Numerical Relativity in higher dimensional spacetimes – Part 2

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Recap – what have we learned last time

- formulation of Einstein's equations

$$^{(D)}R_{MN} - \frac{1}{2} g_{MN}\, {}^{(D)}R = 8\pi G_D T_{MN}$$

  as time evolution problem

- $(D-1)+1$ split of spacetime $(\mathcal{M}, g_{MN})$

$$\mathcal{M} = \Sigma_t \times \mathbb{R}$$

- line element

$$ds^2 = -\left(\alpha^2 - \beta_{\bar{m}}\beta^{\bar{m}}\right) dt^2 + 2\gamma_{\bar{m}\bar{n}}\beta^{\bar{m}} dt dx^{\bar{n}} + \gamma_{\bar{m}\bar{n}} dx^{\bar{m}} dx^{\bar{n}}$$

  with

  - coordinate degrees of freedom: α – lapse, $\beta^{\bar{m}}$ – shift vector
  - dynamical variables:
    $\gamma_{\bar{m}\bar{n}}$ – (D-1)-metric, $K_{\bar{m}\bar{n}} = -\frac{1}{2}\mathcal{L}_n \gamma_{\bar{m}\bar{n}}$ – extrinsic curvature

Recap – what have we learned last time

$(D-1)+1$ split of Einstein's equations

- constraint subsystem

$$\mathcal{H} = {}^{(D-1)}R - K_{\bar{m}\bar{n}}K^{\bar{m}\bar{n}} + K^2 - 16\pi G_D \rho = 0$$
$$\mathcal{M}_{\bar{m}} = D_{\bar{n}}K^{\bar{n}}{}_{\bar{m}} - D_{\bar{m}}K - 8\pi G_D j_{\bar{m}} = 0$$

- evolution subsystem

$$\partial_t \gamma_{\bar{m}\bar{n}} = -2\alpha K_{\bar{m}\bar{n}} + \mathcal{L}_\beta \gamma_{\bar{m}\bar{n}}$$
$$\partial_t K_{\bar{m}\bar{n}} = -D_{\bar{m}}D_{\bar{n}}\alpha + \alpha\left({}^{(D-1)}R_{\bar{m}\bar{n}} - 2K^{\bar{k}}{}_{\bar{m}}K_{\bar{k}\bar{n}} + KK_{\bar{m}\bar{n}}\right)$$
$$+ 4\pi G_D \alpha\left(\gamma_{\bar{m}\bar{n}}(S-\rho) - 2S_{\bar{m}\bar{n}}\right) + \mathcal{L}_\beta K_{\bar{m}\bar{n}}$$

- change of variables and constraint addition
  $\Rightarrow$ numerically stable evolution scheme, e.g., BSSN
- gauge choice – moving punctures

$$\partial_t \alpha = -2\eta_\alpha \alpha K$$
$$\partial_t \beta^{\bar{m}} = \frac{D-1}{2(D-2)}\zeta_\Gamma \tilde{\Gamma}^{\bar{m}} - \eta_\beta \beta^{\bar{m}} + \beta^{\bar{k}}\partial_{\bar{k}}\beta^{\bar{m}}$$

Outline

- $(D-1)+1$ formulation numerically too expensive for $D>4$
- number of gridpoints $\sim N^{D-1}$
- code dealing with generic spacetime dimension D
  vs. individual code for each dimension
- consider symmetric problems
  $\rightarrow$ reduction to $(3+1)$ or $(2+1)$ problems

1. **Approach I – The Cartoon method**

2. **Approach II – Dimensional reduction by isometry**

The Cartoon method

Cartoon method in $D = 4$

- goal: evolution of highly symmetric problems, e.g., with axis-symmetry (2+1)
- natural choice: cylindrical coordinates
  - coordinate singularity at axis
  - sometimes tricky to handle
- Cartesian coordinates
  - avoid coordinate singularities
  - requires 3+1 $\rightarrow$ more expensive: N^3 instead of N^2 grid points

  $\Rightarrow$ combine advantages

$\Rightarrow$ Cartoon method ("CARTesian TWOdimensional") (Alcubierre et al '99)

Cartoon method in $D = 4$

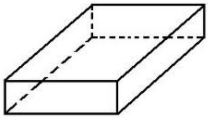
- for now: consider axissymmetric problem in $x - z$-plane ($y = 0$), e.g., head-on collision along z-axis
- in $3d$: symmetry not explicit
  - spatial derivatives in all directions; in general: $\partial_y u \neq 0$
  - e.g., 4^{th} order finite difference stencils

$$\partial_y u_j = \frac{1}{12dy}\left(u_{j-2} - 8u_{j-1} + 8u_{j+1} - u_{j+2}\right)$$

$\Rightarrow$ require $2g + 1 = 5$ points in y-direction

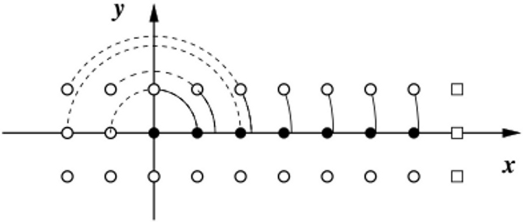
- reduce $3d$ cube with N^3 grid points to $3d$ slab with $N^2 \cdot (2g+1)$ grid points

$\Longrightarrow$



Cartoon method in $D = 4$

- in interior of slab $\rightarrow$ centered finite difference stencils
- at outer boundary $\rightarrow$ physical boundary conditions
- evaluate functions at grid points $(x, y = 0, z)$
- U(1) symmetry around z
  $\Rightarrow$ obtain values at $y \neq 0$ by rotation and interpolation from field at $y = 0$
- required point $(\rho, 0, z) = (\sqrt{x^2 + y^2}, 0, z)$ might not be a grid point
  $\Rightarrow$ interpolation



(Alcubierre et al '99)

Cartoon method in $D = 4$

- cylindrical coordinates (ρ, ϕ, z) with

$$x = \rho\cos\phi\,, \quad y = \rho\sin\phi\,, \quad z = z\,, \quad \rho = \sqrt{x^2 + y^2}$$

- consider rotation by angle ϕ_0

$$\rho' = \rho\,, \quad \phi' = \phi + \phi_0\,, \quad z' = z$$

- define diffeomorphism
  $R :\, {}^{(3)}\Sigma_t \to {}^{(3)}\Sigma_t$ with $R : T \to R^* T$, for $T \in {}^{(3)}\Sigma_t$

$$\left(R(\phi_0)^i{}_j\right) = \begin{pmatrix} \cos\phi_0 & -\sin\phi_0 & 0 \\ \sin\phi_0 & \cos\phi_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{x}{\rho} & -\frac{y}{\rho} & 0 \\ \frac{y}{\rho} & \frac{x}{\rho} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- R is symmetry if $R^* T = T$

Cartoon method in $D = 4$

- transformation of tensor $T(p) \to R^* T(R(p))$ (with $R^* T = T$)

$$T^{i_1 \ldots i_p}{}_{j_1 \ldots j_q}(x, y, z) = R^{i_1}{}_{k_1} \ldots R^{i_p}{}_{k_p} (R^{-1})^{l_1}{}_{j_1} \ldots (R^{-1})^{l_q}{}_{j_q} T^{k_1 \ldots k_p}{}_{l_1 \ldots l_q}(\rho, 0, z)$$

$\Rightarrow$ evaluate functions @ $p' = R(p) = (x, y, z)$ from functions @ $p = (\rho, 0, z)$

- examples ($\diamondsuit$):
  - scalar

$$\Psi(x, y, z) = \Psi(\rho, 0, z)$$

  - vector $V^i(x, y, z) = R^i{}_j V^j(\rho, 0, z)$

$$\begin{aligned} V^x(x, y, z) &= \frac{x}{\rho} V^x(\rho, 0, z) - \frac{y}{\rho} V^y(\rho, 0, z) \\ V^y(x, y, z) &= \frac{y}{\rho} V^x(\rho, 0, z) + \frac{x}{\rho} V^y(\rho, 0, z) \\ V^z(x, y, z) &= V^z(\rho, 0, z) \end{aligned}$$

  - tensor

$$S_{ij}(x, y, z) = R^k{}_i R^l{}_j S_{kl}(\rho, 0, z)$$

Cartoon method in $D \geq 4$

- so far:
  considered 3d space with $U(1)$ symmetry, e.g., head-on collision along z-axis
- further examples in $D > 4$:
  (Yoshino & Shibata '09, Shibata & Yoshino '10,
  Lehner & Pretorius '10, Okawa et al '11)
  - $D = 5$ with $U(1)$ symmetry, e.g., BH collisions with impact parameter

Cartoon method in $D = 5$

$D = 5$ with $U(1)$ symmetry, e.g., BH collisions with impact parameter

- consider Cartesian coordinates (x, y, z, w) with $(x, y, z = w)$
    - $x - y$-plane as symmetry plane
    - Killing vector: $\partial_\psi = z\partial_w - w\partial_z$
- Cartoon method
    - simulate $D = 5$ problem in $3 + 1$
      $\Rightarrow (2g + 1)N^3$ grid points instead of N^4
    - consider cylindrical coordinates

$$x = x\,, \quad y = y\,, \quad z = \rho \cos\psi\,, \quad w = \rho \sin\psi$$

Cartoon method in $D = 5$

$D = 5$ with $U(1)$ symmetry (continued)

- strategy:
  1. prepare data @ $(x, y, z, w = 0)$
  2. generate data @ $(x, y, z, w \neq 0)$ by rotation of data @ $(x, y, \rho, 0)$
- Killing vector: $\partial_\psi = z\partial_w - w\partial_z$
- rotation by $\psi = \psi_0$ with rotation matrix:

$$(R(\psi_0)^{\bar{m}}{}_{\bar{n}}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\psi_0 & -\sin\psi_0 \\ 0 & 0 & \sin\psi_0 & \cos\psi_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{z}{\rho} & -\frac{w}{\rho} \\ 0 & 0 & \frac{w}{\rho} & \frac{z}{\rho} \end{pmatrix}$$

- construct all necessary derivatives / quantities
  $\Rightarrow$ happy number crunching...

Cartoon method

- so far:
  considered 3d space with $U(1)$ symmetry, e.g., head-on collision along z-axis
- further examples in $D > 4$:
  (Yoshino & Shibata '09, Shibata & Yoshino '10,
  Lehner & Pretorius '10, Okawa et al '11)
  - $D = 5$ with $U(1)$ symmetry, e.g., BH collisions with impact parameter
  - $D = 5$ with $U(1) \times U(1)$ symmetry, e.g., Myers-Perry BH with 2 spin parameters ($\diamondsuit$)
  - $D = 5$ with $SO(3)$ symmetry, e.g., head-on collision along z-axis ($\diamondsuit$)
  - $D \geq 5$ spacetime with $SO(D-3)$ symmetry ("modified Cartoon method") ($\diamondsuit$)
- note: see lecture notes for examples

Modified Cartoon method in $D \geq 5$

consider D dimensional vacuum spacetime with $SO(D-3)$ symmetry (Shibata & Yoshino '10)

- coordinates $(t, x, y, z, w_1, ..., w_{D-4})$
- line element
  $ds^2 = -(\alpha^2 - \beta^k \beta_k)dt^2 + 2\gamma_{kl}\beta^k dt dx^l + \gamma_{ab} dx^a dx^b + \gamma_{nn} d\Omega^2_{D-4}$
- $x^a = (x, y, \rho)$ with $\rho = \sqrt{z^2 + \sum_{q=1}^{D-4} w_q^2}$
  $\Rightarrow$ symmetry around $z, w_1, ..., w_{D-4}$
  $\Rightarrow$ geometric quantities depend only on t, x, y, ρ
- strategy:
  1. solve for data in (x, y, z) hyperplane, with $w_1 = ... = w_{D-4} = 0$
  2. rotate data around hyperplane $\Rightarrow$ data at $(x, y, z, w_1 \neq 0, ..., w_{D-4} \neq 0)$

Modified Cartoon method in $D \geq 5$

consider D dimensional vacuum spacetime with $SO(D-3)$ symmetry (continued)

- straightforward Cartoon method
  - becomes numerically more expensive with increasing dimension
  - $N^3(2g+1)^{D-4}$ grid points
- instead: make use of $SO(D-3)$ symmetry
  $\Rightarrow$ relations for tensor Q_{MN}, vector Q^M, scalar Q quantities ($\diamondsuit$):
  - $Q_{ww} := Q_{w_1 w_1} = ... = Q_{w_{D-4} w_{D-4}}$
  - $Q^{w_q} = 0,\ Q_{a w_q} = 0,\ Q_{w_q w_r} = 0$ for $(r \neq q)$
  - ....

  $\Rightarrow$ all quantities and derivatives in $D-1$ dimensions
  are replaced by quantities and derivatives in the (x, y, z, w) plane

now: happy number crunching ...

Dimensional Reduction by isometry

Dimensional reduction

dimensional reduction well known in physics

- Kaluza - Klein theory (1921)
  $\Rightarrow$ unification of gravity and gauge theories
  e.g.: GR – Maxwell in $D = 4 \leftrightarrow$ GR in $D = 5$
- string theory compactifications
- Here:
  "Killing reduction" – dimensional reduction by isometry
  (Geroch '71, Yang '03, ...)

Geroch decomposition I

- consider spacetime $(\mathcal{M}, g_{MN})$ with $dim\mathcal{M} = D$
- consider existence of Killing vectors ξ^M
- consider ξ^M are either timelike or spacelike everywhere
- denote manifold S with $dimS = D - 1$
  - collection of all trajectories of ξ^a
  - $v \in S$ is a curve in $\mathcal{M}$ with v tangent to ξ^M everywhere
- consider tensor $T \in \mathcal{M}$ and $\tilde{T} \in S$

  if $\quad \xi_{M_1} T^{M_1\ldots M_p}{}_{N_1\ldots N_q} = \xi^{N_q} T^{M_1\ldots M_p}{}_{N_1\ldots N_q} = 0$

  $\qquad \mathcal{L}_\xi T^{M_1\ldots M_p}{}_{N_1\ldots N_q} = 0$

  then $\quad$ there is a one-to-one correspondence between T and $\tilde{T}$
- the entire tensor field algebra on S is completely and uniquely mirrored by tensor field algebra on $\mathcal{M}$

Geroch decomposition II

- metric on (S, h_{MN}): $h_{MN} = g_{MN} - \lambda^{-1}\xi_M\xi_N\,,\quad$ with $\quad \lambda = \xi^M\xi_M$
- projection operator: $h^M{}_N = \delta^M{}_N - \lambda^{-1}\xi^M\xi_N$
- consider twist 2-form (exemplarily in $D = 5$):

$$\omega_{MN} = \epsilon_{MNJKL}\xi^J\nabla^K\xi^L$$
$$\text{with}\quad \omega_{MN} = \omega_{[MN]}\,,\quad \omega_{MN}\xi^M = 0$$

- one can show: $\mathcal{L}_\xi\lambda = 0\,,\quad \mathcal{L}_\xi\omega_{MN} = 0$
  $\Rightarrow \lambda \in S$ and $\omega_{MN} \in S$

Geroch decomposition III

- with above relations derive

$$D_{[M}\omega_{NK]} \sim \epsilon_{MNKL_1L_2}\xi^{L_1}\,{}^{(5)}R^{L_2}{}_{L_3}\xi^{L_3} = 0 \quad \text{(in vacuum)}$$

$\Rightarrow$ locally $\exists$ a 1-form $A_M \in S$ with

$$\omega_{MN} = (dA)_{MN} = 2D_{[M}A_{N]}$$

- construct

$$\begin{aligned} {}^{(4)}R_{MN} \sim & (h_{MN}, (dA)_{MN}, (d\lambda)_M) \\ D^K(dA)_{KM} \sim & (dA)_{KM}D^K\lambda \\ D^M D_M\lambda \sim & D^M\lambda D_M\lambda - (dA)_{MN}(dA)^{MN} \end{aligned}$$

$\Rightarrow$ pure gravity in $D = 5$ is equivalent to
GR in $D = 4$ coupled to scalar field λ and vector field A_M

GREAT!

Are we done?

NOT QUITE!

Dimensional reduction I

Goal:

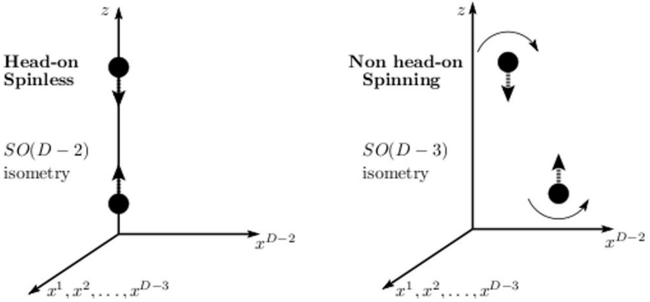
- study problems in $D \geq 5$
- reduction to $3+1$ problems

Why $3+1$?

- approaching limits of computational power
- we want numerical results in reasonable computational time
- we know how to do numerics in $3+1$
- vacuum GR in D dimensions
  $\Rightarrow$ GR coupled to gauge and scalar fields in $D=4$
- all quantities have geometrical interpretation
- straightforward modification of existing $3+1$ codes

Dimensional reduction II

- dimensional reduction by isometry on S^{D-4} sphere
  $\Rightarrow$ has $SO(D-3) \subset SO(D-2)$ isometry group
- possible classes of models
  - in $D \geq 5$:
    head-on collisions of non-spinning BHs
  - in $D \geq 6$:
    BH collisions with impact parameter
    collision of spinning BHs as long as dynamics are restricted to one plane



Zilhão et al '10

Dimensional reduction III

Notation:

$$
\begin{aligned}
M, N, \ldots =& 0, \ldots, D-1 \\
\mu, \nu, \ldots =& 0, \ldots, 3 \\
\bar{\mu}, \bar{\nu}, \ldots =& 4, \ldots, D-1 \\
i, j, \ldots =& 1, 2, 3
\end{aligned}
$$

- dimensional reduction on S^{D-4} sphere
- ansatz for metric

$$
\begin{aligned}
ds^2 =& g_{MN} dx^M dx^N \\
=& g_{\mu\nu}(x^M) dx^\mu dx^\nu + \Omega_{\bar{\mu}\bar{\nu}}(x^M) \left(dx^{\bar{\mu}} - A^{\bar{\mu}}{}_{\mu}(x^M) dx^\mu\right) \left(dx^{\bar{\nu}} - A^{\bar{\nu}}{}_{\nu}(x^M) dx^\nu\right)
\end{aligned}
$$

Dimensional reduction IV

- $\{x^{\mu}\}$ coordinates in base space; $\{x^{\bar{\mu}}\}$ coordinates along the fibre
- consider Killing vector fields ξ^{a} with $a = 1, \ldots, \frac{(D-4)(D-3)}{2}$

$$\mathcal{L}_{\xi} g_{MN} = 0$$

- Lie algebra $[\xi_a, \xi_b] = \epsilon_{ab}{}^{c} \xi_c$ with $\epsilon_{ab}{}^{c}$ structure constants of $SO(D-3)$
- fibre has minimal dimension necessary to host $\frac{(D-4)(D-3)}{2}$ independent KVs
- assume that KV has components exclusively along the fibre

$$\xi_a = \xi^{\bar{\mu}}{}_{a} \partial_{\bar{\mu}}$$

- normalize KVs, such that they only depend on coordinates along the fibre

$$\partial_{\mu} \xi^{\bar{\mu}}{}_{a} = 0$$

Dimensional reduction V

$\mathcal{L}_\xi g_{MN} = 0$ implies

1. $M = \bar{\mu}, N = \bar{\nu}$: $\mathcal{L}_\xi g_{MN} = \mathcal{L}_\xi \Omega_{\bar{\mu}\bar{\nu}} = 0$
   - $\Omega_{\bar{\mu}\bar{\nu}}$ is metric on maximally symmetric space $\to S^{D-4}$ sphere

$$\Omega_{\bar{\mu}\bar{\nu}} = \lambda(x^\mu) h^{S^{D-4}}_{\bar{\mu}\bar{\nu}}\,, \quad \text{with } h^{S^{D-4}}_{\bar{\mu}\bar{\nu}} \text{ metric on } S^{D-4} \text{ sphere}$$

2. $M = \mu, N = \bar{\nu}$: $\mathcal{L}_\xi g_{MN} = -\mathcal{L}_\xi (\Omega_{\bar{\mu}\bar{\nu}} A^{\bar{\mu}}{}_\mu) = 0 \quad \Rightarrow \mathcal{L}_\xi A^{\bar{\mu}}{}_\mu = 0$
   - equivalent to $[\xi_a, A_\mu] = 0 \quad \Rightarrow A^{\bar{\mu}}{}_\mu = 0$
   - for $D \geq 5$ there are no non-trivial vector fields on S^{D-4} that commute with all KVs on the sphere
   - all KVs are orthogonal to hypersurface
3. $M = \mu, N = \nu$: $\mathcal{L}_\xi g_{MN} = \mathcal{L}_\xi g_{\mu\nu} + \mathcal{L}_\xi (\Omega_{\bar{\mu}\bar{\nu}} A^{\bar{\mu}}{}_\mu A^{\bar{\nu}}{}_\nu) = 0 \quad \Rightarrow \mathcal{L}_\xi g_{\mu\nu} = 0$
   - KVs act transitively to the fibre
   - base space metric $g_{\mu\nu}$ is independent of fibre coordinates

$$g_{\mu\nu}(x^M) = g_{\mu\nu}(x^\mu)$$

Dimensional reduction VI

- D dimensional spacetime metric has block diagonal form

$$ds^2 = g_{MN} dx^M dx^N = \underbrace{g_{\mu\nu} dx^\mu dx^\nu}_{D=4} + \underbrace{\lambda(x^\mu) h^{S^{D-4}}_{\bar\mu\bar\nu} dx^{\bar\mu} dx^{\bar\nu}}_{D-4}$$

- components of D dimensional Ricci tensor
  1. $M=\mu, N=\nu$:

$$^{(D)}R_{\mu\nu} = {}^{(4)}R_{\mu\nu} - \frac{D-4}{2\lambda}\left(\nabla_\mu\nabla_\nu\lambda - \frac{1}{2\lambda}\nabla_\mu\lambda\nabla_\nu\lambda\right)$$

  2. $M=\mu, N=\bar\mu$:

$$^{(D)}R_{\mu\bar\mu} = 0$$

  3. $M=\bar\mu, N=\bar\nu$:

$$^{(D)}R_{\bar\mu\bar\nu} = {}^{(D-4)}R_{\bar\mu\bar\nu} - \frac{1}{2} h^{S^{D-4}}_{\bar\mu\bar\nu}\left(\nabla^\mu\nabla_\mu\lambda + \frac{D-6}{2\lambda}\nabla^\mu\lambda\nabla_\mu\lambda\right)$$

Dimensional reduction VII

- consider D dimensional vacuum spacetime $\Rightarrow {}^{(D)}R_{MN} = 0$
- equations of motion:

$$\mathcal{S} = \nabla^\mu \nabla_\mu \lambda + \frac{D-6}{2\lambda} \nabla^\mu \lambda \nabla_\mu \lambda - 2(D-5) = 0$$

$$E_{1,\mu\nu} = {}^{(4)}R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \, {}^{(4)}R - 8\pi T_{\mu\nu} = 0$$

with

$$T_{\mu\nu} = \frac{D-4}{16\pi\lambda} \left(\nabla_\mu \nabla_\nu \lambda - \frac{1}{2\lambda} \nabla_\mu \lambda \nabla_\nu \lambda - (D-5) g_{\mu\nu} + \frac{D-5}{4\lambda} g_{\mu\nu} \nabla^\kappa \lambda \nabla_\kappa \lambda \right)$$

- GR in D dimensions with $SO(D-3)$ isometry
$\Rightarrow$ reduction to GR in $D = 4$ coupled (non-minimally) to scalar field

We are almost there!

$3+1$ decomposition I

- GR in D dimensions with $SO(D-3)$ isometry
  $\Rightarrow$ reduction to GR in $D=4$ coupled (non-minimally) to scalar field
- reformulation as $3+1$ time evolution problem
- $3+1$ variables:
  - coordinates:
    lapse function α
    shift vector β^i
  - dynamical variables:
    3-metric γ_{ij}
    extrinsic curvature $K_{ij} = -\frac{1}{2\alpha}(\partial_t - \mathcal{L}_\beta)\gamma_{ij}$
    scalar field λ
    momentum of scalar field $K_\lambda = -\frac{1}{2\alpha}(\partial_t - \mathcal{L}_\beta)\lambda$
- $3+1$ decomposition of GR coupled to scalar field
  $\rightarrow$ modify our mathematica notebook developed in the last lecture ($\diamondsuit$)

Note: both notebooks will become available with the lecture notes and online

$3+1$ decomposition II

constraint equations

$$
\begin{aligned}
\mathcal{H} =& R - K_{ij}K^{ij} + K^2 \\
& - \frac{D-4}{\lambda}\left(-(D-5) + D^i D_i \lambda + \frac{D-7}{4\lambda} D^i \lambda D_i \lambda - (D-5)\frac{K_\lambda^2}{\lambda} - 2KK_\lambda \right) = 0 \\
\mathcal{M}_i =& D_j K^j{}_i - D_i K - \frac{D-4}{2\lambda}\left(D_i K_\lambda - K^j{}_i D_j \lambda - \frac{K_\lambda}{\lambda} D_i \lambda \right) = 0
\end{aligned}
$$

$3+1$ decomposition III

- solution to the initial data problem representing non-spinning BHs starting from rest
- time symmetric, i.e., modified Brill-Lindquist initial data
- assume maximal slicing ($K=0$), conformal flatness

$$\gamma_{ij} = \psi^{\frac{4}{D-3}}\eta_{ij}\,, \quad K_{ij} = 0$$
$$\lambda = \psi^{\frac{4}{D-3}}y^2\,, \quad K_\lambda = 0$$

with

$$\psi_0 = 1 + \sum_{i=1}^{N} \frac{\mu_{(i)}}{4|r - r_{(i)}|^{D-3}}$$

- mass parameter

$$\mu_{(i)} \equiv r_{S_{(i)}}^{D-3} \equiv \frac{16\pi M_{(i)}}{\mathcal{A}_{D-2}(D-2)}\,,$$

$3+1$ decomposition IV

evolution equations

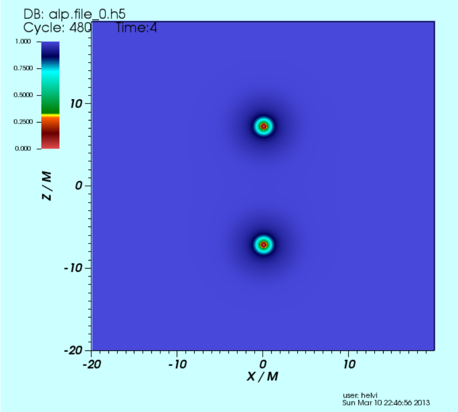
$$
\begin{aligned}
(\partial_t - \mathcal{L}_\beta)\gamma_{ij} =& -2\alpha K_{ij} \\
(\partial_t - \mathcal{L}_\beta)\lambda =& -2\alpha K_\lambda \\
(\partial_t - \mathcal{L}_\beta)K_{ij} =& -D_i D_j \alpha + \alpha\left(R_{ij} - 2K^k{}_i K_{kj} + K K_{ij}\right) \\
& -\alpha\frac{D-4}{2\lambda}\left(D_i D_j\lambda - \frac{1}{2\lambda}D_i\lambda D_j\lambda - 2K_\lambda K_{ij}\right) \\
(\partial_t - \mathcal{L}_\beta)K_\lambda =& -\frac{1}{2}D^i\alpha D_i\lambda \\
& +\alpha\left((D-5) + K K_\lambda + \frac{D-6}{\lambda}K_\lambda^2 - \frac{1}{2}D^i D_i\lambda - \frac{D-6}{4\lambda}D^i\lambda D_i\lambda\right)
\end{aligned}
$$

gauge choice – modified puncture gauge

$$
\begin{aligned}
\partial_t\alpha =& \beta^i\partial_i\alpha - 2\alpha\left(K + (D-4)\mu_\lambda\frac{K_\lambda}{\lambda}\right) \\
\partial_t\beta^i =& \beta^k\partial_k\beta^i - \eta_\beta\beta^i + \xi_\Gamma\tilde{\Gamma}^i + \xi_\lambda\frac{D-4}{2}\frac{\partial^i\lambda}{\lambda}
\end{aligned}
$$

$3+1$ decomposition V

- choose strongly hyperbolic formulation, e.g., BSSN (note: expressions become lengthy, see lecture notes)
- implementation / modification of $3+1$ GR code
- example: head-on collision of two BHs starting from rest



Summary

- consider highly symmetric problems in GR in D dimensions
  $\Rightarrow$ reduction to $3+1$ or $2+1$ system
- Cartoon method
  - use symmetry to reduce numerical grid
  - set up "slab" instead of "cube" $\Rightarrow$ reduce from N^D grid points to $N^3 \cdot (2g+1)$
  - use Cartesian coordinates $\Rightarrow$ avoid coordinate singularities in grid
  - rotation and interpolation of functions from point $(*, X=0, *)$ to $(*, X \neq 0, *)$
- dimensional reduction by isometry
  - consider problems with $SO(D-3)$ isometry $\Rightarrow$ reduction on S^{D-4} sphere
  - GR in D dimensions $\Rightarrow$ GR in $D=4$ coupled to scalar field
  - all quantities have geometric interpretation
  - straightforward modification of existing codes
- tomorrow: introduction to visualization toolkit VisIt