Numerical Relativity in higher dimensional spacetimes – Part 1

Helvi Witek

DAMTP
University of Cambridge, UK

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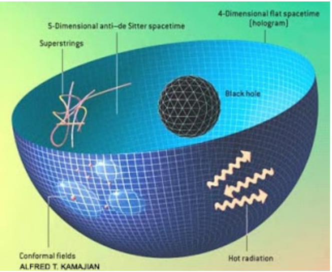
Outline

next time:

Cartoon method
Dimensional reduction

Why higher D gravity?

Gauge / gravity duality (Maldacena '97)



- correspondence between
  gravity in $AdS_D \times S^n$
  $\Updownarrow$
  strongly coupled CFT in $d = D - 1$
  "living" on AdS boundary
- BHs in AdS_D dual to thermal states in CFT
- Applications:

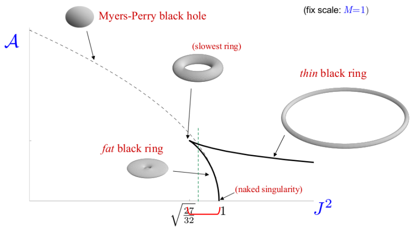
- gravity / hydrodynamics correspondence
  - computing transport coefficients
  - turbulence in hydrodynamics
- (gravitational) turbulent instabilities in AdS (see Andrzej's lecture)
- gravity / condensed matter systems
  - computing conductivity in condensed matter systems
- heavy ion collisions RHIC

Why higher D gravity?

BHs in $D = 4$: uniqueness, spherical topology, dynamical stability, ...

BHs in $D > 4$:

- black objects with non-spherical topology:
  - black string ($S^q \times \mathbb{R}$)
  - black p-branes ($S^q \times \mathbb{R}^p$) (Horowitz & Strominger '91)
  - 5D black ring ($S^2 \times S^1$) (Emparan & Reall '02)
  - black saturn (Elvang & Figueras '07)
  - ...
- no No-Hair theorem
- dynamical stability?

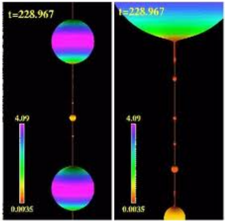


Emparan & Reall. '08

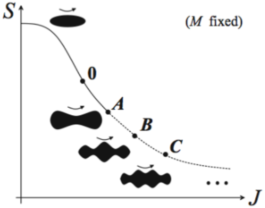
Why higher D gravity?

BHs in $D > 4$: dynamically (un-)stable solutions

- in $D \geq 5$:
  Gregory-Laflamme instability ('93)
- dynamical studies of the GL instability of black strings in $D = 5$
  (Choptuik et al. '03, Lehner & Pretorius '10)
- in $D \geq 6$:
  Myers-Perry BHs with one rotation parameter: ultra-spinning instability
  (Dias et al. '09)
- numerical study of non-axisymmetric perturbation of Myers-Perry BH in $D = 5, ..., 8$: critical spin parameter $a/\sqrt{\mu} \approx 0.87$ in $D = 5$
  (Shibata & Yoshino '09, Shibata & Yoshino '10)



Lehner & Pretorius '10



Dias et al. '09

Why higher D gravity?

High energy collisions and TeV gravity

- Hoop - Conjecture:
  BH formation if $C < 2\pi r_S$ or if $b \leq r_S$
  (Thorne '72, Penrose '74, Eardley & Giddings '02)
- ultra relativistic collision of
  - boson stars: BH formation if boost $\gamma_c \geq 2.9$
    (Choptuik & Pretorius '10)
  - fluid particles: BH formation if boost $\gamma_c \geq 8.5$
    (East & Pretorius '12)

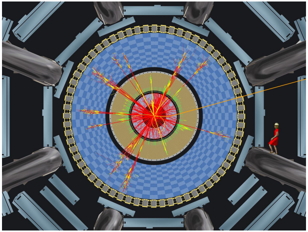
$\Rightarrow$ black hole formation in high energy collisions of particles

- high energy particle collisions with $E = 2\gamma m_0 c^2 > M_{Pl}$
  - gravity is dominant $\rightarrow$ classical description
  - "matter does not matter"

$\Rightarrow$ high energy particle collisions well described by high energy BH collisions

Why higher D gravity?

- TeV gravity scenarios – gravity theories with
  - large extra dimensions (Arkani-Hamed, Dimopoulos & Dvali '98)
  - warped extra dimensions (Randall & Sundrum '99)
- $M_{PL,D>5} \sim \mathcal{O}(TeV)$
  $\Rightarrow$ signatures of BH production at LHC or in cosmic rays?



Atlas experiment @LHC / CERN

- Goal: precise understanding of BH formation in $D > 4$ (energy emission, mass, spin, cross section)
  - lower bound from area theorem (Yoshino & Nambu '02)
  - shock wave collisions: $E = \frac{1}{2} - \frac{1}{D}$ (Coelho et al '12) (see Marco's lecture)
  - PP approximation: $\frac{ME_{rad}}{m_0^2 E} \sim 26\%(D=4)\ldots10\%(D=11)$ (Berti et al '10)
  - numerical simulations in $D=4$: $\frac{E_{rad}}{M} \sim 14\ldots35\%$ (Sperhake et al '08,'09, Shibata et al '09)
  - numerical simulations of BH collisions in $D=5$ (Zilhao et al '10, Witek et al '10, Okawa et al '11)

Numerical Relativity in a nutshell

NumRel in a nutshell I

Goal: time evolution problems in D-dim gravity
$\Rightarrow$ solve Einstein's equations $R_{MN} - \frac{1}{2} g_{MN} R = 8\pi G_D T_{MN}$

- $D(D+1)/2$ coupled PDEs
  - mixed elliptic, hyperbolic, parabolic type
  - in general: non-separable
- (semi-) analytic methods / "soft numerics":
  - weak-field regime – $(v/c)^2 \ll 1$: Post-Newtonian methods
  - perturbative methods – neglect backreaction (see Paolo's lectures)
- here: strong field/ high curvature regime
  - "hard-core" numerical methods
  - supercomputers (see Sérgio's lectures)

Cosmos @ DAMTP/ Cambridge

Note: visit to Baltasar cluster this week

NumRel in a nutshell II

Tasks: formulation of EEs as initial (boundary) value problem

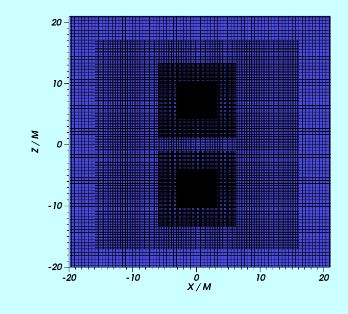
- constraint subsystem
  - construction of initial data (see Hiro's lectures)
- evolution subsystem
  - reduction to effectively $3+1$ problems
    $\Rightarrow$ (modified) Cartoon method
    $\Rightarrow$ dimensional reduction
  - well-posedness of formulation (see David's lectures)
  - here: BSSN formulation of EEs in $D > 4$
- treatment of singularity
  - excision
  - moving punctures
- gauge choice yielding numerically stable evolutions
- analysis tools
  - apparent horizon
  - gravitational waves from metric perturbations

NumRel in a nutshell III

discretization for numerical integration

- formulate EEs as time evolution problem: $\partial_t \mathbf{u} \sim f(\mathbf{u}, \partial\mathbf{u}, \partial\partial\mathbf{u})$
- approximate continuum functions $\mathbf{u}_c$ by discrete functions $\mathbf{u}_{i,j,k}$
- method of lines:
  1. evaluate rhs in spatial domain using, e.g., finite differences
  2. integrate in time using 4^{th} order Runge-Kutta, iCN, ...

- adaptive mesh refinement
- parallelization (e.g. MPI, OpenMP)
- example for international effort to provide powerful GR package:
  Einstein Toolkit (see Miguel's lecture)
  (http://einsteintoolkit.org/)



$(D-1)+1$ decompostion of spacetime

$(D-1)+1$ decomposition of spacetime I

- foliation of D-dimensional spacetime $(\mathcal{M}, g_{MN})$: (Arnowitt, Deser, Misner '62)

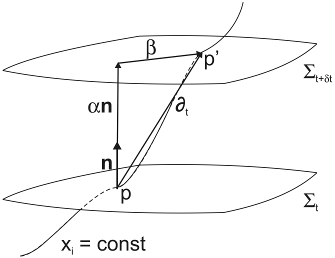
$$\mathcal{M} =^{(D-1)} \Sigma_t \times \mathbb{R}$$

  - ${}^{(D-1)}\Sigma_t$ – $(D-1)$-dim. spatial hypersurface
  - $t \in \mathbb{R}$ – time parameter
  - g_{MN} – metric in $\mathcal{M}$; $\gamma_{\bar{m},\bar{n}}$ – metric in Σ_t

- n^M – unit vector normal to Σ_t
- α – lapse function
  - relates coordinate time t to time measured by an Eulerian observer
- β^M – shift vector
  - relative velocity between Eulerian observer and lines of const. spatial coordinates

Notation

- $M, N, ... = 0, ..., D-1$
- $\bar{m}, \bar{n}, ... = 1, ..., D-1$
- $\mu, \nu, ... = 0, ..., 4$
- $i, j, ... = 1, 2, 3$



- vector pointing from $p \in \Sigma_t$ to $p' \in \Sigma_{t+\delta t}$:

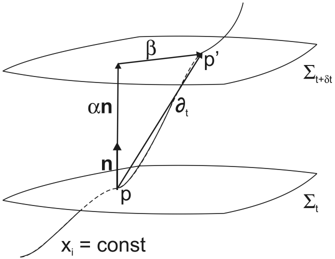
$$t^M = \alpha n^M + \beta^M$$

where

$$\beta^M = (0, \beta^{\bar{m}}) ,$$

$$n_M = -\alpha(1, 0, ..., 0) , \quad n^M = \frac{1}{\alpha}(1, -\beta^{\bar{m}}) ,$$

$$n_M n^M = -1$$



$(D-1)+1$ decomposition of spacetime III

- line element of $(\mathcal{M}, g_{MN})$

$$ds^2 = g_{MN} dx^M dx^N = -\left(\alpha^2 - \beta_{\bar{k}}\beta^{\bar{k}}\right) dt^2 + 2\gamma_{\bar{m}\bar{n}}\beta^{\bar{m}} dt dx^{\bar{n}} + \gamma_{\bar{m}\bar{n}} dx^{\bar{m}} dx^{\bar{n}}$$

- $\gamma_{\bar{m}\bar{n}}$ – induced, $(D-1)$-dimensional, spatial metric on Σ_t

$$\gamma_{MN} = g_{MN} + n_M n_N \,, \quad \gamma^{MN} = g^{MN} + n^M n^N$$
$$\gamma_{MN} n^N = n_M + n_M (n_N n^N) = 0$$

  – determines projection operator

$$\perp = \gamma^M{}_N = \delta^M{}_N + n^M n_N$$

$(D-1)+1$ decomposition of spacetime IV

- any (p,q)-tensor $T^{M_1\ldots M_p}{}_{N_1\ldots N_q} \in \mathcal{M}$ can be decomposed into its normal component, fully spatial projection and mixed projections
- example: rank-2 tensor T_{MN}
  - normal component:
$$\mathcal{N} = T_{MN} n^M n^N$$
  - spatial projection
$$\mathcal{S}_{MN} = \gamma^K{}_M \gamma^L{}_N T_{KL}$$
  - mixed projection
$$\mathcal{T}_M = \gamma^K{}_M T_{KL} n^L$$
- reconstruct full tensor $T_{MN} \in \mathcal{M}$
$$T_{MN} = \mathcal{S}_{MN} + \mathcal{T}_M n_N + \mathcal{T}_N n_M + \mathcal{N} n_M n_N$$

$(D-1)+1$ decomposition of spacetime V

extrinsic curvature K_{MN}

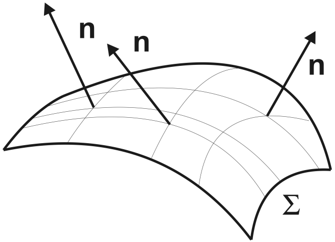
- describes how Σ_t is embedded in $\mathcal{M}$
- describes how the direction of n^M changes as it moves along Σ_t

$$K_{MN} = -\gamma^K{}_M \nabla_K n_N = \ldots = -\nabla_M n_N - n_M a_N \,, \quad \text{with} \quad a_M = n^N \nabla_N n_M$$

- properties ($\diamondsuit$) :

$$\text{symmetric:} \quad K_{MN} = K_{(MN)}$$

$$\text{spatial:} \quad K_{MN} n^M = 0$$



- relation between K_{MN} and metric γ_{MN} ($\diamondsuit$)

$$K_{MN} = -\frac{1}{2\alpha}\mathcal{L}_{\alpha n}\gamma_{MN} = -\frac{1}{2\alpha}(\partial_t - \mathcal{L}_\beta)\gamma_{MN}$$

$\Rightarrow$ momentum of spatial metric γ_{MN}

derivation from geometrical concepts $\Rightarrow$ purely kinematic

$(D-1)+1$ decomposition of Einstein's equations

$(D-1)+1$ decomposition of EEs I

- so far: considered only geometric concepts and relations $\Rightarrow$ kinematics
- dynamical description from Einstein's equations

$$ {}^{(D)}R_{MN} - \frac{1}{2} g_{MN}\, {}^{(D)}R = 8\pi G_D T_{MN} $$

- $(D-1)+1$ split of EEs (York '79)
  1. projections of the Riemann tensor $\rightarrow$ Gauss-Codazzi relations
  2. projections of EEs ($\diamondsuit$hand's-on session with mathematica)
- intrinsic curvature on Σ_t – $(D-1)$-dim Riemann tensor ${}^{(D-1)}R^{\bar{k}}{}_{\bar{l}\bar{m}\bar{n}}$:

$$ (D_{\bar{m}} D_{\bar{n}} - D_{\bar{n}} D_{\bar{m}}) v^{\bar{k}} = {}^{(D-1)} R^{\bar{k}}{}_{\bar{l}\bar{m}\bar{n}} v^{\bar{l}} \,, \qquad \forall v^{\bar{k}} \in \Sigma_t $$

- note: ∇_M corresponding to g_{MN}, $D_{\bar{m}}$ corresponding to $\gamma_{\bar{m}\bar{n}}$

$(D-1)+1$ decomposition of EEs II

projections of the Riemann tensor and Gauss-Codazzi relations
(sketch for calculation (♢))

- consider

$$\begin{aligned}\nabla_M\gamma^A{}_B =& n^A\nabla_M n_B + n_B\nabla_M n^A\\ =& -n^A K_{BM} - n^A n_M a_B - n_B K^A{}_M - n_B n_M a^A\end{aligned}$$

  with $K_{MN} = -\nabla_M n_N - n_M a_N$ and $a_M = n^N\nabla_N n_M = ... = \frac{1}{\alpha}D_M\alpha$
- consider

$$\begin{aligned}D_M D_N v_L =& \gamma^A{}_M\gamma^B{}_N\gamma^C{}_L\nabla_A\nabla_B v_C\\ &+ \gamma^A{}_M\gamma^B{}_N\gamma^C{}_L\nabla_D v_C\nabla_A\gamma^D{}_B + \gamma^A{}_M\gamma^B{}_N\gamma^C{}_L\nabla_B v_D\nabla_A\gamma^D{}_C\\ =& \gamma^A{}_M\gamma^B{}_N\gamma^C{}_L\nabla_A\nabla_B v_C - K_{MN}\gamma^C{}_L n^D\nabla_D v_C - K_{LM}K^C{}_N v_C\end{aligned}$$

- insert into

$$(D_M D_N - D_N D_M)v^K =^{(D-1)} R^K{}_{LMN}v^L$$

projections of the Riemann tensor and Gauss-Codazzi relations ($\diamondsuit$)

1. fully spatial projection

$$\begin{aligned}\perp^{(D)} R^{L}{}_{MNK} =& \gamma^{L}{}_{A}\gamma^{B}{}_{M}\gamma^{C}{}_{N}\gamma^{D}{}_{K}\,{}^{(D)}R^{A}{}_{BCD}\\ =& {}^{(D-1)}R^{L}{}_{MNK} + K_{KM}K^{L}{}_{N} - K_{KN}K^{L}{}_{M}\end{aligned}$$

2. projection once along normal, $3\times$ onto Σ_t

$$\gamma^{A}{}_{M}\gamma^{B}{}_{N}\gamma^{C}{}_{K}\,{}^{(D)}R_{LABC}n^{L} = D_N K_{KM} - D_M K_{KN}$$

3. projection twice along normal, twice onto Σ_t

$$\gamma^{A}{}_{M}\gamma^{B}{}_{N}\,{}^{(D)}R_{LAKB}n^{L}n^{K} = \mathcal{L}_n K_{MN} + K_{ML}K^{L}{}_{N} + \frac{1}{\alpha}D_M D_N\alpha$$

$(D-1)+1$ decomposition of EEs IV

- rewrite EEs as

$$E_{1,MN} = {}^{(D)}R_{MN} - \frac{1}{2} g_{MN} \, {}^{(D)}R - 8\pi G_D T_{MN} = 0$$

$$E_{2,MN} = {}^{(D)}R_{MN} - 8\pi G_D \left(T_{MN} - \frac{1}{2} g_{MN} T \right) = 0$$

- possible projections
  1. Hamiltonian constraint: $\mathcal{H} = E_{1,MN} n^M n^N = 0$
  2. momentum constraint: $\mathcal{M}_M = \gamma^K{}_M E_{1,KN} n^N = 0$
  3. evolution equation: $\mathcal{P} = \gamma^K{}_M \gamma^L{}_N E_{2,KL} = 0$
- projections of the stress-energy tensor
  1. energy density: $T_{MN} n^M n^N =: \rho$
  2. energy current: $\gamma^K{}_M T_{KN} n^N =: -j_M$
  3. spatial components: $\gamma^K{}_M \gamma^L{}_N T_{KL} =: S_{MN}$
- now: lengthy calculation or use mathematica with xtensor package (`http://www.xact.es/xTensor/index.html`, J. M. Martín-García et al '02-'13)

$(D-1)+1$ decomposition of EEs V

derivation of EEs in so-called "ADM" form (Arnowitt, Deser, Misner '62, York '79)
use mathematica with xtensor package

- constraint system

$$\mathcal{H} = {}^{(D-1)}R - K_{\bar{m}\bar{n}}K^{\bar{m}\bar{n}} + K^2 - 16\pi G_D\rho = 0$$
$$\mathcal{M}_{\bar{m}} = D_{\bar{n}}K^{\bar{n}}{}_{\bar{m}} - D_{\bar{m}}K - 8\pi G_D j_{\bar{m}} = 0$$

$\Rightarrow$ solve for initial data (see Hiro's lecture)

- evolution system

$$\partial_t\gamma_{\bar{m}\bar{n}} = -2\alpha K_{\bar{m}\bar{n}} + \mathcal{L}_\beta\gamma_{\bar{m}\bar{n}}$$
$$\partial_t K_{\bar{m}\bar{n}} = -D_{\bar{m}}D_{\bar{n}}\alpha + \alpha\left({}^{(D-1)}R_{\bar{m}\bar{n}} - 2K^{\bar{k}}{}_{\bar{m}}K_{\bar{k}\bar{n}} + KK_{\bar{m}\bar{n}}\right) + 4\pi G_D\alpha\left(\gamma_{\bar{m}\bar{n}}(S-\rho) - 2S_{\bar{m}\bar{n}}\right) + \mathcal{L}_\beta K_{\bar{m}\bar{n}}$$

BSSN formulation of Einstein's equations

BSSN formulation I

- EEs in ADM-like form
  - weakly hyperbolic
  - ill-posed initial value problem (IVP)
  - numerically unstable
- necessary condition for numerical stability: well-posed IVP
  (see David's lecture)
- examples:
  - generalized harmonic formulation (wave eqn.-like – 2^{nd} order in time/space)
  - Z4c (1^{st} order in time / 2^{nd} order in space)
  - BSSN (Baumgarte, Shapiro '99, Shibata, Nakamura '95)
  - plethora of first order reductions
- BSSN – 1^{st} order in time, 2^{nd} order in space formulation
  1. constraint addition
  2. conformal decomposition of variables
  3. gauge choice

  $\Rightarrow$ change character of PDE system

  $\Rightarrow$ strongly hyperbolic

BSSN formulation II

now: computations only in $\Sigma_t \Rightarrow$ drop superscripts ${}^{(D)}X$ and ${}^{(D-1)}X$

constraint addition

- define additional constraint

$$G_{\bar{m}} = \tilde{\gamma}_{\bar{m}\bar{k}}\tilde{\Gamma}^{\bar{k}} - \tilde{\gamma}^{\bar{k}\bar{l}}\partial_{\bar{k}}\tilde{\gamma}_{\bar{m}\bar{l}} = 0\,, \quad \text{with} \quad \tilde{\Gamma}^{\bar{m}} = -\partial_{\bar{k}}\tilde{\gamma}^{\bar{k}\bar{m}}$$

- modify evolution equations

$$\begin{aligned}
\partial_t \gamma_{\bar{m}\bar{n}} =& [ADM] \\
\partial_t K_{\bar{m}\bar{n}} =& [ADM] + \alpha\partial_{(\bar{m}}G_{\bar{n})} - \frac{1}{D-1}\alpha\gamma_{\bar{m}\bar{n}}\left(\mathcal{H} + \gamma^{\bar{k}\bar{l}}\partial_{\bar{k}}G_{\bar{l}}\right) \\
\partial_t \tilde{\Gamma}_{\bar{m}} =& \left(\tilde{\gamma}^{\bar{k}\bar{l}}\partial_{\bar{k}}\tilde{\gamma}_{\bar{m}\bar{l}}\right)_{[ADM]} + 2\alpha\mathcal{M}_{\bar{m}} - 2\alpha G^{\bar{k}}A_{\bar{m}\bar{k}} \\
& + \mathcal{L}_\beta G_{\bar{m}} + \gamma_{\bar{m}\bar{k}}G^{\bar{l}}\partial_{\bar{l}}\beta^{\bar{k}} - \frac{2}{D-1}G_{\bar{m}}\partial_{\bar{k}}\beta^{\bar{k}}
\end{aligned}$$

BSSN formulation III

conformal decomposition of variables

- conformal factor and metric

$$\chi = \gamma^{-\frac{1}{D-1}}, \quad \text{with} \quad \gamma = \det \gamma_{\bar{m}\bar{n}} \quad \text{and} \quad \tilde{\gamma} = \det \tilde{\gamma}_{\bar{m}\bar{n}} = 1$$
$$\tilde{\gamma}_{\bar{m}\bar{n}} = \chi \gamma_{\bar{m}\bar{n}}$$

- splitting of extrinsic curvature in trace and tracefree part

$$K = \gamma^{\bar{m}\bar{n}} K_{\bar{m}\bar{n}}$$
$$\tilde{A}_{\bar{m}\bar{n}} = \chi \left(K_{\bar{m}\bar{n}} - \frac{1}{D-1} \gamma_{\bar{m}\bar{n}} K \right)$$

- conformal connection function

$$\tilde{\Gamma}^{\bar{m}} = \tilde{\gamma}^{\bar{k}\bar{l}} \tilde{\Gamma}^{\bar{m}}{}_{\bar{k}\bar{l}} = -\partial_{\bar{k}} \tilde{\gamma}^{\bar{k}\bar{m}}$$

BSSN formulation IV

gauge choice for $(\alpha, \beta^{\bar{m}})$ – "moving puncture" gauge

- need to specify coordinates to close the system
- coordinate degree of freedom $\Rightarrow$ $(\alpha, \beta^{\bar{m}})$ freely specifiable
- BUT: gauge choice matters for numerical stability!!!
- 1+log slicing for lapse function

$$\partial_t \alpha = -\,2\eta_\alpha \alpha K$$

  singularity avoiding properties
- Γ-driver shift condition for $\beta^{\bar{m}}$

$$\partial_t \beta^{\bar{m}} = \frac{D-1}{2(D-2)} \zeta_\Gamma \tilde{\Gamma}^{\bar{m}} - \eta_\beta \beta^{\bar{m}} + \beta^{\bar{k}} \partial_{\bar{k}} \beta^{\bar{m}}$$

  - $\eta_\beta \beta^{\bar{m}}$ damping term
  - $\beta^{\bar{k}} \partial_{\bar{k}} \beta^{\bar{m}}$ advection term

BSSN formulation V

$$
\begin{aligned}
\partial_t \chi =& \frac{2}{D-1}\alpha\chi K + \beta^{\bar{k}}\partial_{\bar{k}}\chi - \frac{2}{D-1}\chi\partial_{\bar{k}}\beta^{\bar{k}}\,, \\
\partial_t \tilde{\gamma}_{\bar{m}\bar{n}} =& -2\alpha\tilde{A}_{\bar{m}\bar{n}} + 2\tilde{\gamma}_{\bar{k}(\bar{m}}\partial_{\bar{n})}\beta^{\bar{k}} - \frac{2}{D-1}\tilde{\gamma}_{\bar{m}\bar{n}}\partial_{\bar{k}}\beta^{\bar{k}}\,, \\
\partial_t K =& -D^{\bar{k}}D_{\bar{k}}\alpha + \alpha\left[\tilde{A}^{\bar{k}\bar{l}}\tilde{A}_{\bar{k}\bar{l}} + \frac{1}{D-1}K^2\right] + \beta^{\bar{k}}\partial_{\bar{k}}K\,, \\
\partial_t \tilde{A}_{\bar{m}\bar{n}} =& -\chi[D_{\bar{m}}D_{\bar{n}}\alpha]^{\mathrm{tf}} + \alpha\chi R^{\mathrm{tf}}_{\bar{m}\bar{n}} + \alpha\left[K\tilde{A}_{\bar{m}\bar{n}} - 2\tilde{A}_{\bar{m}\bar{k}}\tilde{A}^{\bar{k}}{}_{\bar{n}}\right] \\
& + \beta^{\bar{k}}\partial_{\bar{k}}\tilde{A}_{\bar{m}\bar{n}} + 2\tilde{A}_{\bar{k}(\bar{m}}\partial_{\bar{n})}\beta^{\bar{k}} - \frac{2}{D-1}\tilde{A}_{\bar{m}\bar{n}}\partial_{\bar{k}}\beta^{\bar{k}}\,, \\
\partial_t \tilde{\Gamma}^{\bar{m}} =& -2\alpha\tilde{\gamma}^{\bar{m}\bar{k}}\frac{D-2}{D-1}\partial_{\bar{k}}K - \tilde{A}^{\bar{m}\bar{k}}\left((D-1)\alpha\chi^{-1}\partial_{\bar{k}}\chi + 2\partial_{\bar{k}}\alpha\right) + 2\alpha\tilde{A}^{\bar{k}\bar{l}}\tilde{\Gamma}^{\bar{m}}{}_{\bar{k}\bar{l}} \\
& + \beta^{\bar{k}}\partial_{\bar{k}}\tilde{\Gamma}^{\bar{m}} + \frac{2}{D-1}\tilde{\Gamma}^{\bar{m}}\partial_{\bar{k}}\beta^{\bar{k}} - \tilde{\Gamma}^{\bar{k}}\partial_{\bar{k}}\beta^{\bar{m}} + \frac{D-3}{D-1}\tilde{\gamma}^{\bar{m}\bar{k}}\partial_{\bar{k}}\partial_{\bar{l}}\beta^{\bar{l}} + \tilde{\gamma}^{\bar{k}\bar{l}}\partial_{\bar{k}}\partial_{\bar{l}}\beta^{\bar{m}}\,, \\
\partial_t \alpha =& -2\eta_\alpha\alpha K\,, \\
\partial_t \beta^{\bar{m}} =& \frac{D-1}{2(D-2)}\zeta_\Gamma\tilde{\Gamma}^{\bar{m}} - \eta_\beta\beta^{\bar{m}} + \beta^{\bar{k}}\partial_{\bar{k}}\beta^{\bar{m}}
\end{aligned}
$$

What we have learned today

- major concepts in NumRel
- $(D-1)+1$ decomposition of spacetime
- $(D-1)+1$ decomposition of Einstein's equations
- focus on the evolution sector
- Einstein's equations in ADM-like form
- BSSN formulation of Einstein's equations as one example for strongly hyberpolic formulations

What's next

- $(D-1)+1$ system for $D>4$ computationally too expensive
  $\Rightarrow$ not realizable with current computational power
- consider highly symmetric problems
  $\Rightarrow$ reduction to effectively $(2+1)$ or $(3+1)$ problems
- Cartoon method
- dimensional reduction