures 2 and 3. (In the absence of rotation, $V_+ = -V_-$ is the square-root of the usual effective potential.) These potentials represent a generalization of the quantum-mechanical potential curves with 'allowed' and 'forbidden' regions. In the regions where Σ is above the curve V_+ or below the curve
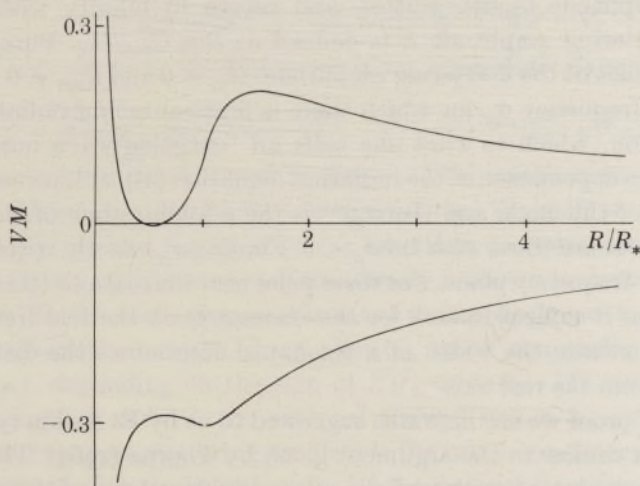

FIGURE 2. Effective potentials for our approximation to a rigidly rotating, uniform density star with a very small e.r. The region where the top curve, V_+, is negative is the e.r. The local maximum in the bottom curve, V_-, shows that there is a stable photon orbit co-rotating with the star. The star has mass M and radius R_*, with $2M/R_* = 0.849$. It has rotational period $47.4M$. The e.r. stretches from $r_0 = 0.454R_* = 1.07M$ to $r_1 = 0.595$ $R_* = 1.40M$; V_+ attains its maximum depth, $-1.67 \times 10^{-3} M^{-1}$ at $R = 0.520R_*$. Other data: $e^{A(R)} = 1.14$, $e^{\Phi(R)} = 0.146$, $V_-(R) = -0.0240 M^{-1}$. Details of the construction of the model are in Schutz & Comins (1978). To convert to ordinary units use $1M_\odot = 4.90$ μs.

V_-, solutions of (9) have oscillatory behaviour (allowed regions). Where Σ lies between V_+ and V_- the solutions have exponential behaviour (forbidden region). Note that the e.r. (if its exists) is the region where $V_+ < 0$. To see this compare V_+ in equation (11) with the boundaries for the e.r. in the equatorial plane, found by using the metric in (2) with (1). The e.r. boundaries are given by

$$r\omega = e^\Phi. \tag{12}$$

It is not hard to show that $V_\pm$ are exactly the effective-potential curves for the motion of photons (i.e. null geodesics) in this geometry, as discussed for the Kerr metric by

Misner *et al.* (1973, ch. 33). The curve V_+ is appropriate to photons with negative angular momentum (that is, in the sense opposite to that of the star) and V_- is for positive angular momentum. The minimum in V_+ shows clearly that an e.r. in a star always contains a *stable* circular photon orbit of negative energy. It is this orbit which the high-frequency, exponentially-growing modes can be expected to populate.

4. OUTGOING WAVE NORMAL MODES

We will now show that the complex eigenfrequencies of the scalar wave modes can be found by examining the scattering of real-frequency waves in these potentials. Consider first the scattering of waves with complex frequency σ, which come in from infinity with amplitude $C_{\mathrm{in}}(\sigma)$, scatter, and return to infinity with amplitude $C_{\mathrm{out}}(\sigma)$. The scattering amplitude S is defined as $S \equiv C_{\mathrm{out}}/C_{\mathrm{in}}$. Purely outgoing waves occur as poles of the scattering amplitude: $C_{\mathrm{in}} = 0$ and $C_{\mathrm{out}} \neq 0$. A pole of S gives a complex frequency σ_p for which there is finite outgoing radiation for zero incoming radiation, which is what one calls an 'outgoing-wave normal mode'. Recalling the time dependence of the radiation (equation (4)), $e^{i\sigma t}$, we see that $\mathrm{Re}\,\sigma_p$ is the frequency of the mode and $(\mathrm{Im}\,\sigma_p)^{-1}$ is the e-folding time of the radiation. The unstable modes are those with $\mathrm{Im}\,\sigma_p < 0$. Finding σ_p exactly requires a search over the complex-frequency plane. For these poles near the real axis ($\mathrm{Re}\,\sigma_p \gg \mathrm{Im}\,\sigma_p$) we shall show that it suffices to look for the *resonances* on the real frequency axis which the poles induce; the width of a resonance determines the distance of the associated pole from the real axis.

The method of proof we use here was suggested to us by R. Sorkin (private communication); it is similar to the argument given by Thorne (1969). The scattering amplitude S is an analytic function of σ in some neighbourhood of the real axis. We define a conjugate function $\bar{S}(\sigma) \equiv [S(\sigma^*)]^*$, where $*$ denotes complex conjugation; $\bar{S}$ is likewise analytic. Since $\bar{S}$ has a pole at σ_p, $\bar{S}$ has a pole at σ_p^*. Consider real values of σ, for which conservation of energy implies $|S| = 1$, or

$$S(\sigma)\,[S(\sigma)]^* = 1.$$

Since σ is real this is equivalent to

$$S(\sigma) = [\bar{S}(\sigma)]^{-1}.$$

Since this is true for all real σ, it is true everywhere where S and $\bar{S}$ are analytic. In particular, since $\bar{S}$ has a pole at σ_p^*, it implies that S has a zero at σ_p^*: an incoming wave normal mode. If the pole is simple (the usual case), it follows that *near* the pole S can be written

$$S(\sigma) = e^{2i\delta_0}(\sigma - \sigma_p^*)/(\sigma - \sigma_p),$$

where δ_0 is a constant phase, and where terms quadratic in $\sigma - \sigma_p$ and $\sigma - \sigma_p^*$ have been neglected. This is the classical resonance behaviour of the scattering phase-shift. We shall need it in the following form

$$S = e^{2i\delta_0}(\sigma - \sigma_r - i/\tau)/(\sigma - \sigma_r + i/\tau), \tag{13}$$

where σ_r and $-1/\tau$ are the real and imaginary parts of σ_p, respectively. In this convention for τ, positive τ represents exponential growth and negative τ, decay.

5. W.K.B.J. ANALYSIS OF RESONANT SCATTERING

In the high m limit, where Friedman (1975) expects the instability to be most easily excited, we shall find it convenient to treat the problem as an eigenvalue problem for $\Sigma = \sigma/m$ rather than for σ itself. To apply the W.K.B.J. analysis (cf. Morse & Feshbach 1953) we must, strictly speaking, transform to a coordinate x which sends the point $r = 0$ to $x = -\infty$: $r = e^x$. Then we define $\psi = e^{-\frac{1}{2}x}\chi$ and keep only the dominant terms in m in (9):

$$\psi_{,xx} + m^2[e^{2(x+\Lambda-\Phi)}(\Sigma+\omega)^2 - e^{2(\Phi-x)}]\psi = 0,$$

where now $-\infty < x < \infty$. To first order in an asymptotic expansion in $1/m$, ψ is given by

$$\psi = A\exp[\pm im\int T^{\frac{1}{2}}\,\mathrm{d}r]/r^{\frac{1}{2}}T^{\frac{1}{4}}, \quad T = e^{2(\Lambda-\Phi)}[(\Sigma+\omega)^2 - e^{2\Phi}/r^2], \tag{14}$$

where we have substituted r back in for e^x. (This substitution should be viewed strictly as a convenient coordinate transformation.) We shall see that the unstable modes are members of a family of modes for which Σ approaches the minimum in V_+ from above. A typical member of this family is shown in figure 3. The four separate regions of the scattering problem are drawn in figure 3: I, the innermost forbidden region ($0 < r < r_0$); II, an allowed region containing or lying entirely within the e.r. depending on the sign of Σ ($r_0 < r < r_1$); III, the potential barrier region ($r_1 < r < r_2$); and IV, the external allowed region ($r_2 < r$). It will become clear that the resonant scattering problem is the usual one: interaction of incoming radiation with the nearly-bound states of the potential well in V_+ by tunnelling through either V_+ or V_-. Of course the whole potential $T(\Sigma, r)$ depends both on V_+ and V_-, but the qualitative features of the problem are well described by V_+, except for the one unusual feature: curves with $\Sigma < 0$ tunnel out through V_-. These will turn out to be the unstable modes.

The W.K.B.J. matching of the wave functions in the different regions is begun in region I with the boundary condition that ψ is finite at $r = 0$. Since Λ, Φ, and ω are finite at $r = 0$, (14) implies that $T \sim r^{-2}$ for small r. Accordingly, ψ must vanish at $r = 0$. The radial wavefunction in region I is

$$\psi_{\mathrm{I}} = C_1\exp\left[-m\int_r^{r_0}\sqrt{|T|}\,\mathrm{d}r\right]\Big/r^{\frac{1}{2}}|T|^{\frac{1}{4}},$$

where the C_1 and all the C_i to follow are constants. The connection equations (Merzbacher 1961, ch. 7) relate ψ_{I} to the wavefunction in other regions and in particular to the incoming and outgoing parts of ψ_{IV} in region IV. (The connection relations are unchanged by the fact that T contains both Σ and Σ^2.) Writing ψ_{IV} as

$$\psi_{\mathrm{IV}} = \frac{C_4}{r^{\frac{1}{2}}T^{\frac{1}{4}}}\exp\left[im\int_{r_0}^r\sqrt{T}\,\mathrm{d}r\right] + \frac{C_5}{r^{\frac{1}{2}}T^{\frac{1}{4}}}\exp\left[-im\int_{r_2}^r\sqrt{T}\,\mathrm{d}r\right], \tag{15}$$
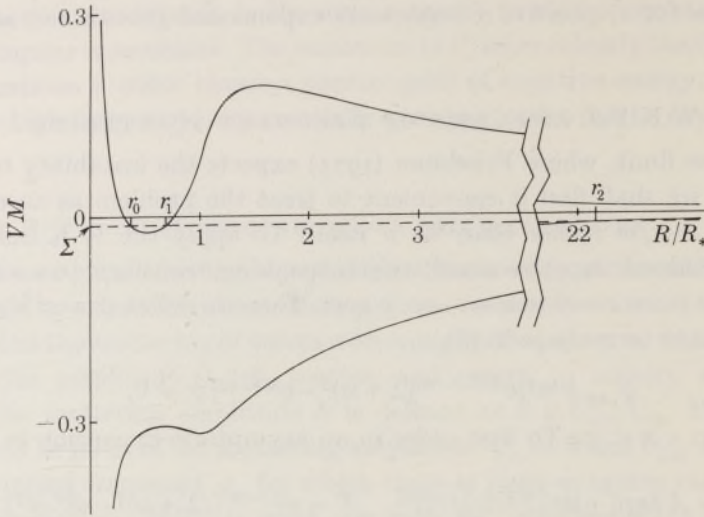
FIGURE 3. Same as figure 2 for a more rapidly rotating star. Any particular value of Σ divides
space into four regions: I $(0 < r < r_0)$ is classically forbidden; II $(r_0 < r < r_1)$ is
allowed; III $(r_1 < r < r_2)$ is again forbidden, and is the wave-tunnelling region; IV
$(r > r_2)$ is the asymptotically free allowed region. The radii r_0 and r_1 in figure 2 corre-
spond in this notation to $\Sigma = 0$. If $\Sigma > 0$, modes corresponding to almost bound states of
the V_+ well are stable and tunnel out of the barrier through V_+. If $\Sigma < 0$ they are unstable
and tunnel through V_-. This star has the same value of M/R_* as in figure 2, but has
rotational period $41.5M$. The e.r. extends from $0.323R_*$ to $0.762R_*$; V_+ attains its
maximum depth, $-1.85 \times 10^{-2} M^{-1}$, at $R = 0.514R_*$. The other parameters are $e^{A(R)} =
1.14$, $e^{\Phi(R)} = 0.145$, $V_-(R) = -0.258M^{-1}$. The V_+ well is no longer accurately described by
a parabola.

it is easy to show that C_1, C_4 and C_5 are related, in matrix notation, by

$$\begin{pmatrix} C_1 e^{i\zeta} \\ C_1 e^{-i\zeta} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2\eta + 1/2\eta & i(2\eta - 1/2\eta) \\ -i(2\eta - 1/2\eta) & 2\eta + 1/2\eta \end{pmatrix} \begin{pmatrix} C_4 \\ C_5 \end{pmatrix}, \tag{16}$$

where

$$\zeta \equiv m \int_{r_0}^{r_1} \sqrt{T}\, dr - \tfrac{1}{4}\pi, \tag{17}$$

$$\ln(\eta) \equiv m \int_{r_1}^{r_2} \sqrt{|T|}\, dr. \tag{18}$$

The identification of the incoming and outgoing amplitudes $C_{\rm in}$ and $C_{\rm out}$ with the
constants C_4 and C_5 depends on the sign of the frequency. If Σ is negative, $C_4 = C_{\rm out}$
and $C_5 = C_{\rm in}$; the opposite is true if $\Sigma > 0$. For $\Sigma < 0$, the scattering amplitude may
be written, using (15)

$$S = \frac{C_4}{C_5} = \frac{(4\eta^2 + 1)\, e^{i\zeta} - i\,(4\eta^2 - 1)\, e^{-i\zeta}}{i(4\eta^2 - 1)\, e^{i\zeta} + (4\eta^2 + 1)\, e^{-i\zeta}}. \tag{19}$$

The imaginary part of the normal-mode frequency σ_p will be small only if the
barrier-penetration integral η is large. In the limit $\eta \to \infty$, we have $S = -i$ unless
$e^{i\zeta} - i\, e^{-i\zeta} = 0$, in which case $S = +i$. So a resonance will clearly occur at a frequency

near that for which $e^{i\zeta} - ie^{-i\zeta}$ vanishes, or for which $\zeta = n\pi + \tfrac{1}{4}\pi$, where n is some integer. We call this frequency σ_n and expand ζ in a Taylor series about it:

$$\zeta(\sigma) = n\pi + \tfrac{1}{4}\pi + \alpha_n(\sigma - \sigma_n) + O(\sigma - \sigma_n)^2, \tag{20}$$

where

$$\alpha_n = \frac{\mathrm{d}}{\mathrm{d}\sigma}\left[m \int_{r_0}^{r_1} \sqrt{T}\, \mathrm{d}r \right]\Bigg|_{\sigma = \sigma_n}. \tag{21}$$

With this, the scattering amplitude S becomes

$$S = \frac{-\alpha(\sigma - \sigma_n) + 1/4\eta^2 + i[\alpha(\sigma - \sigma_n) + 1/4\eta^2]}{-\alpha(\sigma - \sigma_n) + 1/4\eta^2 - i[\alpha(\sigma - \sigma_n) + 1/4\eta^2]} + O(\sigma - \sigma_n)^2, \tag{22a}$$

where from now on we denote $\eta(\sigma_n)$ and α_n simply by η and α.

To put this in the form (13), we rewrite it:

$$S = \frac{1-i}{1+i}\left[\sigma - \sigma_n - \left(\frac{1+i}{1-i}\right)\bigg/ 4\eta^2\alpha \right]\bigg/\left[\sigma - \sigma_n - \left(\frac{1-i}{1+i}\right)\bigg/ 4\eta^2\alpha \right]. \tag{22b}$$

Observe that $(1-i)/(1+i) = -i$, which gives

$$S = -i\,[\sigma - \sigma_n - i/4\eta^2\alpha]/[\sigma - \sigma_n + i/4\eta^2\alpha]. \tag{23}$$

Thus, to this order, the resonant frequency is σ_n and the time τ [see the discussion following (13)] is

$$\tau = 4\eta^2\alpha. \tag{24}$$

It is easy to see that α is positive, since the widening of the well in V_+ with increasing Σ guarantees that the integral for ζ in (17) will increase with Σ. Then the fact that τ is positive means that these negative-Σ modes are unstable. (Had Σ been positive, the above argument would have been identical apart from the identification $C_4 = C_{\mathrm{in}}$ and $C_5 = C_{\mathrm{out}}$. This would change S into S^{-1} and consequently τ into $-\tau$.)

6. Asymptotic behaviour of the instability for large m

Perhaps the most reliable information a W.K.B.J. analysis can give us about τ is its m-dependence for large m. Since we expect the instability to occur through high m modes, it follows that if τ were, say, inversely proportional to m, then the growth time for modes of sufficiently high values of m would be smaller than any other dynamic time scales in the star and the instability could therefore be expected to have a dominant effect on the star. By contrast, if τ increases rapidly with m (as we show to be the case) the time scales for large m would be very long and the instability would be weak. We discover the asymptotic dependence of τ on m by showing the following: (i) that the higher m modes occur deeper in the e.r. well than the lower modes, with Σ approaching the minimum in V_+; (ii) that α is independent of m for large m; and (iii) that $(\ln \eta)/m$ is independent of m for large m. It will follow that to dominant order in m, $\tau \sim \exp(2m\beta)$, where β is a positive constant. We will obtain analytic expressions by approximating the bottom of the V_+ well by a parabola. This will also give useful results for stars with incipient e.rs.

We want to know where the higher-m modes appear in the e.r. well compared to the lower modes. Combining equations (17) and (20) and taking $\sigma = \sigma_n$ (on resonance) gives a relation among η, m and Σ:

$$m \int_{r_0}^{r_1} \sqrt{T}\, \mathrm{d}r = (n + \tfrac{1}{2})\, \pi; \quad n = 0, 1, 2 \ldots . \tag{25}$$

For fixed n, as m increases the integral must decrease. It is clear from (10) or figure 3 that for the integral to decrease Σ must drop deeper in the well. Since $-\Sigma$ is the pattern speed of the wave, the modes for large m approach the behaviour of the stable orbiting photon – not at all a surprising result. Next, observe that the m dependence of η and α can occur in both the integrand and the limits of integration. High in the well the change in α is small for small changes in Σ since the integrand and limits change only slightly. Low in the well this is not so obvious. By approximating the bottom of the well as a parabola we can study the change in α with m (or Σ) in more detail. We let R be the radius at which V_+ attains its minimum value, and we set $a = V_+(R)$, $a < 0$. Then we can write

$$V_+(r) = (r - R)^2/P + a. \tag{26}$$

The boundaries r_0 and r_1 are the roots of $\Sigma - V_+ = 0$: $r_0 = R - \sqrt{(P(\Sigma - a))}$, $r_1 = R + \sqrt{(P(\Sigma - a))}$. To calculate α we need

$$\mathrm{d}\sqrt{T}/\mathrm{d}\Sigma = (\Sigma + \omega)\, \mathrm{e}^{(\Lambda - \Phi)}/\sqrt{T}. \tag{27}$$

Near the bottom of the well we shall take ω, Λ, and Φ to be constants, the only significantly variable term being $\Sigma - V_+$. Then we can write for α

$$\alpha = \frac{\mathrm{d}}{\mathrm{d}\Sigma} \int_{r_0}^{r_1} \sqrt{T}\, \mathrm{d}r = \int_{r_0}^{r_1} \frac{\mathrm{d}}{\mathrm{d}\Sigma} \sqrt{T}\, \mathrm{d}r; \tag{28}$$

the derivatives $\mathrm{d}r_1/\mathrm{d}\Sigma$ and $\mathrm{d}r_0/\mathrm{d}\Sigma$ do not enter because $\sqrt{T}$ vanishes at r_1 and r_0. We have

$$\alpha = \frac{\Sigma + \omega(R)}{[\Sigma - V_-(R)]^{\frac{1}{2}}}\, \mathrm{e}^{\Lambda(R) - \Phi(R)} \int_{R - \sqrt{(P(\Sigma - a))}}^{R + \sqrt{(P(\Sigma - a))}} \left[\Sigma - a - \frac{1}{P}(r - R)^2\right]^{-\frac{1}{2}} \mathrm{d}r. \tag{29}$$

This is easily integrated to give

$$\alpha = \pi P^{\frac{1}{2}} [\Sigma + \omega(R)]\, [\Sigma - V_-(R)]^{-\frac{1}{2}}\, \mathrm{e}^{\Lambda(R) - \Phi(R)}. \tag{30}$$

This is clearly independent of m, since Σ approaches the constant a for large m.

We now turn to the barrier-penetration factor η. We take out the explicit factor of m and define

$$\ln \eta = m\beta, \qquad \beta = \int_{r_1}^{r_2} \sqrt{|T|}\, \mathrm{d}r. \tag{31}$$

For large m the lower limit r_1 approaches R, while r_2 also approaches a constant value independent of m (cf. figure 2). It is clear, then, that β is independent of m for large m.

Combining all of these arguments shows that

$$\tau = 4\eta^2\alpha = 4\alpha e^{2m\beta};$$
(32)

i.e. that τ increases exponentially with m. The instability therefore rapidly becomes less important with short wavelength; the dominant unstable modes have small m.

7. DOMINANT GROWTH RATES FOR MARGINALLY UNSTABLE STARS

We examine now the magnitude of the instability in those stars which have 'small' ergoregions; to do this we must study the smallest unstable values of m. The W.K.B.J. approximation is worst for such modes, but we will use it to get an order of magnitude feeling for the instability. At the end of this section we will discuss its reliability further. The smallest unstable m one can have is obtained by solving (25) for m with $n = 0$ and $\Sigma = 0$: the minimum m is the smallest integer exceeding

$$m_0 = \tfrac{1}{2}\pi \left[\int_{r_0}^{r_1} e^{A-\Phi} (V_+ V_-)^{\frac{1}{2}} \, dr \right]^{-1},$$
(33)

where r_0 and r_1 are the boundaries of the e.r. Of course, if there *is* a mode at $\Sigma = 0$ it will have zero growth rate (be marginally unstable) because the integral for η is infinite. So the fastest-growing mode will have some value of Σ just below zero. Our aim is to find that *maximum* growth rate. We can obtain an analytic expression for it in the limit of a very small ergoregion, where our previous parabolic approximation to the well in V_+ is valid throughout the ergoregion. Then equation (25) for the resonant frequencies becomes

$$\tfrac{1}{2}m\sqrt{P}\,e^{A(R)-\Phi(R)}[\Sigma - V_-(R)]^{\frac{1}{2}}[\Sigma - V_+(R)] = n + \tfrac{1}{2}.$$
(34)

Denoting by A the terms (which are insensitive to Σ because $|\Sigma| \ll |V_+(R)|$)

$$A = [\Sigma - V_-(R)]^{\frac{1}{2}} e^{A(R)-\Phi(R)},$$
(35)

we have for $n = 0$

$$m[\Sigma - V_+(R)] = 1/(A\sqrt{P}).$$
(36)

From (30) we have α, and for this case α is clearly insensitive to changes in Σ as long as Σ is near zero. Finally, we want β for the growth rate,

$$\beta = \int_{r_1}^{r_2} e^{A(r)-\Phi(r)} [(\Sigma - V_-)(V_+ - \Sigma)]^{\frac{1}{2}} \, dr.$$
(37)

For small Σ, r_1 will just be the outer boundary of the e.r. and r_2, where $\Sigma = V_-$, will be very large. Over most of the range of integration, e^{Φ} and e^A will be nearly 1 and ω will be very small compared to $1/r$. The outer limit is

$$r_2 = 1/|\Sigma|,$$

and β becomes

$$\beta \approx \int_{r_1}^{1/|\Sigma|} \left[-\Sigma^2 + \frac{1}{r^2} \right]^{\frac{1}{2}} \, dr.$$
(38)

This is easily integrated to give

$$\beta = -1 - \ln\left(\tfrac{1}{2}|\varSigma|r_1\right). \tag{39}$$

We want the minimum value of τ. Because α is independent of $\varSigma$, this minimum is the minimum of $m\beta$:

$$0 = \frac{1}{m}\frac{dm}{d\varSigma} + \frac{1}{\beta}\frac{d\beta}{d\varSigma}. \tag{40}$$

We use equation (36) for $dm/d\varSigma$,

$$dm/d\varSigma = -(\varSigma - V_+(R))^{-2}/(A\sqrt{P}),$$

Moreover, we can also take $dr_1/d\varSigma = 0$, and, of course, $d|\varSigma|/d\varSigma = -1$. Then (40) becomes

$$-(\varSigma - V_+(R))^{-1} - |\varSigma|^{-1}(1 + \ln\left(\tfrac{1}{2}|\varSigma|r_1\right))^{-1} = 0$$

which is an equation for $\varSigma$. It may be written more conveniently as

$$\exp\left\{-|V_+(R)/\varSigma|\right\} = \tfrac{1}{2}r_1|\varSigma|. \tag{41}$$

Given r_1 (the outer boundary of the e.r. well) and $V_+(R)$ (the depth of the e.r.), solving this equation gives the frequency $\varSigma$ which should have the shortest e-folding time. This $\varSigma$ will generally not be an eigenfrequency, but the eigenfrequency nearest it (for $n = 0$) will have the shortest growth time. The growth rate can be calculated, approximately, from (32) using (30) and (39).

One may be concerned about the accuracy of this 'fastest' growth rate, since the modes concerned have wavelength comparable to the size of their 'allowed' region. A careful discussion of the likely sources of error, however, proves reassuring. There are two numbers one wishes to calculate, the eigenfrequency $\varSigma$ and the growth time τ, and they are determined by integrals over *different* regions. The eigenfrequency comes from an integral over the 'allowed' region (ergoregion), and in this region the W.K.B.J. approximation may be unreliable. (It must be pointed out, however, that the W.K.B.J. approximation *happens* to be good for the simple harmonic oscillator; to the extent that the well in V_+ can be approximated by a parabola and that the deviation of the potential from a parabola at distances far from the ergoregion can be neglected, our approximate eigenfrequencies may in fact be quite good.) But even if we admit substantial uncertainty in the exact eigenfrequencies, the growth rates should be much more accurate. This is because they result from integrals over the barrier-penetration region (III in figure 3), in which the modes *do* have wavelength very short compared to the scale lengths of V_+ and V_-. (This remark would *not* apply to very unstable stars with deep ergoregions.) So it should be possible to trust the W.K.B.J. approximation for calculating the general run of growth time for marginally unstable stars, as in §8 below.

Another source of inaccuracy is the neglect of certain terms in passing from the exact form of the potential in (8) to its approximate form (9). The terms that were dropped are negligible compared to $V_\pm$ if m is sufficiently large and are negligible compared to $\varSigma$ if $|\varSigma|$ is of the same order as ω. In the present case it is certainly true

that $|\Sigma| \ll \omega/m$, so that we should expect the retention of these neglected terms could make a big change in Σ. But again this in itself does not affect the barrier-penetration factor too much. On the other hand, if m is small then these extra terms should be included in $V_{\pm}$ and *could* make a difference in the calculation of the growth rate. But it is hard to believe that for, say, $m = 4$, this would change β by more than say, 20 %. Inasmuch as the values we calculate for β in §8 below are so large, this change would not affect any physical conclusions.

There is another perspective on the problem of whether the potentials $V_{\pm}$ are accurate for small m. The numerical results, in the next section will be obtained for an extremely idealized example of the ergoregion instability. If it should prove possible to have an ergoregion in a realistic star (cf. Schutz & Comins 1978), the 'effective potentials' appropriate to that case will surely differ from the $V_{\pm}$ we use by terms comparable in magnitude to the ones we are neglecting here. Again, in view of the long growth times we shall find, it seems unnecessary to attempt any greater accuracy.

8. Numerical results for e.rs in uniform density stars

In order to obtain some idea of whether these growth rates will be important in stars, it is necessary to use a numerical model of an e.r. We have computed several such models in a slow-rotation approximation (Schutz & Comins 1978) and we shall calculate eigenfrequencies for two of them, both based on a uniform density star of mass M and radius R_*, with $2M/R_* = 0.849$. The first model rotates just rapidly enough to have an e.r., while the second has a comparatively deep e.r. The star with the incipient e.r. has rotational period $47.4M$ (angular velocity $0.133M^{-1}$) and angular momentum J/M^2 equal to 0.549. The potentials for this star are displayed in figure 2. The part of V_+ below the axis can be approximated very closely as a parabola. For a well this shallow we can use the analytic approximation developed in §§6 and 7 for finding the eigenmodes and their corresponding e-folding times. These are presented in table 1. We have checked these against full numerical integrations (within the W.K.B.J. approximation, of course), and find that Σ and $\ln \tau$ are accurate to within 15 %.

The e.r. instability will have a significant effect on stellar evolution if it had an e-folding time on the order of the dynamic time scales of the star. From table 1, we see that the minimum time for the e.r. instability is huge. For such a small e.r., the instability is clearly negligible.

Next, we consider a more rapidly rotating version of the same star, with a much deeper e.r. In this case the star has a rotational period of $41.5M$ and an angular momentum J/M^2 of 0.625. Its potentials are plotted in figure 3; its e.r. well is an order of magnitude deeper than in the previous case. From (25) we see that as the e.r. increases in size, lower values of m should become unstable, and that is just what we find. The lowest unstable mode has decreased from $m = 39$ in the first case to $m = 4$ here. Furthermore, the time scales have become shorter, the shortest ($\ln(\tau/M)$

= 69.8) corresponding to the lowest m mode (but still far too long to be of interest). This does not contradict the previous argument that the very lowest m unstable modes should have long time scales because the mode $m = 4$ occurs relatively deep in the well. Table 2 gives the results (m, Σ, τ) for this star. It is interesting to note that although this e.r. is much deeper and wider than the previous one and almost all values of m have unstable modes, it is still shallow compared to such factors, in the well, as a typical value of $\omega (= 0.138M^{-1})$ and Ω $(= 0.151M^{-1})$.

TABLE 1. EIGENFREQUENCIES OF THE LOWEST UNSTABLE MODES OF A SCALAR
FIELD IN THE GEOMETRY OF THE STAR DESCRIBED IN FIGURE 2

(Modes behave like $Y_m^l(\theta, \phi) \exp (i\sigma t)$, and we define $\Sigma = \sigma/m$; this is the negative of the pattern speed. We calculate only those modes for which $l = m$, and approximate $m \gg 1$. The columns give the mode number m, the eigenfrequency Σ, and the e-folding time τ of the e.r. instability, all deduced from an analytical calculation based on approximating the bottom of V_+ by a parabola (see text, §§6 and 7). (Note that for these data $|\Sigma| \ll \omega/m$, so the eigenfrequencies may not be located well.) The parameters of the fit are: $R = 1.23M$; $P = 16.9$ M^3; $a = 1.68 \times 10^{-3}M^{-1}$; $V_- = -0.240M^{-1}$.)

m	$-\Sigma \times (10^5 M)$	$\ln (\tau/M)$
39	3.62	752
40	7.70	711
41	11.6	695
42	15.3	688
43	18.9	687
44	22.2	689
45	25.5	692
46	28.5	697
47	31.4	703
48	34.4	709
49	37.0	716
50	39.7	724
51	42.1	732
52	44.5	741
53	47.0	749

TABLE 2. EIGENFREQUENCIES OF THE LOWEST UNSTABLE MODES FOR THE STAR
DESCRIBED IN FIGURE 3

(Columns correspond to those in table 1. The only approximation is W.K.B.J. Again we have $|\Sigma| \ll \omega/m$. (Here $\omega = 0.138M^{-1}$ at the bottom of the V_+ well.)

m	$-\Sigma \times (10^3 M)$	$\ln (\tau/M)$
4	1.87	69.8
5	5.03	73.4
6	7.16	80.7
7	8.68	88.8
8	9.85	97.2
9	10.8	106.0
10	11.5	115.0

9. Discussion

We have shown that the time scales for scalar radiation caused by the e.r. instability are much longer than the dynamic time scales of the stars, but are extremely sensitive to the size of the e.r. In particular, for constant density stars which are extremely relativistic, the e.r. time scales can be over 10^6 times longer than the age of the universe. Our calculations of the structure of e.rs (Schutz & Comins 1978) shows that neutron stars will not have e.rs, and even hyperon stars (should they exist) are unlikely to. Even if an e.r. forms briefly in a star (immediately after collapse to compact form, or even during its collapse to a black hole), the instability here described will be far too weak to have any effect. The interest of these calculations, then, lies in the method.

As we noted in the introduction, there are great similarities between the e.r. instability and other general-relativistic instabilities. It is likely that many features of the e.r. instability will be present in the other ones. In outline, we may guess what these features will be. First, the wave equations will have certain rotationally split effective potentials at least for local disturbances in the equatorial plane (for the gravitational-radiation instability found by Friedman & Schutz (1977) these will presumably govern sound waves in the fluid). Second, instabilities will be present if the potential for the counterrotating waves changes sign somewhere inside the system. Third, in such a case the depth of the potential will be the pattern speed of the short-wavelength unstable wave. Our assumption of spherical symmetry in the $t =$ constant slices of the background spacetime certainly has no qualitative effect on these calculations, since in the end only the effective potentials for equatorial photon motion were involved, and these exist independently of spherical symmetry. Where other rotational-dragging instabilities *will* differ from this one is in their coupling to radiation. The e.r. instability is an instability of the radiative field itself. The instability of sound waves inside a rotating star, on the other hand, is not present unless these waves are coupled to a radiation field. Calculations of e-folding times, then, will necessarily vary from one case to another. It is probably safe to assume, however, that in all cases the instability will set in for short wavelengths (large m) and become weaker as $m \to \infty$, probably exponentially.

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After this paper was prepared, we received a preprint from Sato & Maeda (1977) which studies the ergoregion instability for a second-quantized massive scalar field, with similar qualitative conclusions, but with fewer asymptotic results for large m.

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