On the non-radial oscillations of a star
III. A reconsideration of the axial modes

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It is shown that for stars with radii in the range $2.25\,GM/c^2 < R < ca.\ 3GM/c^2$, quasi-normal axial modes of oscillation are possible. These modes are explicitly evaluated for stellar models of uniform energy density.

1. Introduction

The non-radial oscillations of a star are of two kinds: those induced by the incidence of polar gravitational waves and those induced by the incidence of axial gravitational waves. These two classes of oscillations differ in one important respect: the incidence of polar gravitational waves excites fluid motions in the star while the incidence of axial gravitational waves does not. On this latter account, the scattering of axial gravitational waves can be reduced to a simple problem in *pure* scattering by a spherically symmetric static potential (as shown in Chandrasekhar & Ferrari 1991*a*, §11; this paperwill be referred to hereafter as Paper I). That nevertheless the scattering can, under suitable circumstances, exhibit resonances is generally overlooked – it *was* overlooked in Paper I. In this paper this oversight is rectified by specifying the circumstances when such resonances can occur and by illustrating those circumstances by considering the scattering by stars of uniform energy-density.

Since this paper is based exclusively on the analysis of Paper I, §11, familiarity with the methods and notations of that paper will be assumed.

2. A necessary condition for the occurrence of resonance scattering of axial gravitational waves

The scattering of axial gravitational waves by a star described in terms of an energy-density ϵ and an isotropic pressure p is governed by the wave equation (cf. I, equations (146)–(151))

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r_*^2}+\sigma^2\right)Z = VZ, \tag{1}$$

where
$$r_* = \int_0^r \mathrm{e}^{-\nu+\mu_2}\,\mathrm{d}r, \tag{2}$$

$$V = \frac{\mathrm{e}^{2\nu}}{r^3}\,[l(l+1)\,r+r^3\,(\epsilon-p)-6M(r)], \tag{3}$$

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and
$$M(r) = \int_0^r \epsilon r^2 \, \mathrm{d}r \tag{4}$$

is the mass interior to r. In the vacuum, outside the star ($r > r_1$, the radius of the star) $\epsilon = p = 0$ and V given by equation (3) reduces to the 'Regge–Wheeler' potential,

$$V = \frac{1}{r^3}\left(1 - \frac{2M}{r}\right)[l(l+1)\,r - 6M], \tag{5}$$

where M denotes the mass of the star and r_* reduces to the customary variable,

$$r_* = r + 2M \ln[(r/2M) - 1]. \tag{6}$$

It is known that the vacuum potential (5) attains a maximum for $r = r_{\max} \sim 3M$ as can be seen from table 11 and fig. 11 on pages 145 and 146 of *M.T.* (Chandrasekhar 1983). More precisely, we find from equation (5) that

$$r_{\max}/M = (3/2\kappa)\{3 + \kappa + \surd[\kappa^2 - (14\kappa/3) + 9]\}, \quad [\kappa = l(l+1)], \tag{7}$$

in particular,

$$r_{\max} = 3.2801 \quad \text{for} \quad l = 2;\; r_{\max} = 3.1061 \quad \text{for} \quad l = 3;$$
$$\text{and} \quad r_{\max}/M = 3 + \kappa^{-1} + O(\kappa^{-2}) \quad \text{for} \quad l \geqslant 4. \tag{8}$$

Therefore, if a star has a radius

$$r_1 < r_{\max}, \tag{9}$$

the potential (3) will exhibit both a maximum and a minimum.

It has been known for a long time in atomic physics that scattering by a potential which has a minimum (at $r = r_{\min}$, say) followed by a maximum (at $r = r_{\max}$, say) will exhibit resonance (or resonances) if the potential well at $r = r_{\min}$ is sufficiently deep to ensure the occurrence of quasi-stationary states. The occurrence of such quasi-stationary states in the sum of the electrostatic and centrifugal potentials of atoms has played an important role in accounting for atomic 'properties that depend primarily on *different layers* of the atomic structure' (Rau & Fano 1968; see fig. 2 in particular). In our present context, we may conclude that *a necessary condition for the occurrence of resonance scattering of axial gravitational waves by a star is that its radius*

$$r_1 < r_{\max}; \tag{10}$$

and in particular

$$r_1 < 3.2801M \quad \text{for} \quad l = 2. \tag{11}$$

However, a *sufficient condition* may be more stringent: in the illustrative example considered in §3 below, the condition is $r_1/M < 2.6$.

3. An illustrative example

In the example considered in Paper I, §11 (namely, a relativistic polytrope of index 1.5, $\epsilon_0/p_0 = 9$, $M = 0.2054$, and $r_1 = 1.576$) $r_1/M = 7.67$, the potential (exhibited in figure 2) is a monotonic decreasing function; and as to be expected, no resonance was found. Moreover, it appears that for acceptable neutron-star models $r_1/M > 3.4$ and likely to be in the range $4 < r_1/M < 6$. Resonance scattering of axial gravitational waves by neutron stars is, therefore, not to be expected. But the feature predicted by general relativity is sufficiently important that it deserves confirmation. For this purpose, configurations of uniform energy-density provide a suitable sequence of models: for it is known that these models exist for all

$$r_1/M > 2.25 \tag{12}$$

(see equation (16) below). We can therefore explore the domain

$$2.25 < r_1/M < 3.2801 \quad (\text{for} \quad l = 2). \tag{13}$$

The analytic solution for relativistic models of uniform energy density has been known since K. Schwarzschild first considered this problem in 1916. The solution can be written conveniently in the following parametric form (cf. Chandrasekhar (1964), eqs (63)–(66), or Chandrasekhar & Miller (1974), eqs (33)–(38)). Letting

$$y = e^{-2\mu_2} = \surd(1-\tfrac{2}{3}\epsilon r^2) = \surd[1-2M(r)/r], \tag{14}$$

we have

$$\left.\begin{aligned} M(r) = \tfrac{1}{3}\epsilon r^3, \quad p = \epsilon\frac{y-y_1}{3y_1-y}, \\ e^{2\nu} = \tfrac{1}{4}(3y_1-y)^2 \quad \text{and} \quad y_1 = \surd(1-2M/r_1). \end{aligned}\right\} \tag{15}$$

From equations (14) and (15) it follows that

$$y_1 = \surd(1-2M/r_1) > \tfrac{1}{3} \quad \text{or} \quad r_1/M > 2.25, \tag{16}$$

a result due to Schwarzschild.

For homogeneous models described by equations (14) and (15), the potential V_{int}, interior to $r = r_1$, given by equation (3) can be reduced to the form,

$$V_{\text{int}} = \frac{1}{4r^2}(3y_1-y)^2\left[l(l+1)-2\epsilon r^2\frac{y_1}{3y_1-y}\right] \quad (r \leqslant r_1 \quad \text{and} \quad y \leqslant y_1) \tag{17}$$

For $r \to r_1 - 0$, equation (17) gives

$$V_{\text{int}}(r \to r_1 - 0) = \frac{1}{r_1^2}\left(1-\frac{2M}{r_1}\right)[l(l+1)\,r_1 - 3M]; \tag{18}$$

and this is different from the limit $r \to r_1 + 0$ of the external potential (5). This Heaviside discontinuity in V at $r = r_1$ is a consequence of the corresponding discontinuity in ϵ at r_1.

In figure 1, we display the potentials given by equations (5) and (17) for $l = 2$ and for various masses with r_1/M in the range (2.26, 2.80). All these potentials exhibit both a minimum and a maximum. The minima V_{min} and the ordinates r_{min} at which they occur are listed in table 1. The maxima of the potentials V_{max} are always the same ($= 0.1501$) and occur at the same $r_{\text{max}} = 3.2801$.

We observe that even for $r_1/M = 2.6$, the minimum V_{min} ($= 0.1525$) of the interior potential is greater than the maximum V_{max} ($= 0.1501$) of the external potential. Consequently, we cannot expect any resonance scattering of axial gravitational waves for $r_1/M > 2.6$ – considerably smaller than the minimum acceptable radius ($r_1 = 3.4M$) for neutron stars.

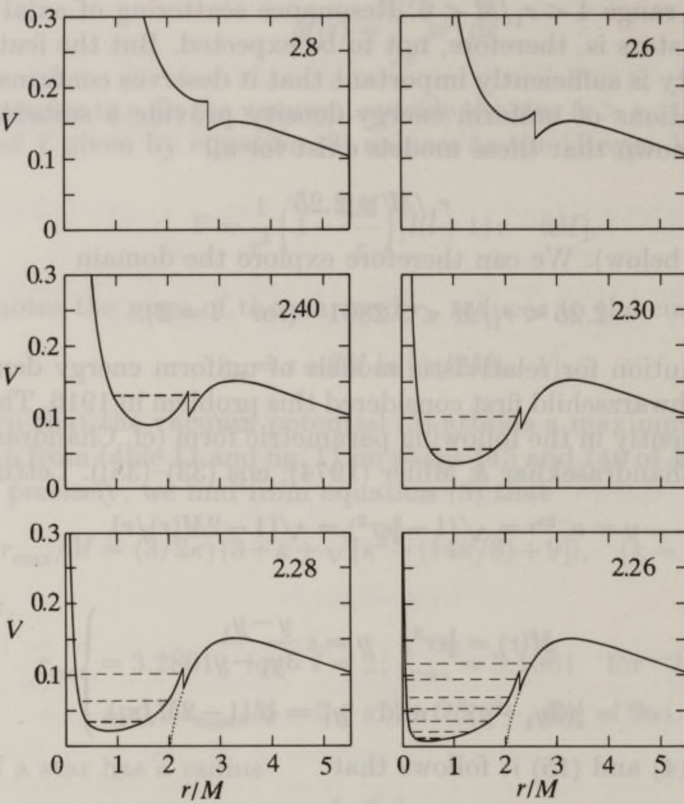

Figure 1. The potential curves (V for $l = 2$) of homogeneous stellar models of different masses (M) and radii (r_1). The curves (in the different panels) are labelled by the values of r_1/M to which they belong. At the boundary r_1/M, the potentials have a discontinuity derived from the discontinuity of ϵ. The vacuum potentials, for $r > r_1$, are the same for all models; and they have a maximum at $r = 3.280\,M$. For $r/M < 2.6$ the potentials are characterized by a maximum and a minimum. In the potential wells, one or more quasi-stationary states occur; they are shown by dashed curves.

Table 1. *The minima* ($V_{\min}$) *of the potentials and the ordinates* ($r_{\min}$) *at which they occur*

(r and M are measured in the unit $\epsilon^{\frac{1}{2}}$; V is dimensionless.)

r_1/M	r_1	M	$r_{\min}$	$V_{\min}$
2.25	1.154701	0.513200	0.241293×10^{-3}	0.448740×10^{-9}
2.26	1.152143	0.509798	0.489396	0.759228×10^{-2}
2.28	1.147079	0.503105	0.821299	0.220045×10^{-1}
2.30	1.142080	0.496557	1.031152	0.354373×10^{-1}
2.40	1.118034	0.465848	1.614285	0.898564×10^{-1}
2.50	1.095445	0.438178	1.958775	0.127347
2.60	1.074172	0.413143	2.227370	0.152513

The quasi-normal modes of axial oscillations for masses in the range (2.26, 2.50) listed in table 1 were determined by the procedure described in Paper I, §§8 and 9 (and also in Chadrasekhar & Ferrari 1991*b*, §6). The method, briefly, is the following. For an assigned real σ, we first integrate the equation (I, eq. (145)),

$$X_{,r,r} - \frac{2}{ry}\left(1 - 2\epsilon r^2 \frac{y_1}{3y_1 - y}\right) X_{,r} - \frac{(l-1)(l+2)}{r^2 y} X + \frac{4\sigma^2}{y(3y_1 - y)^2} X = 0, \tag{19}$$

Table 2. *The complex characteristic values, $\sigma_0 + i\sigma_i$, belonging to quasi-normal modes*
(M and σ are measured in the units $\epsilon^{\frac{1}{2}}$ and $\epsilon^{-\frac{1}{2}}$, respectively.)

r_1/M	M	σ_0	σ_i	$(\sigma_0 M)^2$
2.26	0.509798	0.213863874	0.23×10^{-8}	0.1189×10^{-1}
2.28	0.503105	0.3689962	0.12×10^{-5}	0.3446×10^{-1}
2.30	0.496557	0.473525	0.26×10^{-4}	0.5529×10^{-1}
2.40	0.465858	0.7767	0.92×10^{-2}	0.1309

(which is an alternative form of equations (1) and (3) for $rZ = X$, starting with the series expansion I, eq. (152) at $r = 0$. We continue the integration to the boundary of the star at $r = r_1$. At $r = r_1$ we switch to the integration of the vacuum equation,

$$\left(\frac{d^2}{dr_*^2} + \sigma^2\right) Z = \frac{1}{r^3}\left(1 - \frac{2M}{r}\right)[l(l+1)\,r - 6M], \tag{20}$$

ensuring that Z and $Z_{,r}$ are continuous at $r = r_1$. (The Heaviside discontinuity of V at $r = r_1$ is of no consequence here.) We continue the integration of equation (20) to a sufficiently large value of r_* to enable the determination of the amplitudes, $\alpha_0(\sigma)$ and $\beta_0(\sigma)$ in the asymptotic behaviour (cf. I, eq. (155))

$$Z \to \alpha_0(\sigma)\cos\sigma r_* - \beta_0(\sigma)\sin\sigma r_* \quad \text{for} \quad r_* \to \infty. \tag{21}$$

The existence of a resonance will be manifested by $\alpha_0^2 + \beta_0^2$, as a function of the initially assigned σ, exhibiting a deep minimum at a determinate frequency σ_0 with the behaviour,

$$\alpha_0^2 + \beta_0^2 = \text{const.}\,[(\sigma - \sigma_0)^2 + \sigma_i^2] \quad \text{at} \quad \sigma = \sigma_0. \tag{22}$$

A comparison of the results of the numerical integrations with the foregoing formula will determine both σ_0 and σ_i. Alternative formulae for σ_i derived in Chandrasekhar *et al.* (1991, eq. (24)) are

$$\sigma_i = \alpha_0(\sigma_0)/\beta_0'(\sigma_0) = -\beta_0(\sigma_0)/\alpha_0'(\sigma_0), \tag{23}$$

where primes denote differentiation with respect to σ.

The complex charateristic values belonging to the quasi-normal modes determined in this fashion are listed in table 2.

With respect to the values of σ_0 and σ_i listed in table 2, it should be noted that the number of significant figures retained in σ_0 are needed to locate the minimum of the $(\alpha_0^2 + \beta_0^2, \sigma)$-curve to a sufficient accuracy that the alternative ways of determining σ_i – by fitting a parabola of the form (22) to the numerically determined values of $(\alpha_0^2 + \beta_0^2)$ and by the relations (23) – all give the same value to the number of significant figures retained. To achieve the necessary accuracy all the relevant numerical integrations were performed in double precision.

A further test of the reliability of the values of σ_0 and σ_i listed in table 2 is provided by a search, in the complex σ-plane, for the complex characteristic value $\sigma_0 + i\sigma_i$ by integrating equation (1) directly for the real and the imaginary parts of the complex Z that belongs to $\sigma_0 + i\sigma_i$. (Details of this direct method of determining the quasi-normal modes are given in the Appendix.) By this method it was found that from $r_1/M = 2.4$ and $M = 0.4658$,

$$\sigma_0 = 0.7755 \quad \text{and} \quad \sigma_i = 0.935\times10^{-2}. \tag{24}$$

These values are to be compared with

$$\sigma_0 = 0.7767 \quad \text{and} \quad \sigma_i = 0.92\times 10^{-2}, \tag{25}$$

listed in table 2. We may conclude that the indirect method adopted for determining σ_0 and σ_i provide them to better than a few parts in a thousand and to a few parts in a hundred, respectively.

Returning to the values of σ_i listed in table 2, we observe that σ_i decreases very rapidly as r_1/M approaches its limiting value 2.25 – a fact readily understood in terms of the increasing depth and breadth of the potential barrier that has to be tunnelled.

It was also found that when the potential wells are sufficiently deep, as they are for $r_1/M < 2.30$, one or more *excited* quasi-stationary states emerge. The different quasi-stationary states with 'energies' $(\sigma_0 M)^2$, that were found, are shown in figure 1 as in energy-level diagrams.

4. Concluding remarks

As we have noted in §2, it is unlikely that neutron stars are sufficiently compact ($r_1/M < 2.6$, say) to manifest resonance scattering of the kind that we have demonstrated. But the existence of these modes is one further example of what is possible in the frame-work of general relativity. It is however possible that these modes are excited as transients during gravitational collapse. It is also possible that masses spiralling in the vicinity of a sufficiently compact star with a source term with non-vanishing components $T_{(1)(2)}$ and $T_{(1)(3)}$ of the energy–momentum tensor may excite these resonant modes with long damping times. This matter is currently being investigated by Ferrari and Ibañez.

A related question, that we have not considered, concerns the excitation of additional polar quasi-normal modes, besides those considered in Paper I, for stars with $r_1/M < ca.$ 3, when the vacuum Zerilli-potential will also have a maximum outside the star. Since the scattering of polar gravitational waves is not a simple problem in pure potential scattering (as the scattering of axial gravitational waves is) it is not possible to surmise in any unambiguous way.

Finally, it should perhaps be emphasized that the axial quasi-normal modes that we have found for stable stellar models with radii approaching the minimum allowed radius, 2.25 GM/c^2, are not related, generically or otherwise, to the quasi-normal modes of the Schwarzschild black-hole. The difference arises principally in the *different character of the potentials* that underlie the scattering process in the two cases (as can be seen from a comparison of the potentials illustrated in figure 1 for $r_1/M = 2.28$ and 2.26); and in the *different boundary conditions* that are applicable. In the case of the stellar model, at the boundary, $r = r_1$, we require only that the metric functions and their derivatives are continuous with *no restriction on the direction of the flow of the radiation*; and at $r = 0$, where $V(r) \to \infty$ as r^{-l}, the solution is required to be free of singularity. In contrast, in the case of the black hole, the only boundary condition is that at the horizon ($r = 2GM/c^2$) we can have *no outward directed radiation:* we can have *only inward directed radiation*. The black hole is, accordingly, characterized by a reflexion and an absorption coefficient; in contrast, the stellar model scatters the incident radiation *elastically* with no consumptive absorption.

Since quasi-stationary states can occur in the potential well in the interior of the star, resonant scattering can take place. When r_1/M approaches the limiting value

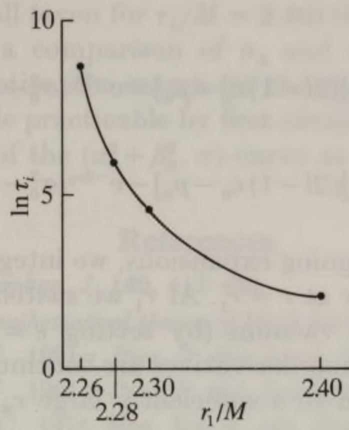

Figure 2. The damping time τ_i (in the unit $1/M$) of the lowest quasi-stationary state as a function of r_1/M. (No stable star can have a radius $r_1 \leqslant 2.25M$.)

2.25, the potential well is so deep that the damping time $\tau_i(=\sigma_i^{-1})$ of the lowest state increases very drastically (see figure 2). The stark difference in the two cases becomes clear when we compare the damping time $\tau_i \sim 4\times10^8$ for a stellar model with $r_1/M = 2.26$ (and $M = 0.5098$) and $\tau_i = 5.73$ for a Schwarzschild hole of the same mass. The effective trapping of the lowest quasi-stationary state as $r_1/M \rightarrow 2.25$ implies that the star cannot, for all practical purposes, radiate in the resonant frequency; in this respect (and only for radiation of the lowest resonant frequency σ_0), the stellar model behaves like the black hole.

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Appendix A. A direct method for determining the complex characteristic values belonging to the quasi-normal modes

While the method to be described is quite general, we shall restrict ourselves to determining the resonant axial modes considered in this paper. Accordingly, by writing

$$Z = Z_R + iZ_I \quad \text{and} \quad \sigma = \sigma_0 + i\sigma_i, \tag{A 1}$$

in equation (1) and separating the real and the imaginary parts, we obtain the pair of equations,

$$\left.\begin{aligned}\frac{d^2Z_R}{dr_*^2} + [(\sigma_0^2-\sigma_i^2)-V]Z_R - 2\sigma_0\sigma_i Z_I = 0,\\ \frac{d^2Z_I}{dr_*^2} + [(\sigma_0^2-\sigma_i^2)-V]Z_I + 2\sigma_0\sigma_i Z_R = 0.\end{aligned}\right\} \tag{A 2}$$

From the requirement that Z_R and Z_I are free of singularities at $r = 0$, we derive the series expansions,

$$\left.\begin{aligned}Z_R &= r^{l+1} + Z_{2R}\, r^{l+3} + O(r^{l+5}),\\ \text{and} \qquad Z_I &= r^{l+1} + Z_{2I}\, r^{l+3} + O(r^{l+5}),\end{aligned}\right\} \tag{A 3}$$

where (cf. I, eq. (152))

$$\left.\begin{aligned} Z_{2\mathrm{R}} &= \frac{1}{2(2l+3)}\{(l+2)\,[\tfrac{1}{3}(2l-1)\,\epsilon_0-p_0]-\mathrm{e}^{-2\nu_0}(\sigma_0^2-\sigma_\mathrm{i}^2)+2\,\mathrm{e}^{-2\nu_0}\sigma_0\,\sigma_\mathrm{i}\},\\ Z_{2\mathrm{I}} &= \frac{1}{2(2l+3)}\{(l+2)\,[\tfrac{1}{3}(2l-1)\,\epsilon_0-p_0]-\mathrm{e}^{-2\nu_0}(\sigma_0^2-\sigma_\mathrm{i}^2)-2\,\mathrm{e}^{-2\nu_0}\sigma_0\,\sigma_\mathrm{i}\}. \end{aligned}\right\} \quad (\mathrm{A}\ 4)$$

Starting at $r = 0$ with the foregoing expansions, we integrate equations (A 2) till we reach the boundary of the star at $r = r_1$. At r_1 we switch over to the integration of equations appropriate to the vacuum (by setting $\epsilon = p = 0$ in the potential V) ensuring that Z_R and Z_I and their derivatives are continuous at $r = r_1$. We integrate the vacuum equations forward to a sufficiently large r_* that comparison with the asymptotic behaviours given by

$$\left.\begin{aligned} Z_\mathrm{R} &= A_0[X\cos\sigma r_* - Y\sin\sigma r_*]-A_1[X\sin\sigma r_*+Y\cos\sigma r_*]\\ &\quad +B_0[R\cos\sigma r_*+T\sin\sigma r_*]+B_1[R\sin\sigma r_*-T\cos\sigma r_*],\\ Z_\mathrm{I} &= A_0[X\sin\sigma r_*+Y\cos\sigma r_*]+A_1[X\cos\sigma r_*-Y\sin\sigma r_*]\\ &\quad +B_0[T\cos\sigma r_*-R\sin\sigma r_*]+B_1[T\sin\sigma r_*+R\cos\sigma r_*], \end{aligned}\right\} \quad (\mathrm{A}\ 5)$$

and

$$\left.\begin{aligned} X &= \left\{1+\frac{(n+1)}{r(\sigma_0^2+\sigma_\mathrm{i}^2)}\sigma_\mathrm{i}-\frac{1}{r^2}\left[\frac{n(n+1)}{2(\sigma_0^2+\sigma_\mathrm{i}^2)^2}(\sigma_0^2-\sigma_i^2)+\frac{3M}{2(\sigma_0^2+\sigma_\mathrm{i}^2)}\sigma_\mathrm{i}\right]\right\}\mathrm{e}^{-\sigma_\mathrm{i}r_*},\\ Y &= \left\{\frac{(n+1)}{r(\sigma_0^2+\sigma_\mathrm{i}^2)}\sigma_0+\frac{1}{r^2}\left[\frac{n(n+1)}{(\sigma_0^2+\sigma_\mathrm{i}^2)^2}\sigma_0\,\sigma_\mathrm{i}-\frac{3M}{2(\sigma_0^2+\sigma_\mathrm{i}^2)}\sigma_0\right]\right\}\mathrm{e}^{-\sigma_\mathrm{i}r_*},\\ R &= \left\{1-\frac{(n+1)}{r(\sigma_0^2+\sigma_\mathrm{i}^2)}\sigma_\mathrm{i}-\frac{1}{r^2}\left[\frac{n(n+1)}{2(\sigma_0^2+\sigma_\mathrm{i}^2)^2}(\sigma_0^2-\sigma_\mathrm{i}^2)-\frac{3M}{2(\sigma_0^2+\sigma_\mathrm{i}^2)}\sigma_\mathrm{i}\right]\right\}\mathrm{e}^{+\sigma_\mathrm{i}r_*},\\ T &= \left\{-\frac{(n+1)}{r(\sigma_0^2+\sigma_\mathrm{i}^2)}\sigma_0+\frac{1}{r^2}\left[\frac{n(n+1)}{(\sigma_0^2+\sigma_\mathrm{i}^2)^2}\sigma_0\,\sigma_\mathrm{i}+\frac{3M}{2(\sigma_0^2+\sigma_\mathrm{i}^2)}\sigma_0\right]\right\}\mathrm{e}^{+\sigma_\mathrm{i}r_*}, \end{aligned}\right\} \quad (\mathrm{A}\ 6)$$

enables us to determine the coefficients A_0, A_1, B_0 and B_1 in equations (A 5). These equations follow from equation (1) by substituting for Z a behaviour at $r_* \to \infty$ of the form (cf. I, eqn. (155)),

$$\begin{aligned} Z \to &+\mathrm{e}^{+\mathrm{i}\sigma r_*}(A_0+\mathrm{i}A_1)\left\{1+\mathrm{i}(n+1)\,\frac{1}{\sigma r}-\tfrac{1}{2}[n(n+1)+3\mathrm{i}M\sigma]\,\frac{1}{\sigma^2 r^2}+\dots\right\}\\ &+\mathrm{e}^{-\mathrm{i}\sigma r_*}(B_0+\mathrm{i}B_1)\left\{1-\mathrm{i}(n+1)\,\frac{1}{\sigma r}-\tfrac{1}{2}[n(n+1)-3\mathrm{i}M\sigma]\,\frac{1}{\sigma^2 r^2}+\dots\right\}. \end{aligned} \quad (\mathrm{A}\ 7)$$

The last remaining boundary condition that at $r_* \to \infty$ we have no incoming waves requires that A_0 and A_1 (i.e. the coefficient $A_0 + \mathrm{i}A_1$ of $\mathrm{e}^{+\mathrm{i}\sigma r_*}$ in equation (A 7)) vanish simultaneously. Satisfying this last boundary condition will complete the solution of the characteristic-value problem and determine the characteristic value, $\sigma_0 + \mathrm{i}\sigma_\mathrm{i}$, that we seek.

The increasing exponential behaviour, $\mathrm{e}^{+\sigma_\mathrm{i} r_*}$, of the terms R and T in equations (A 7) will make the matching of the integrated solution with the required asymptotic behaviour (A 5) difficult if σ_i is of order unity. But for the cases considered in this paper, σ_i is sufficiently small (even for $r_1/M = 2.40$) that the problem of matching is not unduly difficult; and a comparison of σ_0 and σ_i derived by the alternative methods is feasible. In practise, the search for the characteristic value of $\sigma_0 + i\sigma_i$ in the complex σ-plane is made practicable by first obtaining a preliminary value for σ_0 by locating the minimum of the $(\alpha_0^2 + \beta_0^2, \sigma)$-curve as described in §3.

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