Turbulent Instability of Anti-de Sitter Space

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joint work with Piotr Bizoń
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Anti-de Sitter spacetime in $d + 1$ dimensions

Anti-de Sitter spacetime is the maximally symmetric solution of the vacuum Einstein equations

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = 0\,,$$

with negative cosmological constant $\Lambda < 0$.

Anti-de Sitter spacetime in $d+1$ dimensions

Geometrically, AdS_{d+1} is wrapped around hiperboloid

$$-\left(X^0\right)^2+\sum_{k=1}^{d}\left(X^k\right)^2-\left(X^{d+1}\right)^2=-\ell^2$$

embedded in flat, $SO(d,2)$ invariant space

$$ds^2=-\left(dX^0\right)^2+\sum_{k=1}^{d}\left(dX^k\right)^2-\left(dX^{d+1}\right)^2$$

Parametrization

$$X^0=\ell\sqrt{1+(\tan x)^2}\cos t,\quad X^{d+1}=\ell\sqrt{1+(\tan x)^2}\sin t,\quad X^k=\ell\tan x\, n^k$$

$$-\infty<t<+\infty,\quad 0\le x<\pi/2,\quad \sum_{k=1}^{d}\left(n^k\right)^2=1,$$

induces

$$ds^2=\frac{\ell^2}{(\cos x)^2}\left[-dt^2+dx^2+(\sin x)^2\,d\Omega^2_{S^{d-1}}\right]$$

Anti-de Sitter spacetime in $d+1$ dimensions

Induced metric

$$ds^2 = \frac{\ell^2}{(\cos x)^2}\left[-dt^2 + dx^2 + (\sin x)^2\, d\Omega^2_{S^{d-1}}\right],$$

$$-\infty < t < +\infty, \quad 0 \le x < \pi/2\,.$$

is the maximally symmetric solution of the vacuum Einstein equations $R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R + \Lambda g_{\alpha\beta} = 0$ with negative cosmological constant $\Lambda < 0$:

$$\Lambda = -d(d-1)/(2\ell^2)$$

Conformal infinity $x = \pi/2$ is the timelike hypersurface $\mathcal{I} = \mathbb{R} \times S^{d-1}$ with the boundary metric $ds^2_{\mathcal{I}} = -dt^2 + d\Omega^2_{S^{d-1}}$

Poincaré patch

$$\left.\begin{array}{rl} X^0 - X^d & = \ell^2/u > 0 \\ X^{d+1} & = \ell t/z \\ X^i & = \ell x^i/z \end{array}\right\} \Rightarrow X^0 + X^d = z\left(1 + \frac{\vec{x}^2 - t^2}{z^2}\right)$$

induced metric:

$$ds^2 = \frac{-dt^2 + d\vec{x}^2 + dz^2}{z^2},$$

but in this talk we are concerned with *global* AdS:

$$ds^2 = \frac{\ell^2}{(\cos x)^2}\left[-dt^2 + dx^2 + (\sin x)^2\, d\Omega^2_{S^{d-1}}\right],$$

Maximally symmetric solutions of vacuum Einstein's equations and their stability

$$R_{\alpha\beta} - \frac{1}{2} R\, g_{\alpha\beta} + \Lambda\, g_{\alpha\beta} = 0\,.$$

A solution (of a dynamical system) is said to be stable if small perturbations of it at $t = 0$ remain small for all later times

- $\Lambda = 0$: Minkowski (trivial, yet most important)
  asymptotically stable (Christodoulou&Klainerman 1993),
- $\Lambda > 0$: de Sitter (important in cosmology - Nobel Prize 2011)
  asymptotically stable (Friedrich 1986),
- $\Lambda < 0$: anti- de Sitter (most popular on arXiv due to AdS/CFT)
  Stability of AdS seems unexplored

Is AdS stable?

- A solution (of a dynamical system) is said to be stable if small perturbations of it at $t = 0$ remain small for all later times
- The question of stability of AdS is open. Surprisingly, with almost fifteen years of activity on AdS/CFT, this question has been rarely asked (with a notable exception M. Anderson 2005)
- In contrast, Minkowski ($\Lambda = 0$) and de Sitter spacetimes ($\Lambda > 0$) are known to be stable (actually asymptotically stable) – (Christodoulou&Klainerman 1993 and Friedrich 1986)
- The key difference between these solutions and AdS: the main mechanism of stability - dissipation of energy (dispersion in Minkowski, expansion in de Sitter) - is absent in AdS because AdS is effectively bounded (for no flux boundary conditions at $\mathcal{I}$ it acts as a perfect cavity)
- Note that by positive energy theorems both Minkowski and AdS are the unique ground states among asymptotically flat/AdS spacetimes

Model

- To deal with the problem of the stability of AdS we start with spherical symmetry (effectually $1+1$ dimensional problem)
- Spherically symmetric vacuum solutions are static (Birkhoff's theorem) $\Rightarrow$ we need matter to generate dynamics
- Simple matter model: massless scalar field ϕ in d+1 dimensions

$$G_{\alpha\beta} + \Lambda\, g_{\alpha\beta} = 8\pi G \left(\partial_\alpha \phi\, \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} \partial_\mu \phi \partial^\mu \phi\right), \; \Lambda = -d(d-1)/(2\ell^2)$$

$$g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi = 0$$

- In the corresponding asymptotically flat ($\Lambda = 0$) model Christodoulou proved the weak cosmic censorship (dispersion for small data and collapse to a black hole for large data) and Choptuik discovered critical phenomena at the threshold for black hole formation
- Remark: For even $d \geq 4$ there is a way to bypass Birkhoff's theorem (cohomogeneity-two Bianchi IX ansatz, Bizoń, Chmaj, Schmidt 2005)

- Convenient parametrization of asymptotically AdS spacetimes

$$ds^2 = \frac{\ell^2}{(\cos x)^2}\left[-Ae^{-2\delta}dt^2 + A^{-1}dx^2 + (\sin x)^2\, d\Omega^2_{S^{d-1}}\right],$$

where A and δ are functions of (t, x).

- Auxiliary variables $\Phi = \phi'$ and $\Pi = A^{-1}e^{\delta}\dot{\phi}$ $\quad ({}' = \partial_x, \dot{} = \partial_t)$
- Field equations (using units where $8\pi G = d - 1$)

$$A' = (1-A)\frac{d-2+2(\sin x)^2}{(\cos x)(\sin x)} - (\cos x)(\sin x)\, A\left(\Phi^2 + \Pi^2\right),$$

$$\delta' = -(\cos x)(\sin x)\left(\Phi^2 + \Pi^2\right),$$

$$\dot{\Phi} = \left(Ae^{-\delta}\Pi\right)', \qquad \dot{\Pi} = \frac{1}{(\tan x)^{d-1}}\left[(\tan x)^{d-1}\, Ae^{-\delta}\Phi\right]'.$$

- AdS space: $\phi \equiv 0$, $A \equiv 1$, $\delta \equiv 0$; now we want to perturb AdS solving the initial-boundary value problem for this system starting with some small, smooth initial data $(\phi, \dot{\phi})_{|t=0}$

Boundary conditions

- We assume that initial data $(\phi, \dot{\phi})_{|t=0}$ are smooth
- Smoothness at the center implies that near $x = 0$ (Λ irrelevant)

$$\phi(t,x) = f_0(t) + \mathcal{O}(x^2), \quad \delta(t,x) = \mathcal{O}(x^2), \quad A(t,x) = 1 + \mathcal{O}(x^2)$$

- Smoothness at spatial infinity and conservation of the total mass M imply that near $x = \pi/2$ (using $z = \pi/2 - x$)

$$\phi(t,x) = f_\infty(t)\, z^d + \mathcal{O}\left(z^{d+2}\right), \quad \delta(t,x) = \delta_\infty(t) + \mathcal{O}\left(z^{2d}\right),$$

$$A(t,x) = 1 - (M/\ell^{d-2}) z^d + \mathcal{O}\left(z^{d+2}\right)$$

Remark: There is no freedom in prescribing boundary data

- Local well-posedness (Friedrich 1995, Holzegel&Smulevici 2011)
- mass function and asymptotic mass:

$$m(t,x) = (1 - A(t,x))\, (\ell \tan x)^{d-2}\, \left(1 + \tan^2 x\right)$$

$$M = \lim_{x \to \pi/2} m(t,x) = \ell^{d-2} \int_0^{\pi/2} \left(A\Phi^2 + A\Pi^2\right) (\tan x)^{d-1}\, dx$$

Reminder: asymptotically flat ($\Lambda = 0$) self-gravitating scalar field

- Christodoulou (1986-1993): dispersion for small data and collapse to a black hole for large data (proof of the weak cosmic censorship)
- Consider a family of initial data $\Phi(p)$ which interpolates between dispersion and collapse (Choptuik 1993)
- There exists a critical value of the parameter p^* such that
  - $p < p^* \Rightarrow$ dispersion
  - $p > p^* \Rightarrow$ black hole
- Universal behavior in the near-critical region $|p - p^*| \ll 1$
  - $m_{BH} \sim (p^* - p)^\gamma$ with universal exponent γ
  - discretely self-similar attractor with universal period Δ
- Critical solution ($p = p^*$) is a non-generic naked singularity

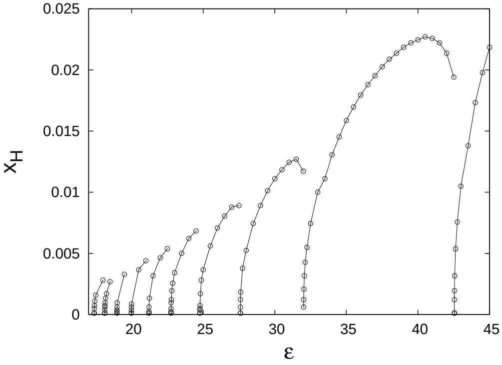
Movie

- numerical results to be presented are obtained for $d = 3$,
- but qualitatively the same behaviour in any $d \geq 3$;
- on the other hand the $d = 2$ case is very special (Pretorius&Choptuik 2000, Jałmużna in preparation)

Critical behavior

Initial data: $\Phi(0,x)=0\,,\Pi(0,x)=\varepsilon\left[\exp\left(-\frac{\tan x}{\sigma}\right)^2\right]$

We fix $\sigma=1/16$ and vary ε.



BH size vs. amplitude

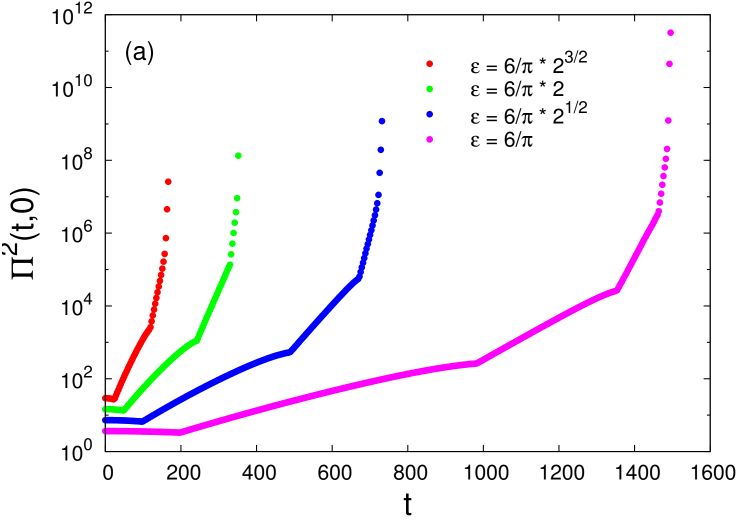
There is a decreasing sequence of critical amplitudes ε_n for which the evolution, after making n reflections from the AdS boundary, locally asymptotes Choptuik's solution. In each small right neighborhood of ε_n

$$m_{BH}(\varepsilon)\sim(\varepsilon-\varepsilon_n)^\gamma$$

with $\gamma\simeq 0.37$. It seems that $\lim_{n\to\infty}\varepsilon_n=0$

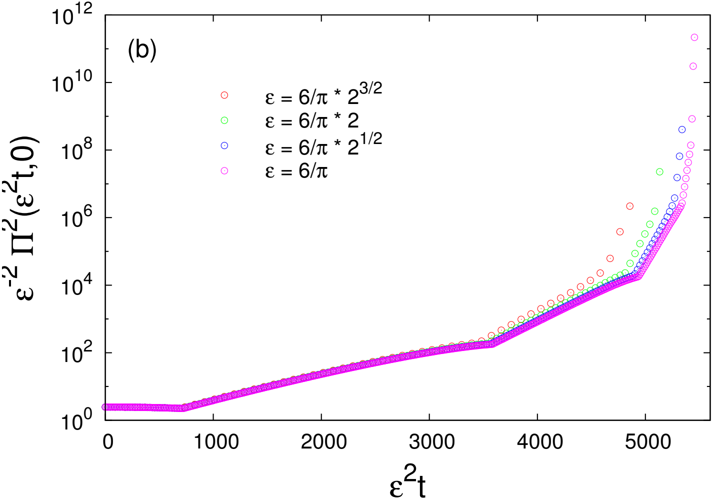
Remark: The generic endstate of evolution is the Schwarzschild-AdS BH of mass M (in accord with Holzegel&Smulevici 2011)

Key evidence for instability



Ricci scalar $R = 2\left(\Phi^2 - \Pi^2\right)/\ell^2 - 12/\ell^2$

Key evidence for instability



Onset of instability at time $t = \mathcal{O}(\varepsilon^{-2})$

Weakly nonlinear perturbations

- We seek an approximate solution starting from small initial data $(\phi,\, \dot{\phi})\Big|_{t=0} = (\varepsilon f(x), \varepsilon g(x))$
- Perturbation series

$$\phi = \varepsilon \phi_1 + \varepsilon^3 \phi_3 + \ldots$$
$$\delta = \varepsilon^2 \delta_2 + \varepsilon^4 \delta_4 + \ldots$$
$$1 - A = \varepsilon^2 A_2 + \varepsilon^4 A_4 + \ldots$$

where $(\phi_1, \dot{\phi}_1)\Big|_{t=0} = (f(x), g(x))$ and $(\phi_j, \dot{\phi}_j)\Big|_{t=0} = (0, 0)$ for $j > 1$.

- Inserting this expansion into the field equations and collecting terms of the same order in ε, we obtain a hierarchy of linear equations which can be solved order-by-order.

First order

- Linearized equation (Ishibashi&Wald 2004)

$$\ddot{\phi}_1 + L\phi_1 = 0, \quad L = -\frac{1}{\tan^{d-1} x}\,\partial_x \left(\tan^{d-1} x\,\partial_x\right)$$

The operator L is essentially self-adjoint on $L^2([0,\pi/2), \tan^{d-1} x\,dx)$.

- Eigenvalues and eigenvectors of L are $(j = 0, 1, \dots)$

$$\omega_j^2 = (d+2j)^2, \quad e_j(x) = N_j\,(\cos x)^d\,P_j^{d/2-1,d/2}(\cos 2x)$$

$\Rightarrow$ AdS is linearly stable

- Linearized solution

$$\phi_1(t,x) = \sum_{j=0}^{\infty} a_j \cos(\omega_j t + \beta_j)\,e_j(x)$$

where amplitudes a_j and phases β_j are determined by the initial data.

Second order (back-reaction on the metric)

$$A_2' + \frac{d-2+2\sin^2 x}{\sin x \cos x} A_2 = \sin x \cos x \left(\dot{\phi}_1^2 + {\phi_1'}^2\right)$$

$$\delta_2' = -\sin x \cos x \left(\dot{\phi}_1^2 + {\phi_1'}^2\right)$$

so

$$A_2(t,x) = \frac{(\cos x)^d}{(\sin x)^{d-2}} \int_0^x \left(\dot{\phi}_1(t,y)^2 + \phi_1'(t,y)^2\right) (\tan y)^{d-1}\, dy$$

$$\delta_2(t,x) = -\int_0^x \left(\dot{\phi}_1(t,y)^2 + \phi_1'(t,y)^2\right) \sin y \cos y \, dy$$

quadratic in ϕ_1!

(1)

It follows that

$$M = \frac{\varepsilon^2}{2} \int_0^{\pi/2} \left(\dot{\phi}_1(t,y)^2 + \phi_1'(t,y)^2\right) (\tan y)^{d-1}\, dy + \mathcal{O}(\varepsilon^4)$$

Third order

- $\ddot{\phi}_3 + L\phi_3 = S(\phi_1, A_2, \delta_2)\,, \qquad (\star)$
  where $S := 2(A_2+\delta_2)\ddot{\phi}_1 + (\dot{A}_2+\dot{\delta}_2)\dot{\phi}_1 + (A_2'+\delta_2')\phi_1'$.

- Let $\phi_3(t,x) = \sum_j f_j(t)\, e_j(x)$. Projecting Eq.$(\star)$ on the basis $\{e_j\}$ we obtain an infinite set of decoupled forced harmonic oscillators for the generalized Fourier coefficients $f_j(t) := (e_j\,|\phi_3\,)$

$$\ddot{f}_j + \omega_j^2 f_j = S_j := (e_j\,|S\,) \quad \text{and} \quad (f_j, \dot{f}_j)\Big|_{t=0} = 0$$

  If S_j has a part oscilating with a resonant frequency ω_j: $\cos\omega_j t$ or $\sin\omega_j t$, it will give rise to a secular term $\sim t\sin\omega_j t$ or $\sim t\cos\omega_j t$.

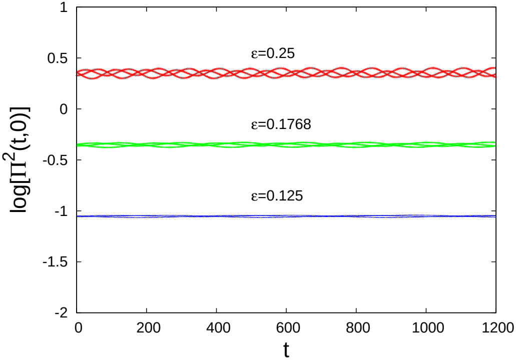
- S cubic in ϕ_1 $\Rightarrow$ contains all frequencies $|\pm\omega_1 \pm\omega_2 \pm\omega_3|$, where $\omega_k \in \Omega_1$ and $\phi_1(t,x) = \sum_k\,[\omega_k\in\Omega_1]\, a_k \cos(\omega_k t + \beta_k)\, e_k(x)\,,$
  ω_k - odd integers $\Rightarrow$ all frequencies in S potentially resonant!
  Not all resonances survive the projection $(e_j\,|S\,)$. Some of those, which do survive can be compensated with frequency shifts in ϕ_1 and are harmless for stability, but the others put stability in question!

Example 1: single-mode data $(\phi, \dot\phi)|_{t=0} = (\varepsilon\, e_0, 0)$

- First order $\phi_1(t,x) = \cos(\omega_0 t) e_0(x), \quad \omega_0 = 3 \qquad (\omega_j = 3 + 2j)$
- Third order $\phi_3(t,x) = \sum_{j=0}^{\infty} f_j(t) e_j(x), \quad (f_j, \dot f_j)\Big|_{t=0} = 0$ and

$$\ddot f_j + \omega_j^2 f_j = b_{j,0} \cos(\omega_0 t) + b_{j,3} \cos(\omega_3 t).$$

But $b_{3,3} = 0$ (!) and only $j = 0$ is resonant.



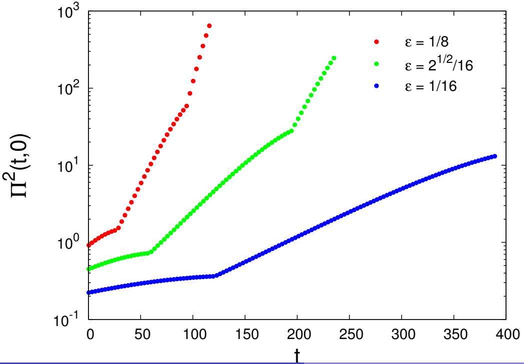
The $j = 0$ resonance can be easily removed by the two-scale method (slow-time phase modulation) which gives $\phi_1 = \cos((\omega_0 + \frac{153}{4\pi}\varepsilon^2)\, t)\, e_0(x)$. This suggests that there are non-generic initial data which may stay close to AdS solution

Example 2: two-mode data $(\phi, \dot{\phi})|_{t=0} = (\varepsilon\,(e_0 + e_1), 0)$

- First order $\phi_1(t,x) = \cos(\omega_0 t)e_0(x) + \cos(\omega_1 t)e_1(x), \;\; \omega_0 = 3, \omega_1 = 5$
- Third order $\phi_3(t,x) = \sum_{j=0}^{\infty} f_j(t)e_j(x), \quad (f_j, \dot{f}_j)\Big|_{t=0} = 0 \quad$ and

$$\ddot{f}_j + \omega_j^2 f_j = \sum_k [\omega_k \in \Omega_3] b_{j,k} \cos(\omega_k t), \text{ where } \Omega_3 = \{|\pm\omega_{0,1} \pm\omega_{0,1} \pm\omega_{0,1}|\}$$

Here $\Omega_3 = \{1, 3, 5, 7, 9, 11, 13, 15\}$, but the resonance $(b_{j,j} \neq 0)$ only if $\omega_j \in \{3, 5, 7\}$.



$\omega_0 \to \omega_0 + (87/\pi)\epsilon^2$,
$\omega_1 \to \omega_1 + (413/\pi)\epsilon^2$ shifts remove the resonances $\omega_j = 3, 5$, but the resonance $\omega_j = 7$ cannot be removed. Thus we get the secular term $c_2(t) \sim t \sin(7t)$. We expect this term to be a progenitor of the onset of exponential instability.

Conjectures

Our numerical and formal perturbative computations lead us to:

Conjecture 1

Anti-de Sitter space is unstable against the formation of a black hole under arbitrarily small generic perturbations

Crucial ingredients: no dissipation, resonant frequencies.
Proof (and a precise formulation) is still left as a challenge.
Note that we do not claim that all perturbed solutions end up as black holes.

Conjecture 2 'NR/HEP2

There are (non-linearly) stable periodic solutions in Einstein-AdS-massless scalar field system. They form stability islands in the ocean of instability

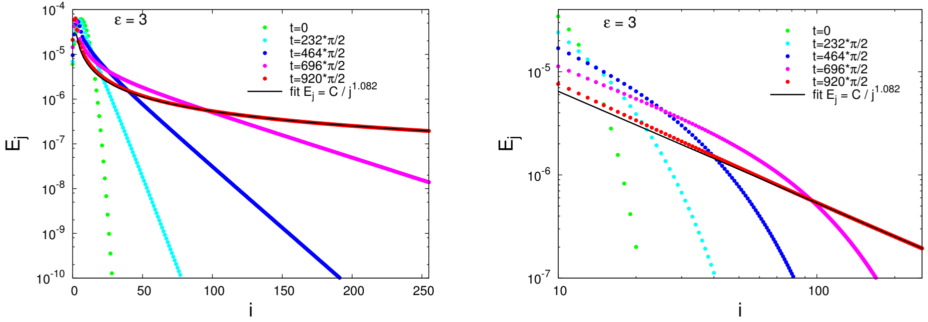
Strong evidence (still not a proof) - tomorrow's lecture.
Analogous conjecture for vacuum Einstein's equations by Dias,Horowitz&Santos (2011), Dias,Horowitz,Marolf&Santos (2012) (existence of geons).

Turbulence: transfer of energy from low to high frequencies

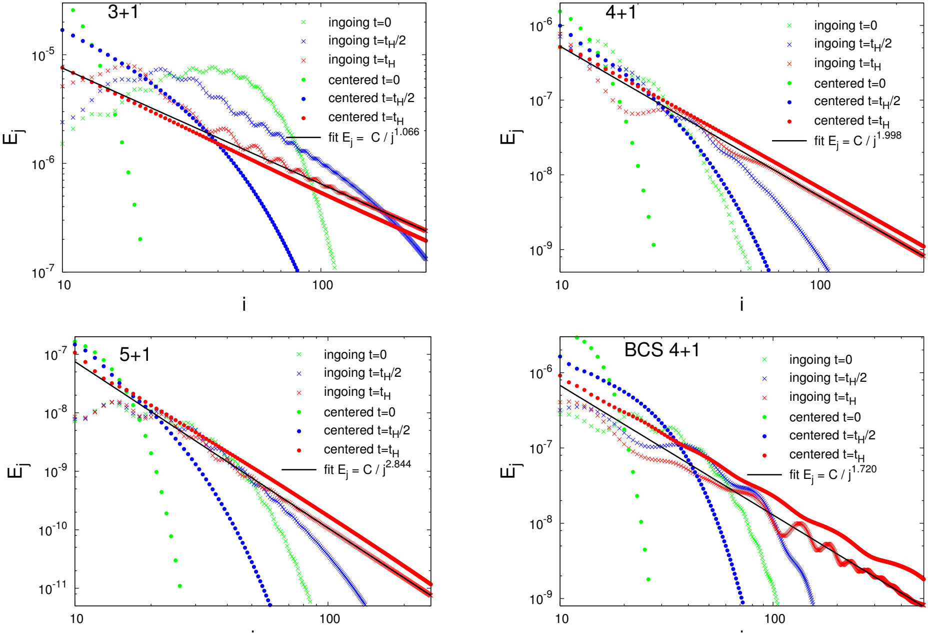
Let $\Pi_j := (\sqrt{A}\,\Pi, e_j)$ and $\Phi_j := (\sqrt{A}\,\Phi, e_j')$. Then

$$M = \ell \int_0^{\pi/2} \left(A\Phi^2 + A\Pi^2\right) (\tan x)^2 \, dx = \sum_{j=0}^{\infty} E_j(t)\,,$$

where $E_j := \Pi_j^2 + \omega_j^{-2}\Phi_j^2$ can be interpreted as the j-mode energy.



Power-law scaling



Final remarks

- Weakly turbulent behavior seems to be common for (non-integrable) nonlinear wave equations on bounded domains (e.g. NLS on torus, Colliander&Keel, Staffilani,Takaoka&Tao 2008, Carles&Faou 2010) and our work shows that Einstein's equations are not an exception.
- For Einstein's equations the transfer of energy to high frequencies cannot proceed forever because concentration of energy on smaller and smaller scales inevitably leads to the formation of a black hole.
- The role of negative cosmological constant is purely kinematical, that is the only role of Λ is to confine the evolution in an effectively bounded domain. Similar turbulent dynamics has been observed for small perturbations of Minkowski in a box (Maliborski 2012)