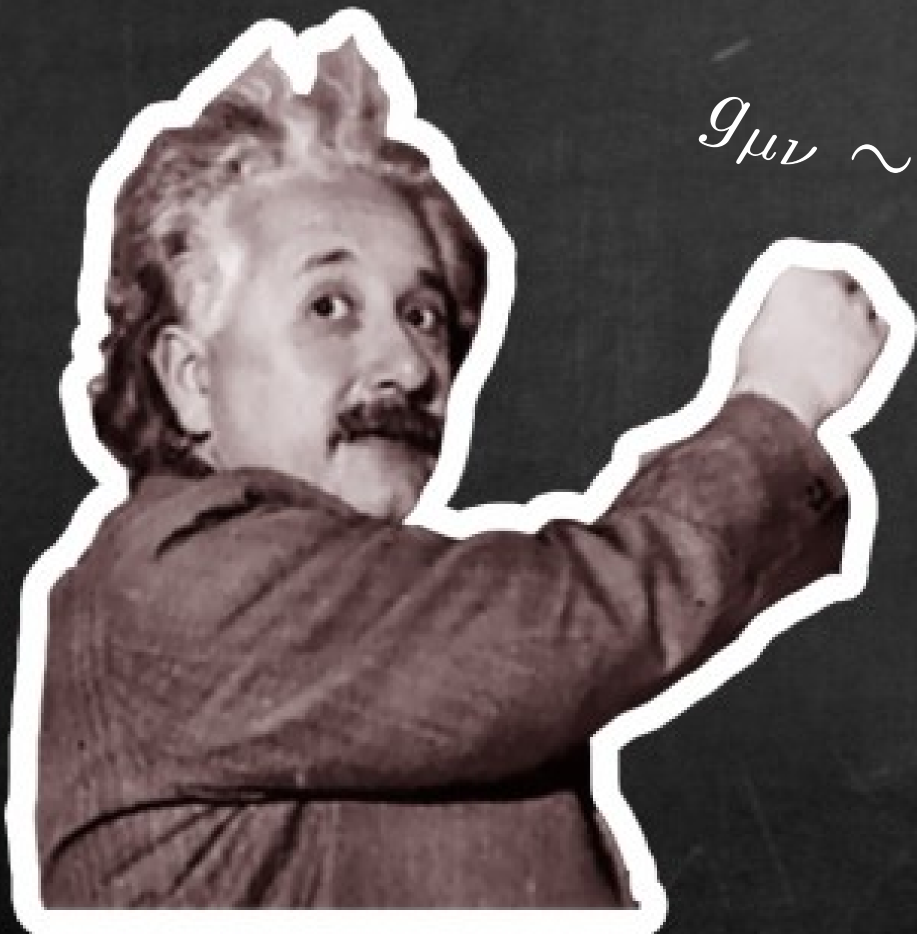
NR/HEP2: Spring School

11-14 March 2013 - IST, Lisbon (PT)

# Advanced Methods in Black-Hole Perturbation Theory



$$g_{\mu\nu} \sim g^{(0)}_{\mu\nu} + h_{\mu\nu}$$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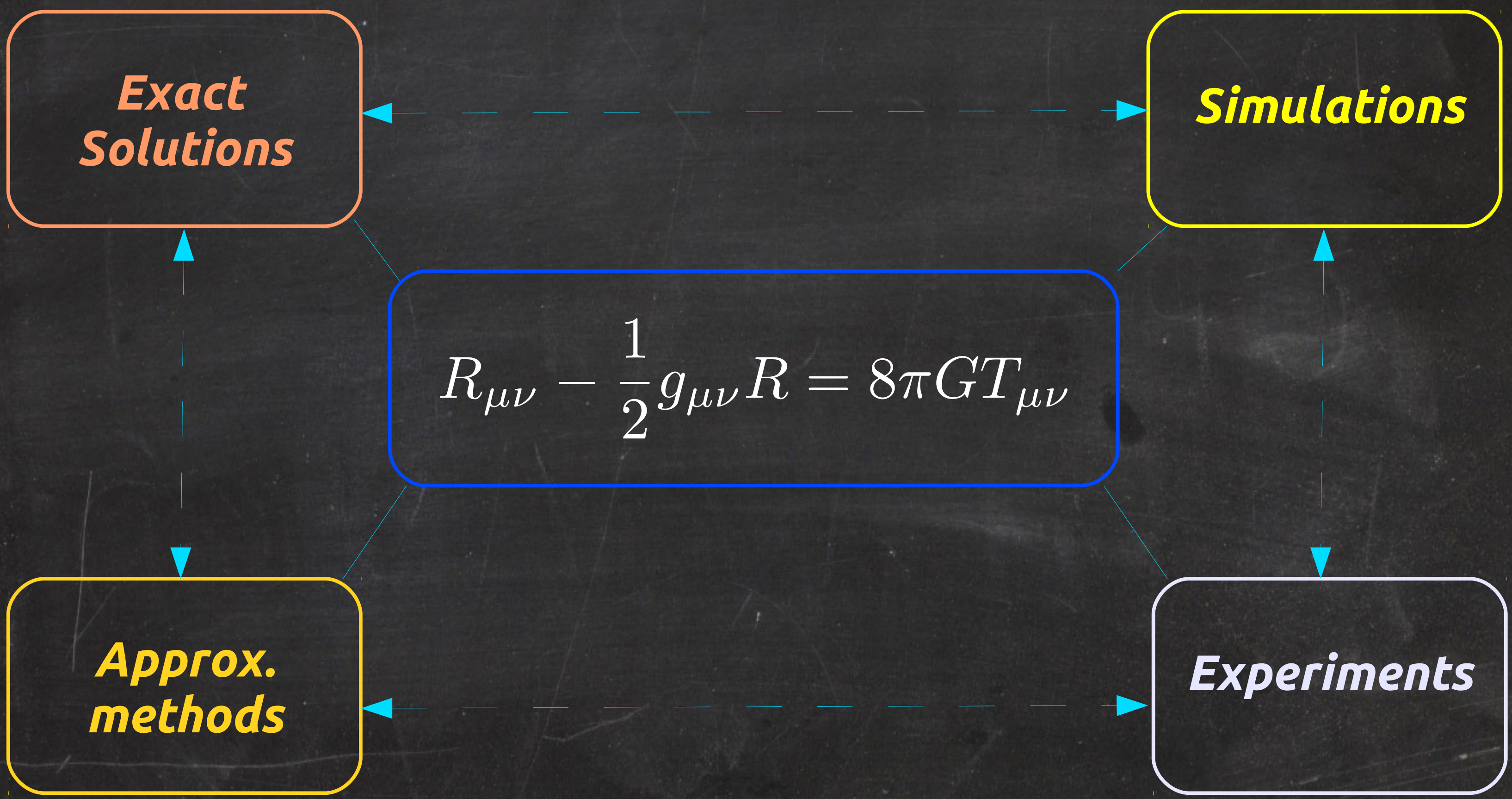
http://blackholes.ist.utl.pt


DYBHO

# Challenging GR



# Outline

- (I) BH dynamics at linear level: why is important?
- (I) Perturbations of nonspinning BHs: formalism and methods

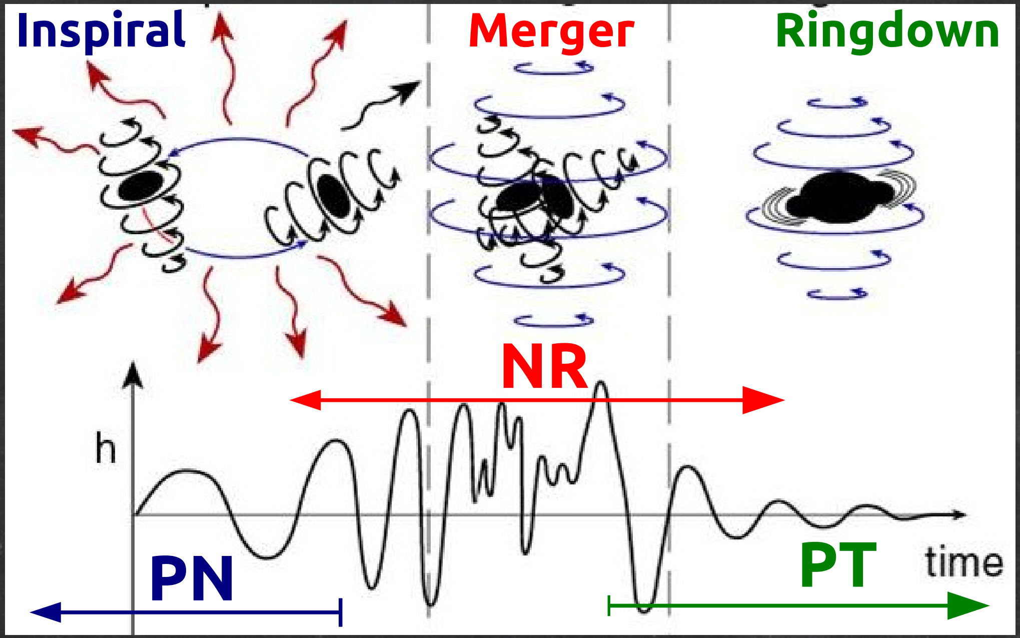
- (II) Stationary and axisymmetric BHs: seminanalytical techniques
- (II) Applications

# "Soft numerics" VS "Hard numerics"

*Idealized situations*

*Physical insights*

*"Easy" to perform*




*Adapted from Thorne*

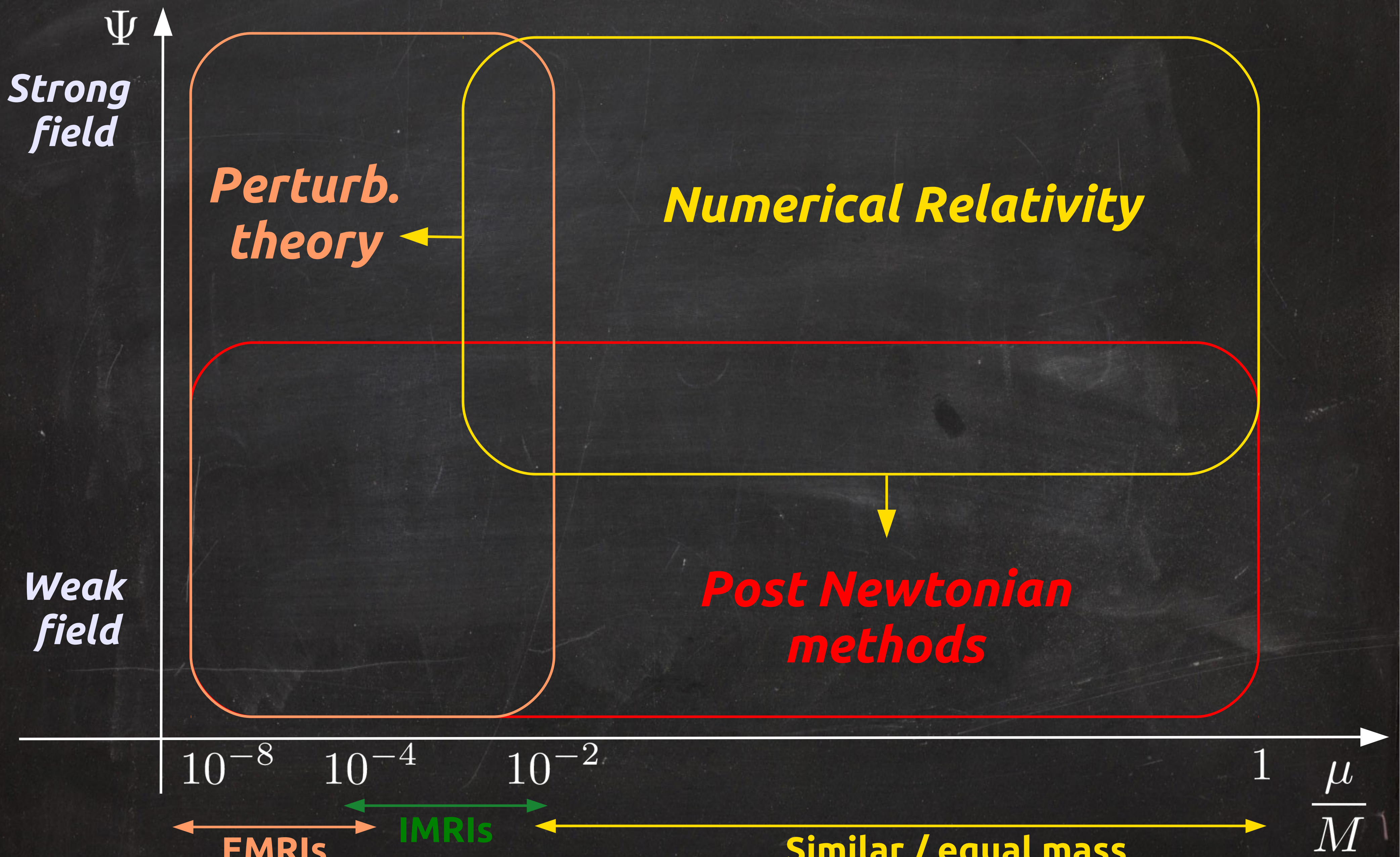
*Realistic situations*

*Numerics → Physics*

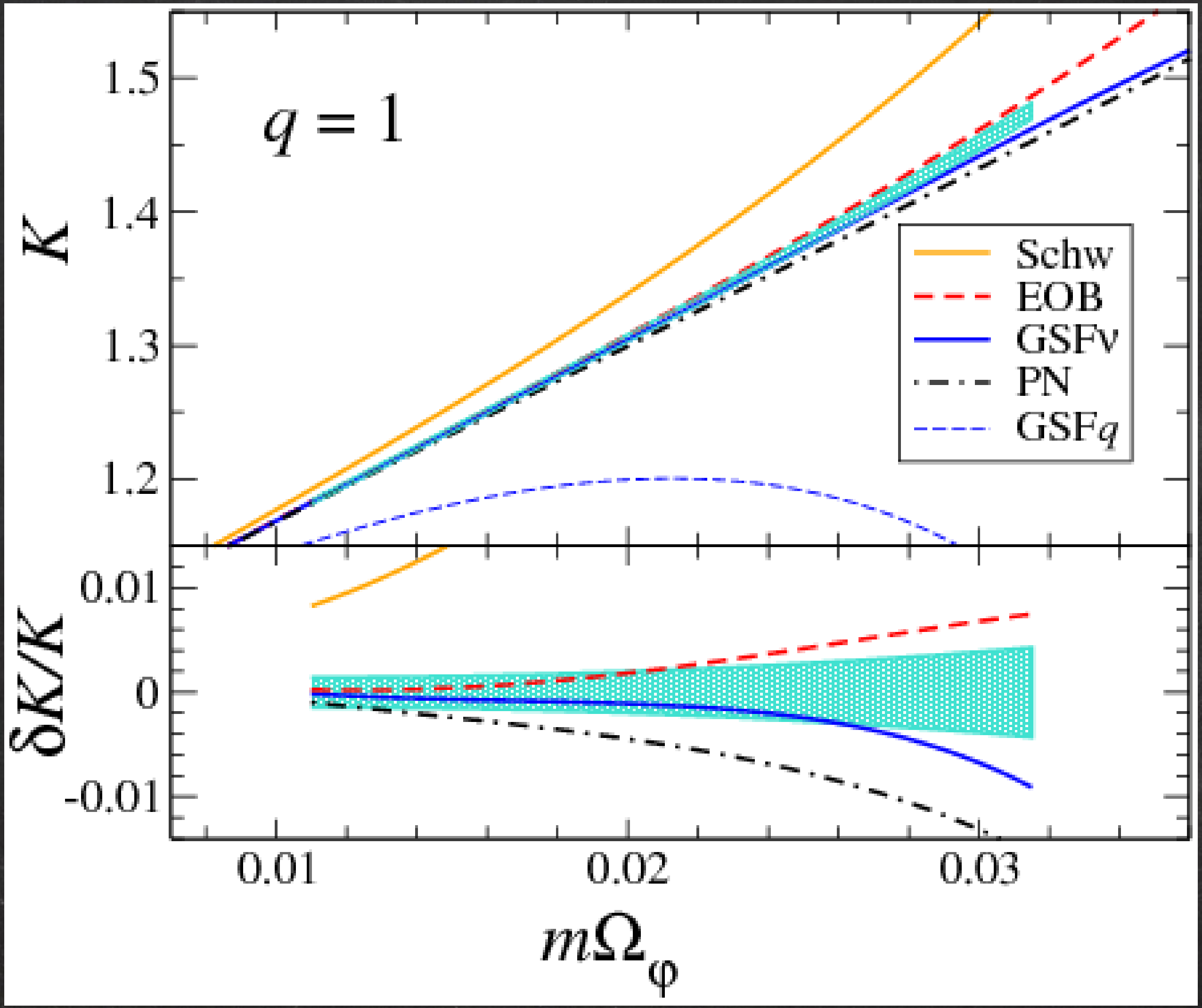
*Supercomputers*

- Approximate doesn't mean worse!
- Complementary approach
- Synergy between approx. and NR

# Two body problem in General Relativity



# Periastron Advance in BH Binaries




[Le Tiec et al. 2011]

## Disclaimer:

# Notebooks are online

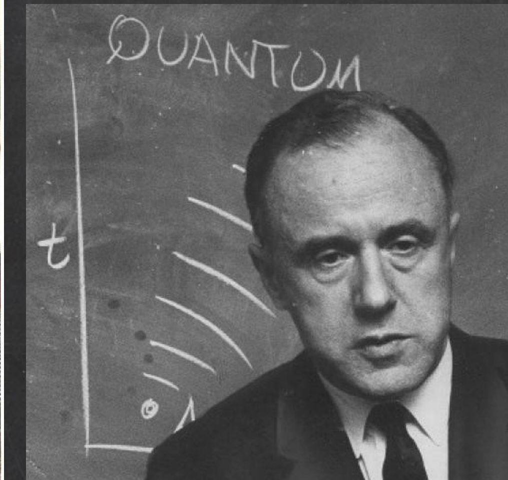
## test them, use them, extend them*

*and acknowledge :-)

## Introduction

# BH perturbations:

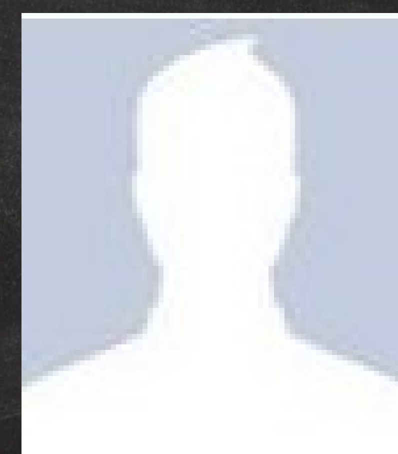
## a 40-year-long tale





**Regge-Wheeler**
**(1957)**

**Teukolsky-Press**
**(70s)**



**Vishveshwara**
**(1970)**

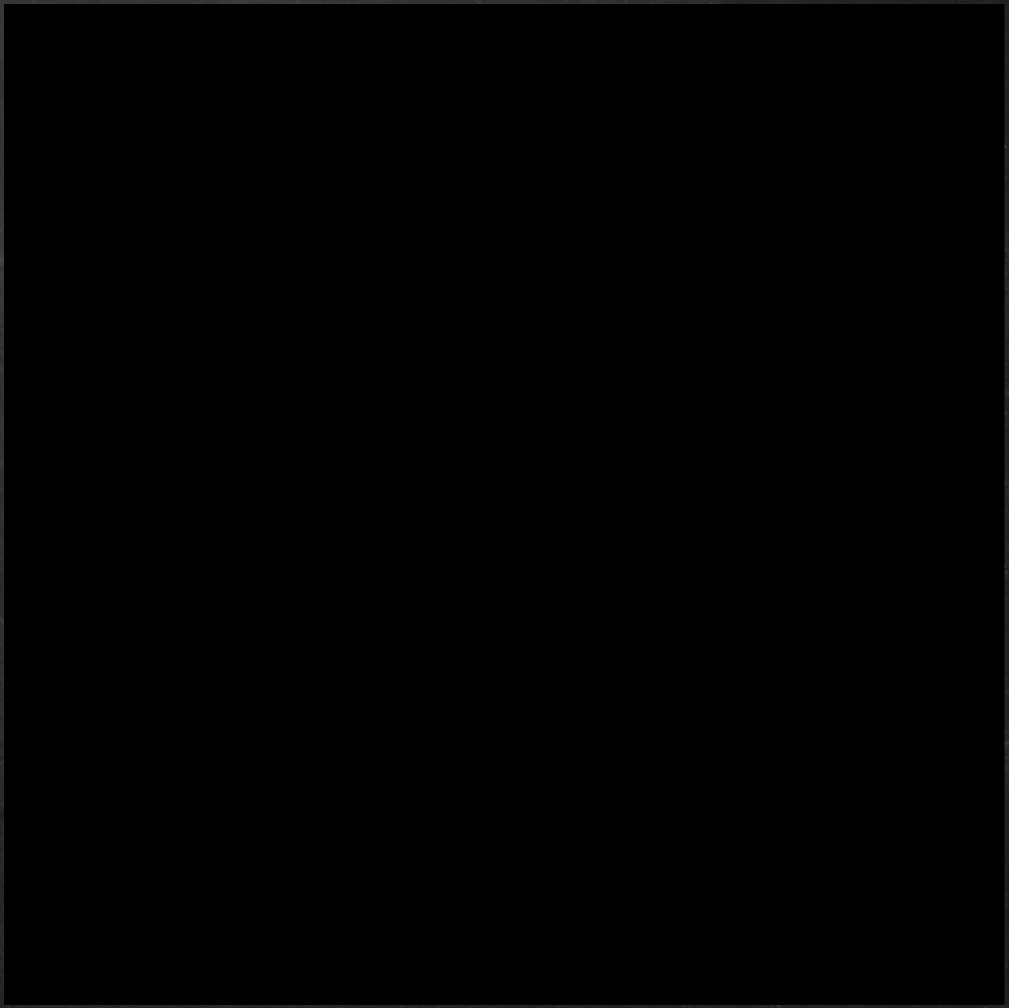
**Chandra**
**(70s-80s)**

**Zerilli, Moncrief, Price, Leaver, ...**

## Open problems

# Test of no-hair theorems

- Kerr BHs?
- Quadrupole moment?
- GW emission by nonKerr BHs
- GW in modified gravity

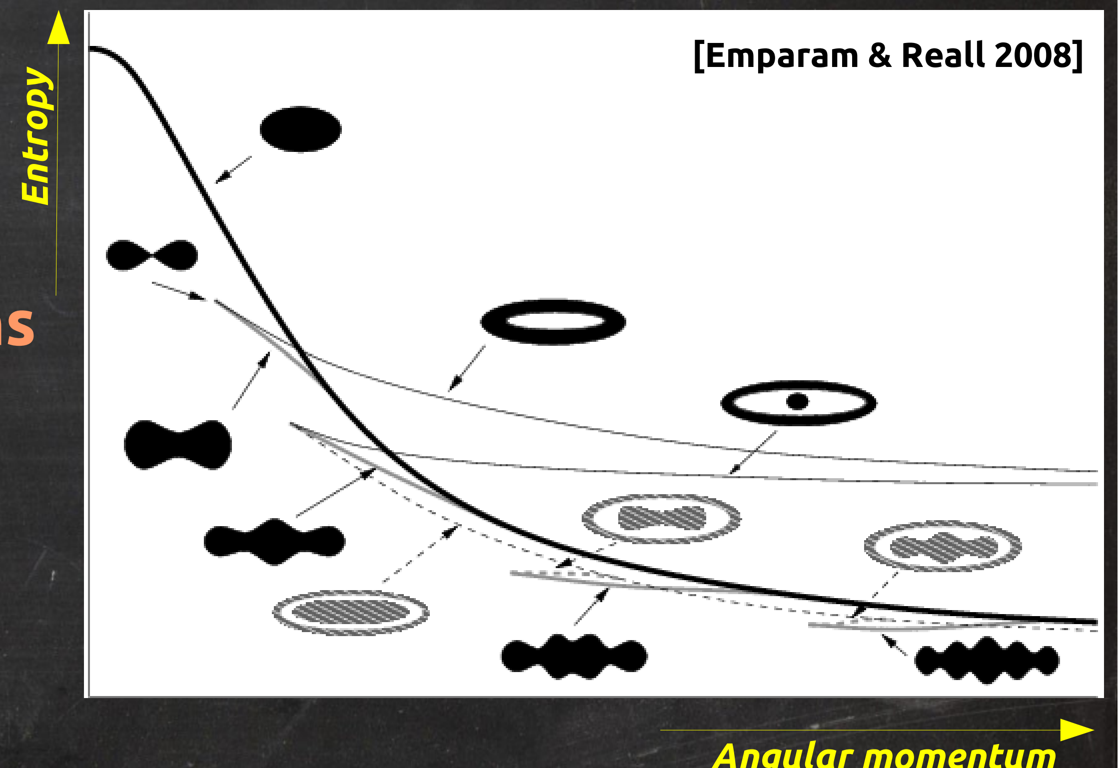


# Do NOT underestimate hairs!

# Open problems

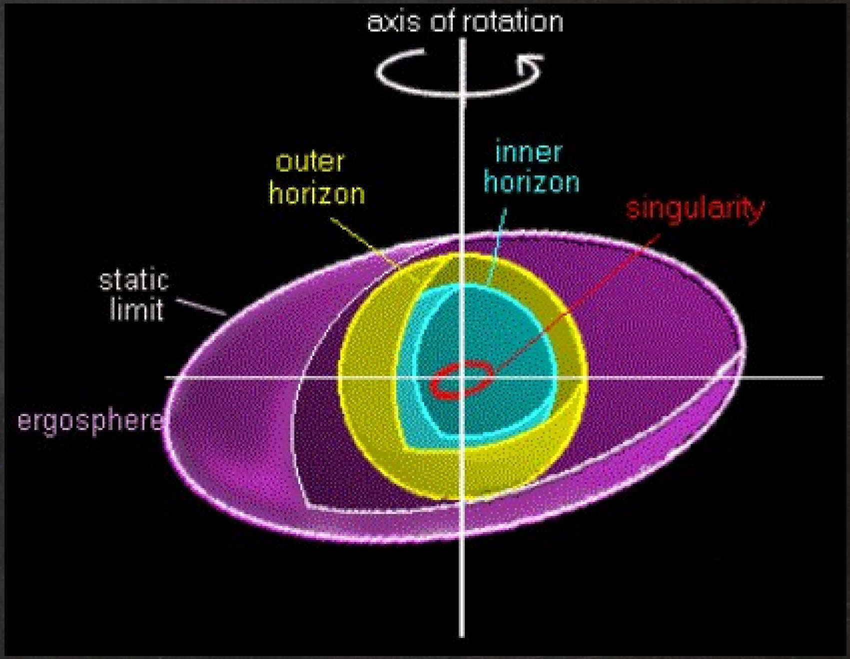
# Spinning BHs in D>4

- Nonuniqueness
- Multiple angular momenta
- Phase transitions, bifurcations
- Very rich parameter space
- Instabilities for large spin
  - Cambridge group, Shibata & Yoshino, ...
- Greybody factors



# Open problems
# Kerr-Newman BHs

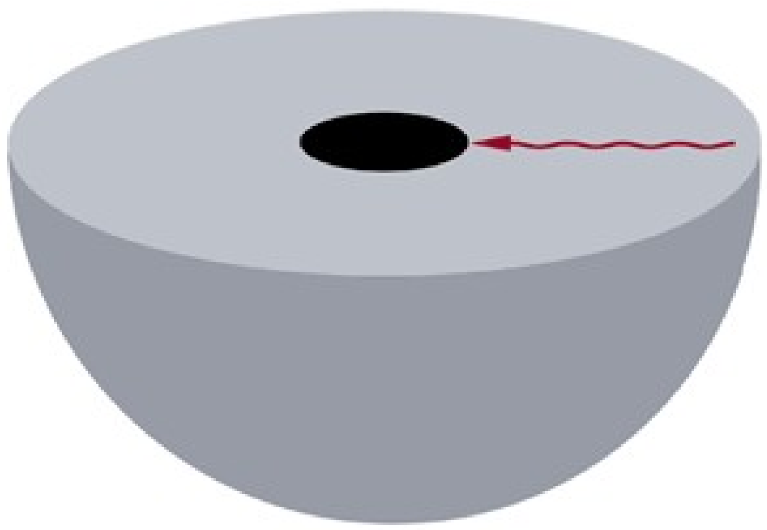
- Most general stationary BH in GR
- Charged and rotatating
- Electric and magnetic field
- Is it stable?



# Open problems

# Massive bosons

- Superradiant instability
- Observational imprints
- Bounds on particle physics
- Massive spin-1, spin-2?

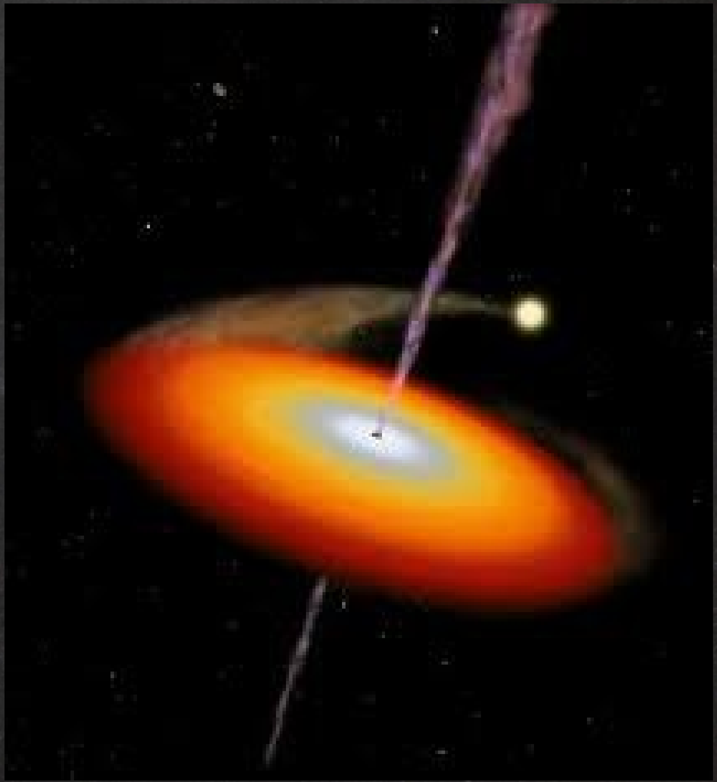


[Credit: Ana Sousa]

# Open problems

# Astrophysical BHs

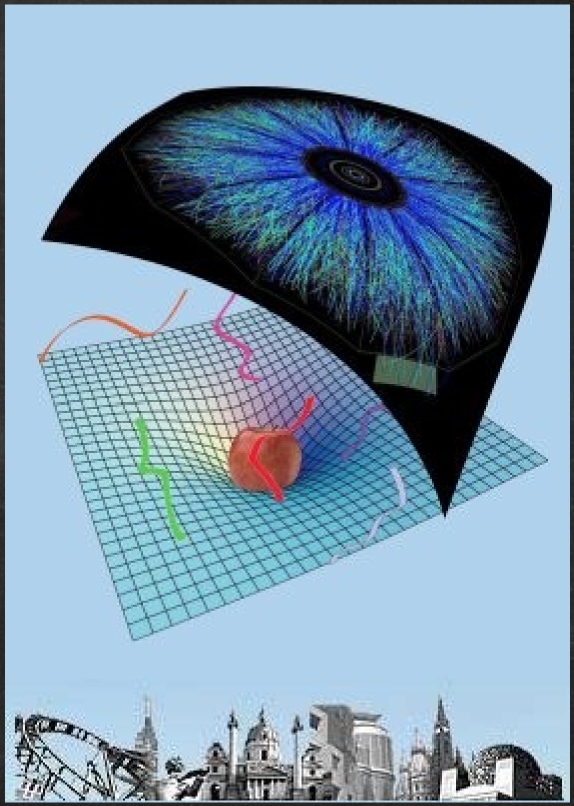
- BHs are surrounded by matter
- Accretion disk
- Magnetic fields → jets
- Perturbative problem?



## Open problems

# Hairy BHs in AdS/CM

- Gravity/gauge duality
- AdS BHs → thermal dual theory
- Linear response ↔ thermalization
- Hairy BHs → coupled perturbations
- Spin?



# *How do small fields propagate in the spacetime of a stationary and axisymmetric BH?*

$$\Box\phi = \mu^2\phi$$

# Part 0

# Field equations

## in Lagrangian theories of gravity

# Field equations

**In modified gravity, field equations can be very ugly:**

$$\mathcal{L} = \mathcal{L}(g_{ab}, \partial_c g_{ab}, ..., \phi, \partial_c \phi, ...) \quad \Rightarrow \quad \delta\mathcal{L} = \delta g \frac{\delta \mathcal{L}}{\delta g} + \delta\phi \frac{\delta \mathcal{L}}{\delta \phi} + .. = 0$$

**Symbolic manipulation software (e.g. Mathematica + xTensor)**

**Example: gravity with scalar field and quadratic curvature corrections:**

$$\mathcal{L}_{\rm DCS} = \sqrt{-g}\left(\frac{R}{16\pi} - \frac{1}{2}\nabla_a\phi\nabla^a\phi + \frac{\alpha}{4}\phi\,{}^*RR - V(\phi)\right) + \mathcal{L}_{\rm matter}$$


$$G_{ab} = 8\pi T_{ab} + 8\pi\left[\partial_a\phi\partial_b\phi - \frac{g_{ab}}{2}(\partial\phi)^2 - g_{ab}V(\phi)\right] - 16\pi\alpha C_{ab}$$

$$\Box\phi = V'(\phi) - \frac{\alpha}{4}{}^*RR$$

# Part I

# Perturbations of nonspinning BHs:

## Framework & Techniques

# *Harmonic decomposition.* *Static case*

[Kokkotas & Schmidt 1998] [Berti et al. 2009] [Konoplya & Zhidenko 2011]

$$ds^2 = \underbrace{-f(r)dt^2 + B(r)^{-1}dr^2 + r^2 d\Omega_2}_{\text{background}} + \underbrace{(\delta g_{\mu\nu})dx^\mu dx^\nu}_{\text{perturbations}}$$

$$\delta g_{\mu\nu}(t,r,\vartheta,\varphi) = \delta g^{\rm odd}_{\mu\nu}(t,r,\vartheta,\varphi) + \delta g^{\rm even}_{\mu\nu}(t,r,\vartheta,\varphi)$$

**Perturbations naturally divide into two classes, accordingly to parity transformations:**

**Axial sector: 3 functions**

$$\delta g^{\rm odd}_{\mu\nu} = \begin{pmatrix} 0 & 0 & h_0^\ell S_\vartheta^\ell & h_0^\ell S_\vartheta^\ell \\ * & 0 & h_1^\ell S_\vartheta^\ell & h_1^\ell S_\vartheta^\ell \\ * & * & -h_2^\ell \frac{X^\ell}{\sin\vartheta} & h_2^\ell \sin\vartheta W^\ell \\ * & * & * & h_2^\ell \sin\vartheta X^\ell \end{pmatrix}$$

$$(S_\vartheta^\ell, S_\varphi^\ell) \equiv \left(-\frac{Y^\ell_{,\varphi}}{\sin\vartheta}, \sin\vartheta Y^\ell_{,\vartheta}\right) \qquad (X^\ell, W^\ell) \equiv \left(2(Y^\ell_{,\vartheta\varphi} - \cot\vartheta Y^\ell_{,\varphi}), Y^\ell_{,\vartheta\vartheta} - \cot\vartheta Y^\ell_{,\vartheta} - \frac{Y^\ell_{,\varphi\varphi}}{\sin^2\vartheta}\right)$$

# Harmonic decomposition. Static case

[Kokkotas & Schmidt 1998] [Berti et al. 2009] [Konoplya & Zhidenko 2011]

$$ds^2 = \underbrace{-f(r)dt^2 + B(r)^{-1}dr^2 + r^2 d\Omega_2}_{\text{background}} + \underbrace{(\delta g_{\mu\nu})dx^\mu dx^\nu}_{\text{perturbations}}$$

$$\delta g_{\mu\nu}(t,r,\vartheta,\varphi) = \delta g_{\mu\nu}^{\rm odd}(t,r,\vartheta,\varphi) + \delta g_{\mu\nu}^{\rm even}(t,r,\vartheta,\varphi)$$

**Perturbations naturally divide into two classes, accordingly to parity transformations:**

**Polar sector: 7 functions**

$$\delta g_{\mu\nu}^{\rm even} = \begin{pmatrix} g_{tt}^{(0)} H_0^\ell Y^\ell & H_1^\ell Y^\ell & \eta_0^\ell Y^\ell_{,\vartheta} & \eta_0^\ell Y^\ell_{,\varphi} \\ * & g_{rr}^{(0)} H_2^\ell Y^\ell & \eta_1^\ell Y^\ell_{,\vartheta} & \eta_1^\ell Y^\ell_{,\varphi} \\ * & * & r^2\left[K^\ell Y^\ell + G^\ell W^\ell\right] & r^2 G^\ell X^\ell \\ * & * & * & r^2\sin^2\vartheta\left[K^\ell Y^\ell - G^\ell W^\ell\right] \end{pmatrix}.$$

$$(S^\ell_\vartheta, S^\ell_\varphi) \equiv \left(-\frac{Y^\ell_{,\varphi}}{\sin\vartheta}, \sin\vartheta Y^\ell_{,\vartheta}\right) \qquad (X^\ell, W^\ell) \equiv \left(2(Y^\ell_{,\vartheta\varphi} - \cot\vartheta Y^\ell_{,\varphi}), Y^\ell_{,\vartheta\vartheta} - \cot\vartheta Y^\ell_{,\vartheta} - \frac{Y^\ell_{,\varphi\varphi}}{\sin^2\vartheta}\right)$$

# Harmonic decomposition. Example

$$\mathcal{L}_{\rm DCS} = \sqrt{-g}\left(\frac{R}{16\pi} - \frac{1}{2}\nabla_a\phi\nabla^a\phi + \frac{\alpha}{4}\phi\,{}^*RR - V(\phi)\right) + \mathcal{L}_{\rm matter}$$

$$G_{ab} = 8\pi T_{ab} + 8\pi\left[\partial_a\phi\partial_b\phi - \frac{g_{ab}}{2}(\partial\phi)^2 - g_{ab}V(\phi)\right] - 16\pi\alpha C_{ab}$$

$$\Box\phi = V'(\phi) - \frac{\alpha}{4}{}^*RR$$


$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V_{RW}(r)\right]Q^\ell(t,r) = T_{RW}(r)\Theta^\ell(t,r)$$

$$\left[\frac{d^2}{dr_*^2} + \omega^2 - V_S(r)\right]\Theta^\ell(t,r) = T_S(r)Q^\ell(t,r)$$

# Perturbation equations. *Static case*

- **Axial and polar sector decoupled**

$$\left[-\frac{d^2}{dt}+\frac{d^2}{dr_*^2}\right]\mathbf{Y}_\ell-\mathbf{V}_\ell(r)\mathbf{Y}_\ell=0$$

Time domain

- **1+1 system of second-order PDEs**
- **Each harmonic index (l,m) decoupled**
- **Stationarity and axisymmetry:** $\tilde{\mathbf{Y}}_\ell(r)=\int d\omega e^{-i\omega t}\mathbf{Y}(t,r)$

$$\left[\frac{d^2}{dr_*^2}+\omega^2-\mathbf{V}_\ell(r)\right]\tilde{\mathbf{Y}}_\ell=0$$

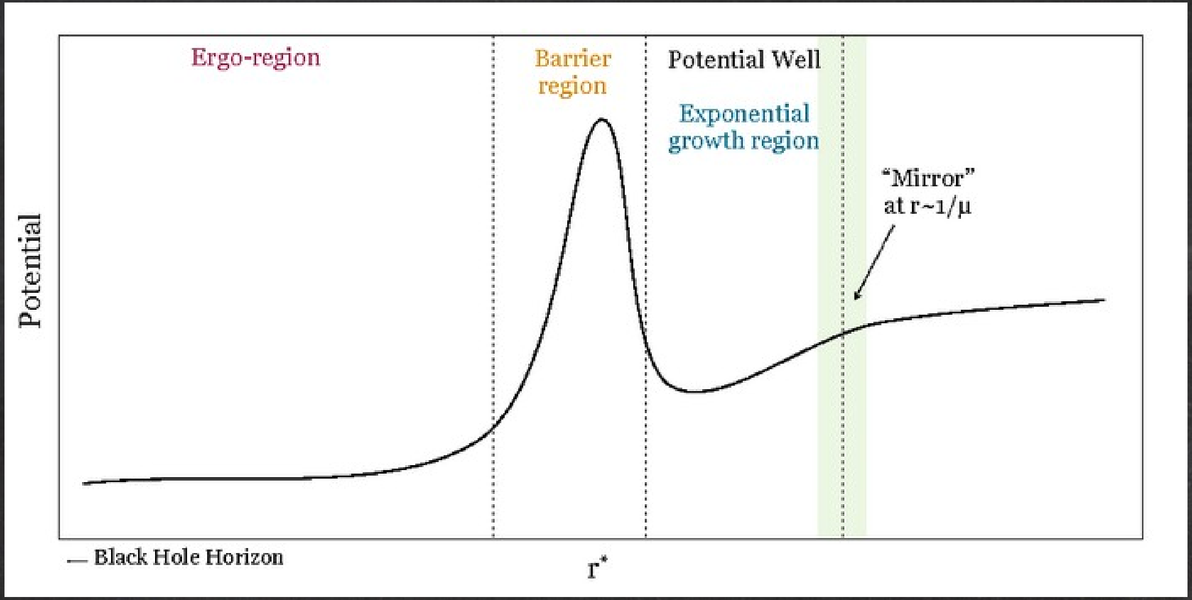
Frequency domain

# Frequency domain. Static case

$$\left[\frac{d^2}{dr_*^2}+\omega^2-\mathbf{V}_\ell(r)\right]\tilde{\mathbf{Y}}_\ell=0$$

Coupled ODEs

- **Boundary conditions → eigenvalue problem**




$Y_i(r)\sim e^{-i\omega r_*}$ $\quad r\to r_+$

Horizon: purely ingoing waves (regularity)

$Y_i\sim B_{(i)}e^{-k_\infty r_*}+C_{(i)}e^{k_\infty r_*}$ $\quad r\to\infty$

Infinity: purely outgoing waves or localized states

$k_\infty=\sqrt{\mu^2-\omega^2}$

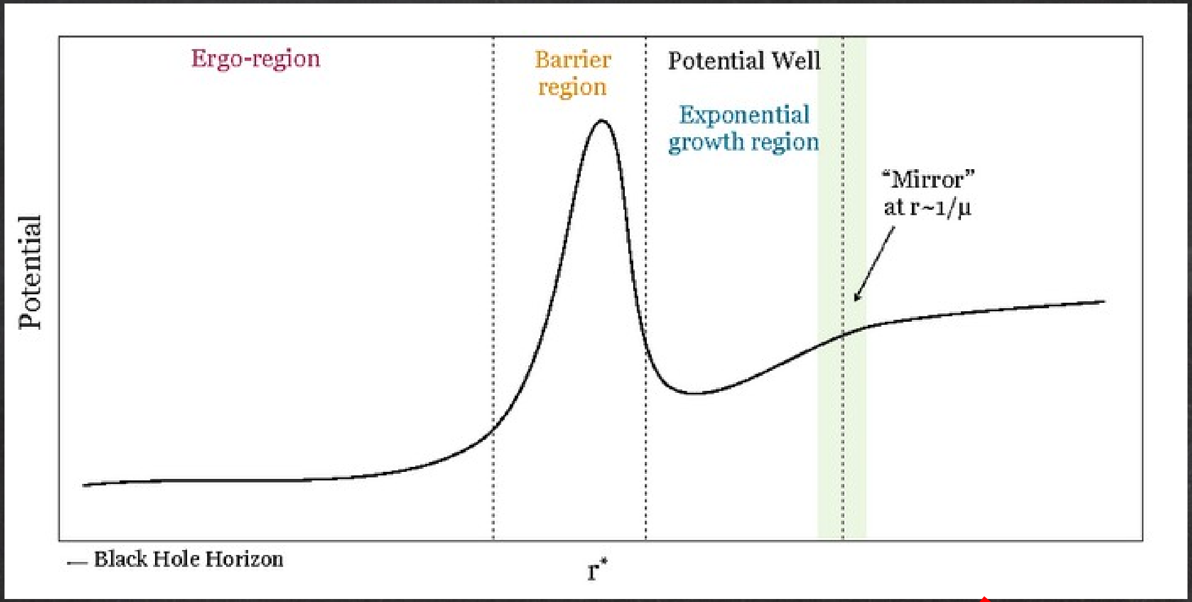
- **(Infinite) set of complex eigenfrequencies**

# Frequency domain. Static case

$$\left[\frac{d^2}{dr_*^2}+\omega^2-+\mathbf{V}_\ell(r)\right]\tilde{\mathbf{Y}}_\ell=0$$

Coupled ODEs

- **Boundary conditions → eigenvalue problem**




Quasi-bound states

$Y_i(r)\sim e^{-i\omega r_*}$ $\qquad r\to r_+$

Horizon: purely ingoing waves (regularity)

$Y_i\sim B_{(i)}e^{-k_\infty r_*}+\cancel{C_{(i)}}e^{k_\infty r_*}$ $\qquad r\to\infty$

Infinity: ~~purely outgoing waves~~ or localized states

$k_\infty=\sqrt{\mu^2-\omega^2}$

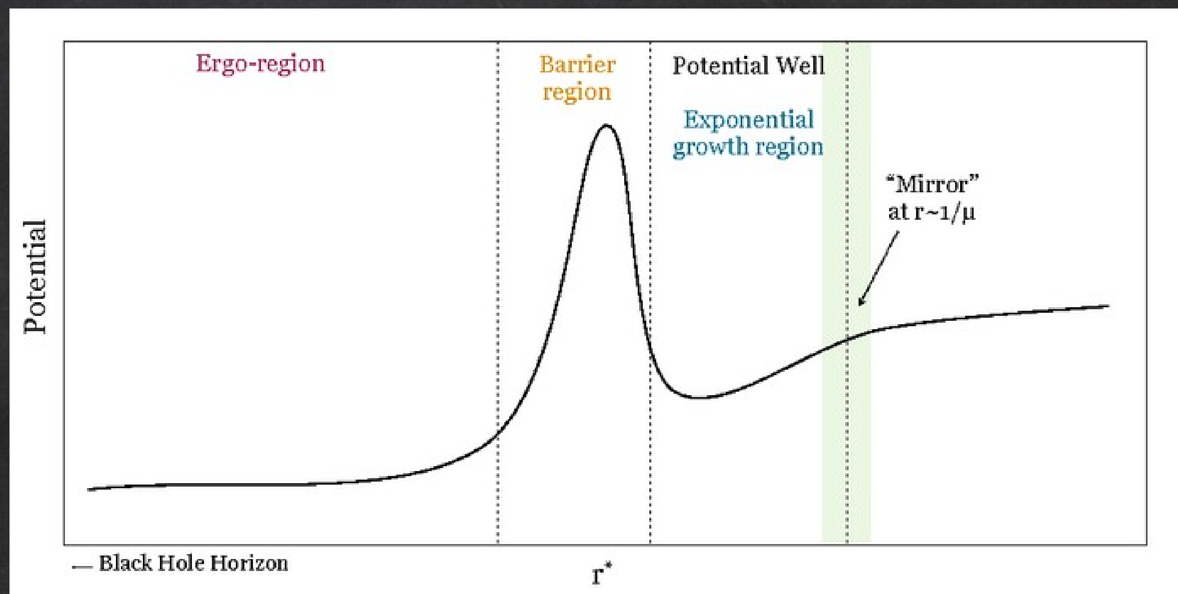
- **(Infinite) set of complex eigenfrequencies**

# Frequency domain. Static case

$$\left[\frac{d^2}{dr_*^2}+\omega^2-+\mathbf{V}_\ell(r)\right]\tilde{\mathbf{Y}}_\ell=0$$

Coupled ODEs

- **Boundary conditions → eigenvalue problem**




**Quasinormal modes**

$Y_i(r)\sim e^{-i\omega r_*}\qquad r\to r_+$

Horizon: purely ingoing waves (regularity)

$Y_i\sim \cancel{B_{(i)}e^{-k_\infty r_*}}+C_{(i)}e^{k_\infty r_*}\qquad r\to\infty$

Infinity: purely outgoing waves or localized states

$k_\infty=\sqrt{\mu^2-\omega^2}$

- **(Infinite) set of complex eigenfrequencies**

# Matrix-valued Continued Fractions

$$Y_i = e^{-i\omega r_*} r^{-\nu} e^{qr} \sum_n a_n^{(i)} F(r)^n \qquad q = \pm k_\infty$$

- **Matrix-valued recurrence relation:**

$$\boldsymbol{\alpha}_0 \mathbf{a}_1 + \boldsymbol{\beta}_0 \mathbf{a}_0 = 0 \qquad n = 0$$

$$\boldsymbol{\alpha}_n \mathbf{a}_{n+1} + \boldsymbol{\beta}_n \mathbf{a}_n + \boldsymbol{\gamma}_n \mathbf{a}_{n-1} = 0\,, \qquad n > 0$$

- **Ladder matrix:** $\mathbf{a}_{n+1} = \mathbf{R}_n^+ \mathbf{a}_n$

$$\mathbf{R}_n^+ = -\left[\boldsymbol{\beta}_{n+1} + \boldsymbol{\alpha}_{n+1} \mathbf{R}_{n+1}^+\right]^{-1} \boldsymbol{\gamma}_{n+1}$$

$$\mathbf{M} \equiv \boldsymbol{\beta}_0 - \boldsymbol{\alpha}_0 \left[\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1 (\boldsymbol{\beta}_2 + \boldsymbol{\alpha}_2 \mathbf{R}_2^+) \boldsymbol{\gamma}_2\right]^{-1} \boldsymbol{\gamma}_1$$

$$\pi = \cfrac{4}{1 + \cfrac{1}{3 + \cfrac{4}{5 + \cfrac{9}{7 + \cfrac{16}{9 + \cfrac{25}{11 + \cfrac{36}{13 + \cfrac{49}{\ddots}}}}}}}}$$

- **We seek for the roots:**

$$\det \mathbf{M} = 0$$

CF_axial_scalar_3terms.nb

# *Matrix-valued Continued Fractions*

- **Four-terms matrix-valued recurrence relation:**

$$\boldsymbol{\alpha}_0\mathbf{a}_1+\boldsymbol{\beta}_0\mathbf{a}_0=0\,,\qquad n=0$$

$$\boldsymbol{\alpha}_1\mathbf{a}_2+\boldsymbol{\beta}_1\mathbf{a}_1+\boldsymbol{\gamma}_1\mathbf{a}_0=0\,,\qquad n=1$$

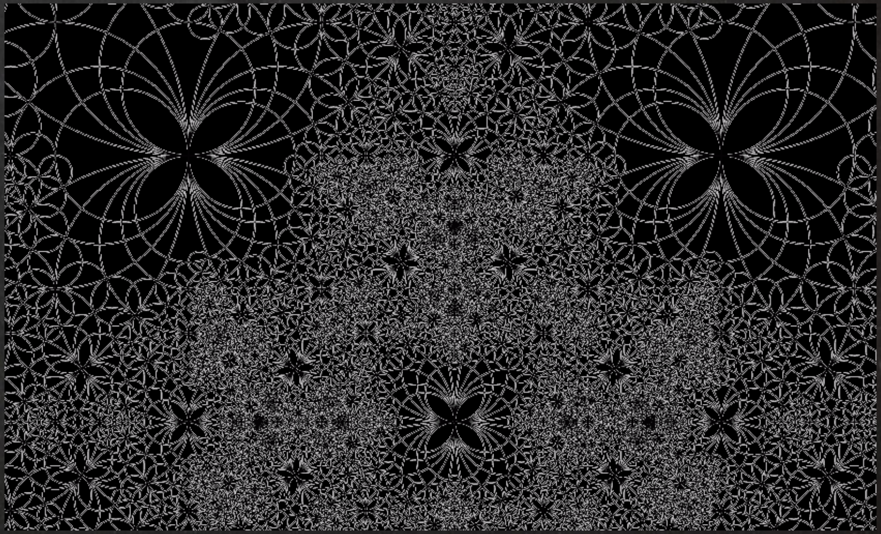
$$\boldsymbol{\alpha}_n\mathbf{a}_{n+1}+\boldsymbol{\beta}_n\mathbf{a}_n+\boldsymbol{\gamma}_n\mathbf{a}_{n-1}+\boldsymbol{\delta}_n\mathbf{a}_{n-2}=0\,,\qquad n>1$$

- **Guassian elimination**

$$\tilde{\boldsymbol{\alpha}}_n=\boldsymbol{\alpha}_n\qquad\tilde{\boldsymbol{\beta}}_0=\boldsymbol{\beta}_0\qquad\tilde{\boldsymbol{\gamma}}_0=\boldsymbol{\gamma}_0$$

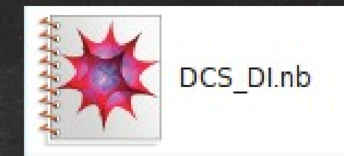
$$\tilde{\boldsymbol{\beta}}_n=\boldsymbol{\beta}_n-\boldsymbol{\delta}_n\left[\tilde{\boldsymbol{\gamma}}_{n-1}\tilde{\boldsymbol{\alpha}}_{n-1}\right]^{-1}\qquad n>0$$

Requires invertible matrices

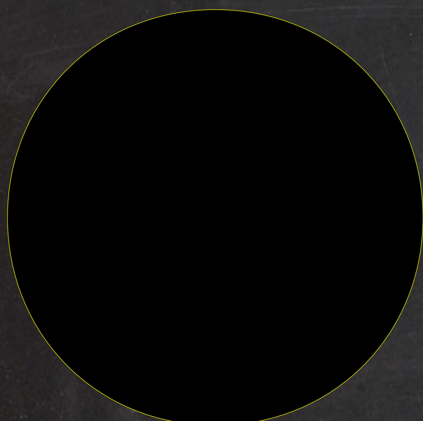


The 8 Moebius Transforms for Complex Continued Fractions

# Matrix-valued Direct Integration




$$Y_i \sim e^{-i\omega r_*} \sum_n b_n^{(i)} (r - r_+)^n \qquad r \to r_+$$



Horizon

$$r \to \infty$$

$$Y_i \sim B_{(i)} e^{-k_\infty r_*} + C_{(i)} e^{k_\infty r_*}$$

N integrations

$$(1,0,0,...,0)$$
$$(0,1,0,...,0)$$
$$\ldots$$
$$(0,0,0,...,1)$$

$$\mathbf{S}_m(\omega) = \begin{pmatrix} A_1^{(1)} & A_1^{(2)} & \ldots & A_1^{(N)} \\ A_2^{(1)} & A_2^{(2)} & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ A_N^{(1)} & \ldots & \ldots & A_N^{(N)} \end{pmatrix}$$

$$\det \mathbf{S}_m(\omega_0) = 0$$

# Breit-Wigner resonances

Approximation of the direct integration in case of slowly-damped modes

$$\det \mathbf{S}_m(\omega_0) = 0 \qquad \omega_I \ll \omega_R$$

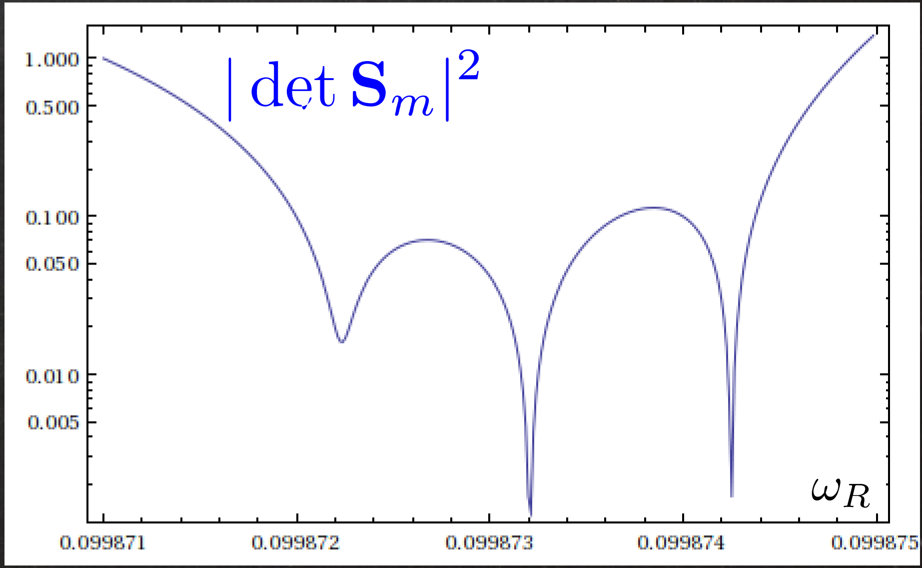
BW_Proca_2nd_order.nb

$$\det \mathbf{S}_m(\omega_0) \simeq \det \mathbf{S}_m(\omega_R) + i\omega_I \left.\frac{d\left[\det \mathbf{S}_m(\omega)\right]}{d\omega}\right|_{\omega_R} = 0$$

$$\det \mathbf{S}_m(\omega) \simeq \det \mathbf{S}_m(\omega_R)\left[1 - \frac{\omega - \omega_R}{i\omega_I}\right] \propto \omega - \omega_R - i\omega_I$$

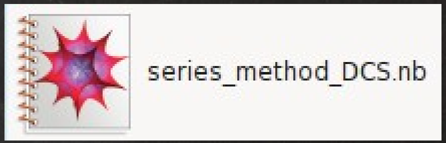
$$|\det \mathbf{S}_m(\omega)|^2 \propto (\omega - \omega_R)^2 + \omega_I^2$$

Quadratic fit around the minimum



# Matrix-valued series methods

Black holes and branes in Anti de Sitter




$$\left[\frac{d^2}{dr_*^2} + \omega^2 - \mathbf{V}_\ell(r)\right]\tilde{\mathbf{Y}}_\ell = 0$$

$$Y_i \to A_i r^{\alpha_i} + B_i r^{\beta_i}$$

Asymptotic behavior at infinity

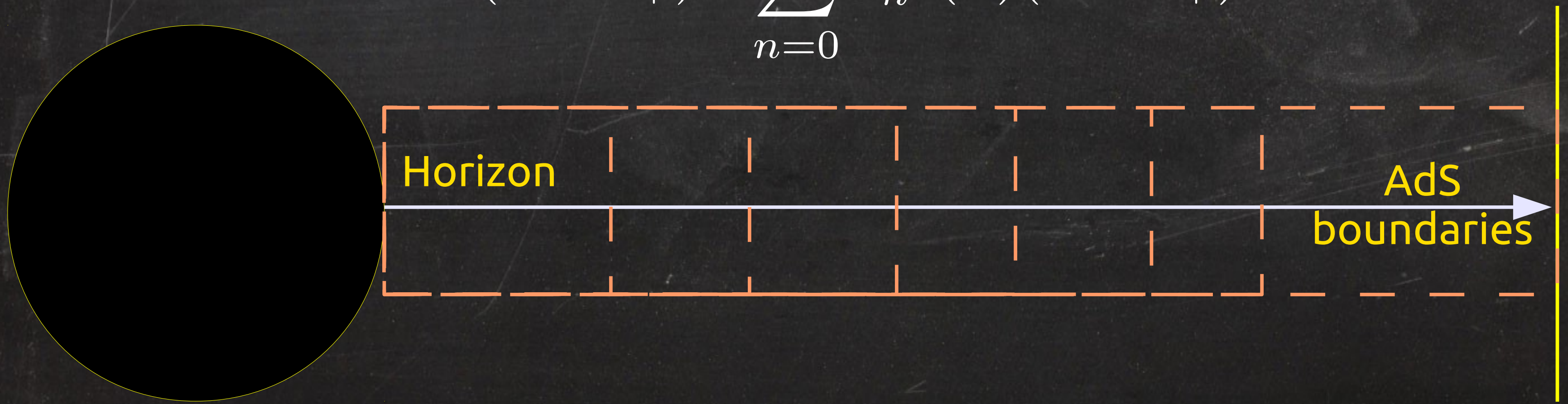
$$Y_i(x) = e^{-i\omega r_*} Z_i(x)$$

$$x = 1/r$$

$$\Rightarrow \quad (x - x_+)s(x)\frac{d^2\mathbf{Z}}{dx^2} + t(x)\frac{d\mathbf{Z}}{dx} + \frac{\mathbf{u(x)}}{x - x_+}\mathbf{Z} = 0$$

Frobenius series:

$$Z_i = (x - x_+)^{\gamma_i} \sum_{n=0}^{\infty} a_n^{(i)}(\omega)(x - x_+)^n$$



# Comparison of the methods

| Continued Fractions | Direct Integration | Breig-Wigner | Series Methods |
|---|---|---|---|
| Robust | Not optimal for QNMs | Only slowly-damped modes | Anti de Sitter |
| QNMs & bound states | Bound states & fundamental QNMs | Typically only bound states | Large BHs and branes |
| Limitations on V(r) | Generic V(r) | V(r) must support bound states | ~ |
| Overtones | No overtones | ~ | Overtones |
| Convergence? | Requires higher-order series expansion | Optimal for large systems of ODEs | Convergence radius |

# What we learnt so far:

- Decomposing any perturbation around generic static BHs
- 1+1 time evolution problems
- 1-dimensional problems in the frequency domain
  - BH spectrum: computing the eigenfrequencies
  - Asymptotically flat, (A)dS
  - GR, modified gravities, extra dof

## Part II

# Perturbations of spinning BHs:

## Framework & Techniques

# Spinning BHs

- Kerr metric: Teukolsky formalism
- Separability is a miracle!

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*NATURE* | LETTER

日本語要約

A rapidly spinning supermassive black hole at the centre of NGC 1365

G. Risaliti F. A. Harrison K. K. Madsen D. J. Walton S. E. Boggs F. E. Christensen W. W. Craig B. W. Grefenstette C. J. Hailey E. Nardini Daniel Stern W. W. Zhang

Affiliations | Contributions | Corresponding authors

*Nature* **494**, 449–451 (28 February 2013) | doi:10.1038/nature11938
Received 29 October 2012 | Accepted 16 January 2013 | Published online 27 February 2013

- NonKerr, nonGR, D>4?
- Time: (1+2) problem
- Frequency: 2D eigenvalue problem

# Eigenfrequencies of spinning BHs

# Spectral methods

- **Spinning BH in higher dimensions** [Dias et al. 2010-2011]
- **Subclass of perturbations →** $\delta g_{\mu\nu} = e^{ikz} e^{-i\omega t} h_{\mu\nu}$

$$(\Delta_L h)_{\mu\nu} \equiv -\nabla_\rho \nabla^\rho h_{\mu\nu} - 2R_{\mu\nu\rho\sigma} h^{\rho\sigma} = -k^2 h_{\mu\nu}$$

Steven A. Orszag

- **Eigenvalue problem for k² →** $\mathcal{D}\mathbf{Y} = -k^2 \mathbf{V}\mathbf{Y}$

$$Y_i(y) = \sum_{j=0}^{n} a_j^{(i)} y^j \qquad y_l = \frac{y_f + y_i}{2} + \frac{y_f - y_i}{2} \cos\left(\frac{(2l-1)\pi}{2n}\right)$$

Finite sum of polynomials

Chebyshev nodes

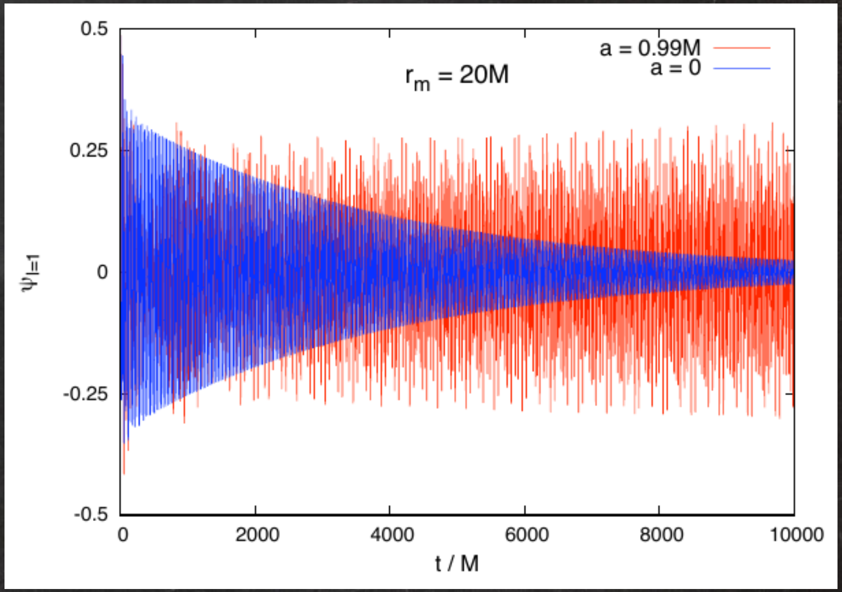
- **Algebraic system for** $a_j^{(i)}$

# Small fields on spinning metrics

# Time domain

$$\Box\phi = \mu^2\phi$$

- **Massive KG eq. on a Kerr metric**
- **Instability time scale ~ $10^6$ M**
- **Time evolution: 2+1 problem**
- **Reduction to 1+1?**




[Dolan, 2012]

# Ultralong evolutions on spinning BHs

[Dolan, 2012]

$$\phi(t,r,\vartheta,\varphi)=\frac{\Psi(t,r,\vartheta)}{r}e^{im\varphi}$$

$$\Box\phi=\mu^2\phi \longrightarrow \left[\mathcal{D}_{tr}-\Delta\left(\mathcal{D}_{\vartheta\tilde{\varphi}}-V-V_\mu\right)\right]\left(\Psi e^{im\tilde{\varphi}}\right)=0$$

Mass term

Radial potential

Laplace Op. on the sphere

Time and radial differential operator

$$V_\mu\equiv\mu^2(r^2+\tilde{a}^2M^2\cos^2\vartheta)$$

- **Not separable in spherical harmonics →** $\Psi(t,r,\vartheta)e^{im\tilde{\varphi}}=\psi_\ell(t,r)Y^\ell$
- **However, the angular dependence can be eliminated by**

$$\cos\vartheta Y^\ell=\mathcal{Q}_{\ell+1}Y^{\ell+1}+\mathcal{Q}_\ell Y^{\ell-1} \qquad \mathcal{Q}_\ell=\sqrt{\frac{\ell^2-m^2}{4\ell^2-1}}$$

$$\cos^2\vartheta Y^\ell=\left(\mathcal{Q}_{\ell+1}^2+\mathcal{Q}_\ell^2\right)Y^\ell+\mathcal{Q}_{\ell+1}\mathcal{Q}_{\ell+2}Y^{\ell+2}+\mathcal{Q}_\ell\mathcal{Q}_{\ell-1}Y^{\ell-2}$$

# Ultralong evolutions on spinning BHs

[Dolan, 2012]

**2+1 evolution equation for the scalar field**

$$\left[\mathcal{D}_{tr}-\Delta\left(\mathcal{D}_{\vartheta\tilde{\varphi}}-V-V_{\mu}\right)\right]\left(\Psi e^{im\tilde{\varphi}}\right)=0$$

$$\Psi(t,r,\vartheta)e^{im\tilde{\varphi}}=\psi_\ell(t,r)Y^\ell$$

$$\cos^2\vartheta Y^\ell=\sum_\ell c_\ell Y^\ell$$

$$\int Y^\ell Y^{*\,\ell'}d\Omega=\delta^{\ell\ell'}$$

Expansion in spherical harm.

Clebsh-Gordan coefficients

Orthogonality properties

$$\mathcal{D}_{tr}\psi_\ell-\Delta\left\{\ell(\ell+1)+V+\mu^2\left[r^2+\tilde{a}^2M^2\left(\mathcal{Q}_{2+\mathcal{Q}_\ell^2}\right)\right]\right\}\psi_\ell=$$

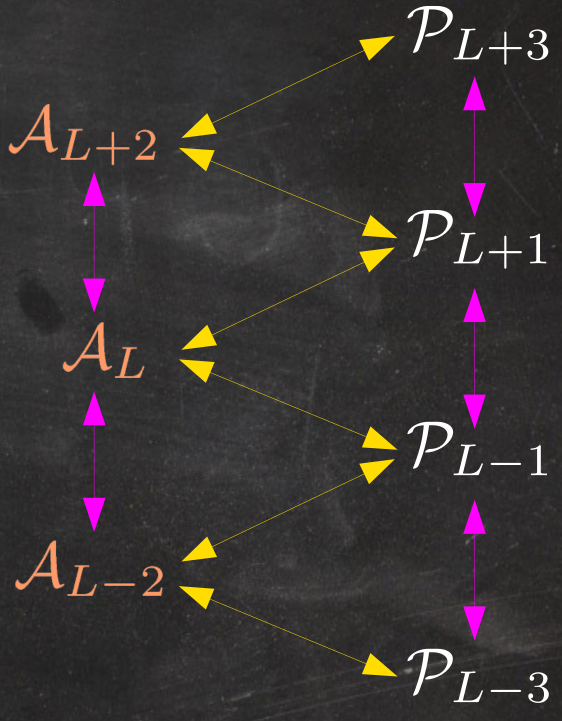
$$=\mu^2\tilde{a}^2M^2\Delta\left[\mathcal{Q}_\ell\mathcal{Q}_{\ell-1}\psi_{\ell-2}+\mathcal{Q}_{\ell+2}\mathcal{Q}_{\ell+1}\psi_{\ell-2}\right]$$

**1+1 system of ODEs with couplings to nearest-neighborhood multipoles**

# Small fields on spinning metrics

# Slow-rotation approx.

- Slow-spin approximation
- Expansion in spherical harmonics
- Couplings to different multipoles
- Only first couplings contribute




[Kojima 1992, 1993, 1997]

[Pani et al., 2012]

# Slow-rotation approximation

Stationary and axisymmetric background

$$ds_0^2 = -H^2dt^2 + Q^2dr^2 + r^2K^2\left[d\vartheta^2 + \sin^2\vartheta(d\varphi - Ldt)^2\right]$$

to 1st order in the angular momentum:

$$ds_0^2 = -F(r)dt^2 + B(r)^{-1}dr^2 + r^2d^2\Omega - 2\varpi(r)\sin^2 d\varphi dt$$

to 2nd order:

$$ds_0^2 = -F(r)\left[1+F_2\right]dt^2 + B(r)^{-1}\left[1+\frac{2B_2}{r-2M}\right]dr^2 + r^2(1+k_2)\left[d\vartheta^2 + \sin^2(d\varphi - \varpi dt)^2\right]$$

$$F_2(r,\vartheta) = F_r(r) + F_\vartheta(r)P_2(\vartheta) \qquad B_2(r,\vartheta) = B_r(r) + B_\vartheta(r)P_2(\vartheta)$$

$$k_2(r,\vartheta) = k_r(r) + k_\vartheta(r)P_2(\vartheta)$$

$$\varpi(r,\vartheta) = \varpi_r(r) + \mathcal{O}(\tilde{a}^3)$$

# Slow-rotation approximation

Expansion of any field perturbation in a complete basis of spherical harmonics

$$0 = \mathcal{A}_\ell$$

**Zeroth order: decoupled**

$$0 = \mathcal{P}_\ell$$

$\mathcal{P}_{L+3}$

$\mathcal{A}_{L+2}$

$\mathcal{P}_{L+1}$

$\mathcal{A}_{L}$

$\mathcal{P}_{L-1}$

$\mathcal{A}_{L-2}$

$\mathcal{P}_{L-3}$

# Slow-rotation approximation

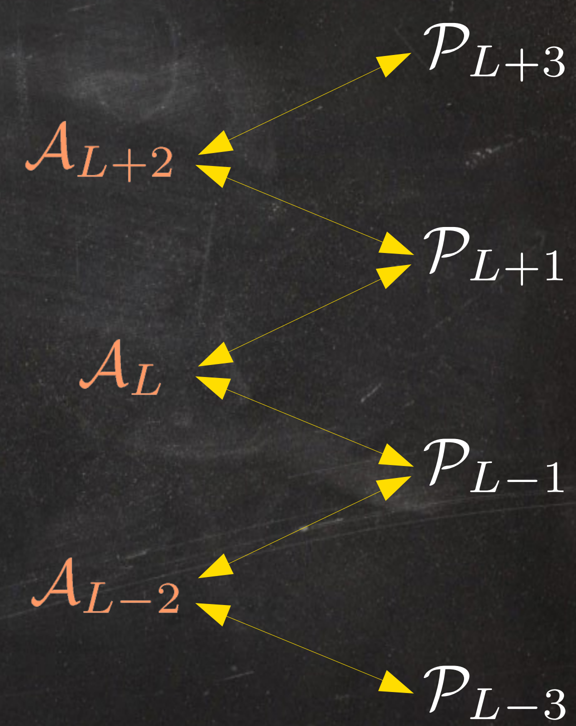
Expansion of any field perturbation in a complete basis of spherical harmonics

$$0 = \mathcal{A}_\ell + \tilde{a} m \bar{\mathcal{A}}_\ell + \tilde{a}\left(\mathcal{Q}_\ell \tilde{\mathcal{P}}_{\ell-1} + \mathcal{Q}_{\ell+1} \tilde{\mathcal{P}}_{\ell+1}\right)$$

**Zeroth order: decoupled**

**First order: polar-axial l±1**

$$0 = \mathcal{P}_\ell + \tilde{a} m \bar{\mathcal{P}}_\ell + \tilde{a}\left(\mathcal{Q}_\ell \tilde{\mathcal{A}}_{\ell-1} + \mathcal{Q}_{\ell+1} \tilde{\mathcal{A}}_{\ell+1}\right)$$



# Slow-rotation approximation

Expansion of any field perturbation in a complete basis of spherical harmonics

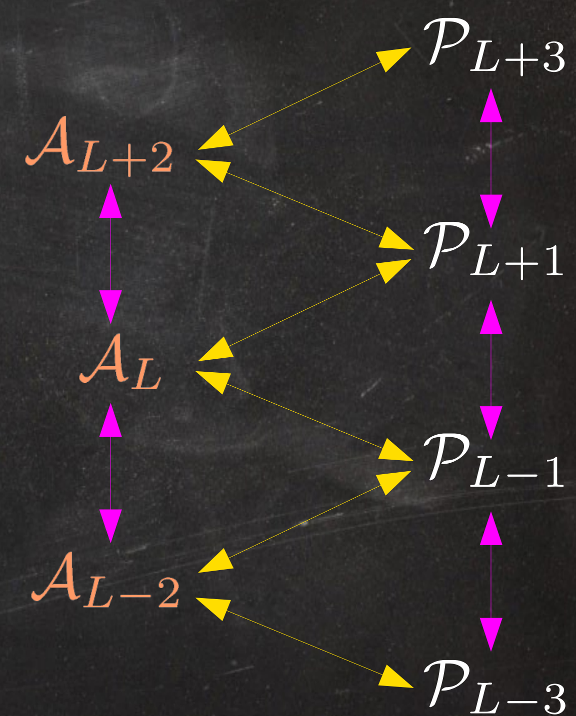
$$0 = \mathcal{A}_\ell$$
$$+\tilde{a}m\bar{\mathcal{A}}_\ell + \tilde{a}(\mathcal{Q}_\ell\tilde{\mathcal{P}}_{\ell-1} + \mathcal{Q}_{\ell+1}\tilde{\mathcal{P}}_{\ell+1})$$
$$+\tilde{a}^2\left[\hat{\mathcal{A}}_\ell + \mathcal{Q}_{\ell-1}\mathcal{Q}_\ell\breve{\mathcal{A}}_{\ell-2} + \mathcal{Q}_{\ell+2}\mathcal{Q}_{\ell+1}\breve{\mathcal{A}}_{\ell+2}\right]$$

**Zeroth order: decoupled**

**First order: polar-axial l±1**

**Second order: l±2**

$$0 = \mathcal{P}_\ell$$
$$+\tilde{a}m\bar{\mathcal{P}}_\ell + \tilde{a}(\mathcal{Q}_\ell\tilde{\mathcal{A}}_{\ell-1} + \mathcal{Q}_{\ell+1}\tilde{\mathcal{A}}_{\ell+1})$$
$$+\tilde{a}^2\left[\hat{\mathcal{P}}_\ell + \mathcal{Q}_{\ell-1}\mathcal{Q}_\ell\breve{\mathcal{P}}_{\ell-2} + \mathcal{Q}_{\ell+2}\mathcal{Q}_{\ell+1}\breve{\mathcal{P}}_{\ell+2}\right]$$




- **Zeeman rule**, **Laporte rule**, **propensity rule**
- **Generic: any metric, any perturbation, any theory, any order**

# Eigenvalues at first order

$$0 = \mathcal{A}_\ell + \tilde{a}m\bar{\mathcal{A}}_\ell + \tilde{a}(\mathcal{Q}_\ell\tilde{\mathcal{P}}_{\ell-1} + \mathcal{Q}_{\ell+1}\tilde{\mathcal{P}}_{\ell+1})$$

$$0 = \mathcal{P}_\ell + \tilde{a}m\bar{\mathcal{P}}_\ell + \tilde{a}(\mathcal{Q}_\ell\tilde{\mathcal{A}}_{\ell-1} + \mathcal{Q}_{\ell+1}\tilde{\mathcal{A}}_{\ell+1})$$

The field equations and the boundary conditions are invariant under:

$$a_{\ell m} \to \mp a_{\ell -m}\,, \qquad p_{\ell m} \to \pm p_{\ell -m}$$

$$\tilde{a} \to -\tilde{a}\,, \qquad m \to -m$$

Therefore, the eigenfrequencies have the schematic form:

$$\omega = \omega_0 + m\,\omega_1\tilde{a} + \omega_2\tilde{a}^2 + \mathcal{O}(\tilde{a}^3)$$

To first order, only perturbations with
the same harmonic index “l” and same parity contribute

$$0 = \mathcal{A}_\ell + \tilde{a}m\bar{\mathcal{A}}_\ell \qquad 0 = \mathcal{P}_\ell + \tilde{a}m\bar{\mathcal{P}}_\ell$$

# Slow-rotation approximation

To a given order, the problem can be always reduced to N coupled ODEs which involve parity-mixing and multipole-mixing terms

These equations are “easy” to solve once the radial coefficients have been computed

# Part III

# Applications of the slow-rotation approach

# #1: Massive scalar fields on Kerr BHs

slow_rot_scalar.nb

$$\Box\phi = \mu^2\phi$$

Expansion of the scalar field in spherical harmonics

$$\phi = \sum_{\ell m} \frac{\Psi_\ell(r)}{\sqrt{r^2+\tilde{a}^2M^2}} e^{-i\omega t} Y^\ell(\vartheta,\varphi)$$

Equation to second order

$$A_\ell Y^\ell + D_\ell \cos^2\vartheta Y^\ell = 0$$

The angular dependence can be eliminated by

$$\cos\vartheta Y^\ell = \mathcal{Q}_{\ell+1} Y^{\ell+1} + \mathcal{Q}_\ell Y^{\ell-1}$$

$$\cos^2\vartheta Y^\ell = \left(\mathcal{Q}_{\ell+1}^2 + \mathcal{Q}_\ell^2\right) Y^\ell + \mathcal{Q}_{\ell+1}\mathcal{Q}_{\ell+2} Y^{\ell+2} + \mathcal{Q}_\ell \mathcal{Q}_{\ell-1} Y^{\ell-2}$$

$$\int Y^\ell Y^{*\,\ell'} d\Omega = \delta^{\ell\ell'}$$

$$\mathcal{Q}_\ell = \sqrt{\frac{\ell^2-m^2}{4\ell^2-1}}$$

# #1: Massive scalar fields on Kerr BHs

slow_rot_scalar.nb

$$A_\ell Y^\ell + D_\ell \cos^2\vartheta Y^\ell = 0$$

$$\cos^2\vartheta Y^\ell = \sum_\ell c_\ell Y^\ell \qquad \int Y^\ell Y^{*\,\ell'} d\Omega = \delta^{\ell\ell'}$$

$$\Big\downarrow$$

$$A_\ell + (\mathcal{Q}_{\ell+1}^2 + \mathcal{Q}_\ell^2) D_\ell + \mathcal{Q}_{\ell-1}\mathcal{Q}_\ell D_{\ell-2} + \mathcal{Q}_{\ell+2}\mathcal{Q}_{\ell+1} D_{\ell+2} = 0$$

To second order, the radial equation can be recast in the form

$$\frac{d^2 Z_\ell}{dr_*^2} + V_\ell Z_\ell = 0$$

The potential is precisely the one obtained within the Teukolsky approach to second order in the angular momentum.

Easy to extend to any order and to any background metric.

# #2: Gravitational pert. of Kerr BHs

In the Regge-Wheeler gauge, perturbations naturally separate into three groups:

$$\delta\mathcal{E}_{(I)} \equiv (A_\ell^{(I)} + \tilde{A}_\ell^{(I)} \cos\vartheta) Y^\ell + B_\ell^{(I)} \sin\vartheta Y^\ell_{,\vartheta} + C_\ell^{(I)} Y^\ell_{,\varphi} = 0$$

(t,t), (t,r), (r,r) and (+) components

$$\delta\mathcal{E}_{(L\vartheta)} \equiv (\alpha_\ell^{(L)} + \tilde{\alpha}_\ell^{(L)} \cos\vartheta) Y^\ell_{,\vartheta} - (\beta_\ell^{(L)} + \tilde{\beta}_\ell^{(L)} \cos\vartheta) \frac{Y^\ell_{,\varphi}}{\sin\vartheta} + \eta_\ell^{(L)} \sin\vartheta Y^\ell + \xi_\ell^{(L)} X^\ell + \chi_\ell^{(L)} \sin\vartheta W^\ell = 0$$

$$\delta\mathcal{E}_{(L\varphi)} \equiv (\beta_\ell^{(L)} + \tilde{\beta}_\ell^{(L)} \cos\vartheta) Y^\ell_{,\vartheta} + (\alpha_\ell^{(L)} + \tilde{\alpha}_\ell^{(L)} \cos\vartheta) \frac{Y^\ell_{,\varphi}}{\sin\vartheta} + \zeta_\ell^{(L)} \sin\vartheta Y^\ell + \chi_\ell^{(L)} X^\ell - \xi_\ell^{(L)} \sin\vartheta W^\ell = 0$$

(t,theta) and (r,theta)

$$\delta\mathcal{E}_{(\vartheta\varphi)} \equiv f_\ell \sin\vartheta Y^{\ell,\vartheta} + g_\ell Y^\ell_{,\varphi} + s_\ell \frac{X^\ell}{\sin\vartheta} + t_\ell W^\ell = 0$$

$$\delta\mathcal{E}_{(-)} \equiv g_\ell \sin\vartheta Y^{\ell,\vartheta} - f_\ell Y^\ell_{,\varphi} - t_\ell \frac{X^\ell}{\sin\vartheta} + s_\ell W^\ell = 0$$

(theta,phi) and (-) components

slow_rot_grav_Kerr.nb

# #2: Gravitational pert. of Kerr BHs

Using the Clebsh-Gordan coefficients for the trigonometric functions and the orthogonality properties of the spherical harmonics

$$\delta\mathcal{E}_{(I)} \equiv (A_\ell^{(I)} + \tilde{A}_\ell^{(I)} \cos\vartheta) Y^\ell + B_\ell^{(I)} \sin\vartheta Y^\ell_{,\vartheta} + C_\ell^{(I)} Y^\ell_{,\varphi} = 0$$

$$\sin\vartheta Y^\ell_{,\vartheta} = \mathcal{Q}_{\ell+1} \ell Y^{\ell+1} - \mathcal{Q}_\ell (\ell+1) Y^{\ell-1} \qquad \Big\downarrow \qquad \int Y^\ell Y^{*\,\ell'} d\Omega = \delta^{\ell\ell'}$$

$$0 = \int d\Omega Y^{*\,\ell} \delta\mathcal{E}_{(I)} = A_\ell^{(I)} + imC_\ell^{(I)} + \mathcal{L}_0^{\pm 1} \tilde{A}_\ell^{(I)} + \mathcal{L}_1^{\pm 1} B_\ell^{(I)}$$

Each group can be reduced to a set of ODEs with couplings to the nearest-neighbour multipoles

slow_rot_grav_Kerr.nb

# #2: Gravitational pert. of Kerr BHs

slow_rot_grav_Kerr.nb

Axial modes to first order (neglecting the couplings)

$$\frac{d^2\psi^\ell}{dr_*^2} + \left[\omega^2 - \frac{4m\tilde{a}M^2\omega}{r^3} - V_{\rm axial}\right]\psi^\ell = 0$$

Polar modes to first order (neglecting the couplings)

$$\frac{d^2 Z^\ell}{dr_*^2} + \left[\omega^2 - \frac{4m\tilde{a}M^2\omega}{r^3} - V_{\rm polar}\right] Z^\ell = 0$$

Suitable for standard treatment (CFs, direct integration, WKB)
Check isospectrality
Can be extended to any order and to other metrics

# Conclusions & Questions

- Many open problems in BH perturbation theory
- Plenty of room for interesting projects
- Developing more versatile semianalytical methods
- Complementary to full-fledge numerical simulations
- Expansion in spherical harmonics: is it always possible/convenient?
- Ultra-long 1+1 evolutions for generic fields/background?
- Separation of variables in nonKerr/nonGR/D>4?

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MARIE CURIE ACTIONS

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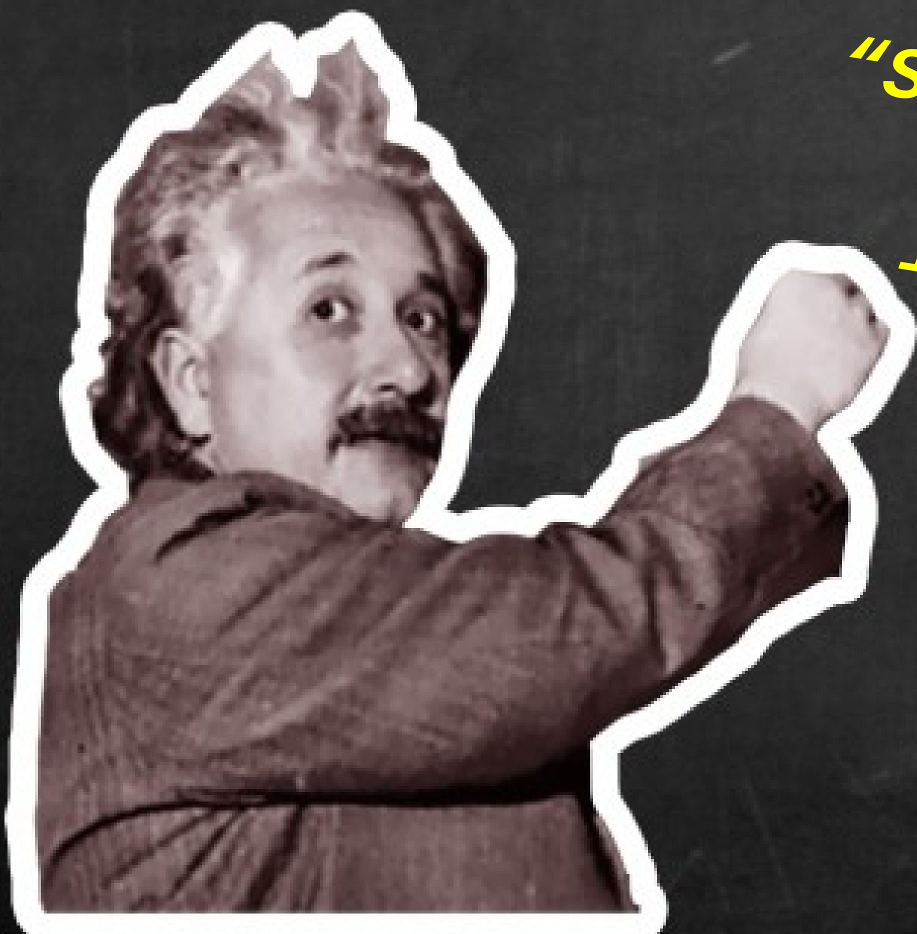

"Seriously, are you really planning to spend your life on my equations?!?"

***Obrigado!***