Free evolution formulations of electromagnetism

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Outline

Recap

In the first lecture we saw several notions of hyperbolicity, and that they are useful in different contexts. The moral of the story was three-fold:

- Strong hyperbolicity is good enough for the initial value problem, and is easy to check– there is no excuse not to!
- Symmetric hyperbolicity, or the energy method, is good for the IBVP, and is the preferred approach whenever it applies.
- The Laplace-Fourier method can be used to analyze well-posedness of the IBVP for PDEs that are strongly hyperbolic of constant multiplicity.

Gauge freedom

In this class we will apply these notions to electromagnetism, and will see that new complications arise. Model for GR.

- Qualitative difference between the Maxwell equations and yesterdays is gauge freedom. Work before definitions.
- Dirac's theory of constrained Hamiltonian systems says there is a relationship between the gauge freedom and constraints.
- Today: discuss Maxwell, but I want to impress upon you that the structure we discover in the equations of motion falls out *because of the Hamiltonian form*.

Outline

Hamiltonian and equations of motion

Hamiltonian for Electromagnetism,

$$H = \int_\Omega \tfrac{1}{2}\left[(\partial \times A)_i(\partial \times A)^i + E_i E^i\right] - \Phi\,\partial^i E_i\,\mathrm{d}V\,.$$

Canonical positions A_i, momenta $\pi^i = -E^i$. Curl is,

$$(\partial \times A)^i = \epsilon^{ijk}\partial_j A_k\,,$$

with ϵ^{ijk} the Levi-Cevita tensor. Hamilton's equations give,

$$\partial_t A_i = -E_i - \partial_i \Phi\,, \qquad \partial_t E^i = \big(\partial \times\ [\partial \times A]\big)^i\,,$$

Could work with the magnetic field $B^i = (\partial \times A)^i$, but would lose the analogy between GR and electromagnetism.

Constraints I

Momentum constraint: Divergence of the electric field vanishes,

$$M = -\partial_i \pi^i = \partial_i E^i = 0,$$

obtained by varying with respect to Φ, the gauge field. Time derivative of momentum constraint

$$\partial_t M = 0.$$

If start with constraint satisfying data, remain satisfied.

- Numerics? Specify initial data satisfying the constraint, and integrate up the equations of motion in time.
- Free-evolution: Main method in NR.
- Evolution equations can be treated with methods of last lecture.

Constraints II

- With free-evolution, constraint violation is inevitable because of numerical error.
- Numerical analyst: As we throw more computational power at the problem can we make errors arbitrarily small?
- Computational physicist: Are there enough computers around?
- Analyze the PDE properties of the system without assuming constraints satisfied. Computing in the larger phase space with violations.
- Different approach is to resolve the constraints after every time-step; constrained-evolution. Elliptic-hyperbolic, can't be treated with the methods of last lecture.

Gauge freedom and the pure gauge system I

EoMs invariant under

$$A_i \to A_i - \partial_i \psi \,,$$

with ψ some arbitrary scalar function. What is the difference in the time development with and without applying this to initial data? Pure gauge field ψ evolves with

$$\partial_t \psi = \Delta[\Phi] \,,$$

where $\Delta[\Phi]$ difference in Φ induced by the gauge change in the initial data.

Gauge freedom and the pure gauge system II

Can't quite start well-posedness analysis; the Hamiltonian does not determine the field Φ. Gauge choice:

$$\partial_t \Phi = -\mu \, \partial^i A_i ,$$

for Φ. Could take Φ apriori function, or satisfy elliptic equation. How will $\Delta[\Phi]$, evolve in time? Time derivative of the difference of the two Φ's

$$\partial_t \Delta[\Phi] = -\mu \partial^i \partial_i \psi \, .$$

Pure gauge system: closed subsystem for the evolution of the change! Relationship between hyperbolicity of this and the Maxwell equations? One complication first.

Expanded phase space

New constraint Z. EoM (somewhat) arbitrary, choose:

$$\partial_t Z = \partial_i E^i - \kappa\, Z\,,$$

If constraints Z and the M are initially satisfied then Z stay satisfied, provided Z doesn't break M.

- Strange. Why expand with *more* freedom to be wrong? PDE properties of problem affected favorably. Well-posedness imperative.
- What is Z? *Roughly* canonical momentum of the gauge field Φ. Normally see that construction only for Lorenz gauge $\mu = 1$. No reason to restrict like that.

Fully expanded EoMs

Full EoMs taken to be

$$\partial_t A_i = -E_i - \partial_i \Phi \,, \qquad \partial_t E^i = \left(\partial \times [\partial \times A]\right)^i + \partial^i Z \,,$$
$$\partial_t \Phi = -\mu \, [\partial^i A_i + Z] \,, \qquad \partial_t Z = M - \kappa \, Z \,.$$

Think about consequences of choices after well-posedness analysis! Constraints,

$$Z = 0 \,, \qquad M = \partial^i E_i = 0 \,.$$
$$\partial_t M = \partial^i \partial_i Z \,.$$

still closed! Free-evolution justified. Hyperbolicity of PG and Constraint subsystems inherited by full EoMs?

Outline

Strong hyperbolicity I

Use unit vector s^i to $2+1$ decompose the vectors,

$$\partial_s A_i = s_i[\partial_s^2\psi] - s_i Z + \perp_i^A [\partial_s A_A]\,, \qquad E_i = s_i E_s + \perp_i^A E_A\,,$$

with projection operator $\perp^i_j = \delta^i{}_j - s^i s_j$. P^s splits into blocks. Read off,

$$\partial_t[\partial_s^2\psi] \simeq -\partial_s[\partial_s\Phi]\,, \qquad \partial_t[\partial_s\Phi] \simeq -\mu\,\partial_s[\partial_s^2\psi]\,,$$
$$\partial_t Z \simeq \partial_s E_s\,, \qquad \partial_t E_s \simeq \partial_s Z\,,$$
$$\partial_t[\partial_s A_A] \simeq \partial_s E_A\,, \qquad \partial_t E_A \simeq \partial_s[\partial_s A_A]\,,$$

Both PG and C principal symbols inherited! No coincidence. Can be shown for constrained Hamiltonian systems. Could have “constraint” variables in gauge.

Strong hyperbolicity II

Principal symbol of each block,

$$P^s_{\mathcal{G}} = \begin{pmatrix} 0 & -1 \\ -\mu & 0 \end{pmatrix}, \qquad P^s_{\mathcal{C}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad P^s_{\mathcal{P}} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

- PG block has eigenvalues $= \pm\sqrt{\mu}$. Weak hyperbolicity if $\mu \geq 0$. Strong if $\mu > 0$.
- Others eigenvalues ± 1, are diagonalizable.
- Characteristic variables,

$$[\partial_s \Phi] \mp [\partial_s^2 \psi], \qquad E_s \pm Z, \qquad E_A \mp [\partial_s A_A].$$

Notebook `Maxwell_Strong.nb`.

Symmetric hyperbolicity I

- Which gauge conditions result in symmetric hyperbolic system? Every gauge that is strongly hyperbolic. Special to Maxwell!
- Given symmetric hyperbolic pure gauge, not known that there is a symmetric hyperbolic formulation.
- Are strongly, not symmetric hyperbolic formulations worse than symmetric ones? Sometimes choice that “works” might be the mathematically weaker one.
- With bigger gauge freedom, could be that choice of useful pure gauge that is is not symmetric hyperbolic. Expectation is that we can't use that gauge to build symmetric formulation.

Symmetric hyperbolicity II

Fully second order form,

$$\partial_t^2\Phi \simeq \mu\gamma^{ij}\partial_i\partial_j\Phi\,, \qquad \partial_t^2 A_i \simeq \gamma^{jk}\partial_j\partial_k A_i + \left(\tfrac{1}{\mu}-1\right)\partial_t\partial_i\Phi\,.$$

In Lorenz gauge, each variable satisfies wave equation. Principal part matrix

$$A^p{}_{i\,k}{}^{j\,l} = \begin{pmatrix} 0 & 0 & \delta^p{}_i & 0 \\ 0 & 0 & 0 & \delta^p{}_i\delta^l{}_k \\ \mu\,\gamma^{pj} & 0 & 0 & 0 \\ 0 & \gamma^{pj}\delta^l{}_k & \left(\frac{1}{\mu}-1\right)\delta^p{}_k & 0 \end{pmatrix},$$

Ansatz for energy density ϵ with

$$\epsilon = u^\dagger_{jm}H^{ij\,km}u_{ik}\,, \qquad u_{ik} = (\partial_i\Phi\,,\partial_i A_k\,,\partial_0\Phi\,,\partial_0 A_k)^\dagger\,,$$

Symmetric hyperbolicity III

... with parametrized ansatz for $H^{ij\,km} =$

$$\begin{pmatrix} h^1_{11}\gamma^{ij} & 0 & 0 & h^1_{14}\gamma^{ik} \\ 0 & h^1_{22}\gamma^{ij}\gamma^{km} + 2\,h^2_{22}\gamma^{k(i}\gamma^{j)m} & h^1_{23}\,\gamma^{im} & 0 \\ 0 & h^1_{23}\,\gamma^{jk} & h^1_{33} & 0 \\ h^1_{14}\gamma^{jm} & 0 & 0 & h^1_{44}\gamma^{km} \end{pmatrix}$$

Imposing Hermiticity of $S_i\,H^{ij\,mn}\,A^p{}_{j\,m}{}^{k\,l}\,s_p\,S_k =$

$$\begin{pmatrix} 0 & h^1_{14}s^l & h^1_{11} + h^1_{14}(\frac{1}{\mu} - 1) & 0 \\ h^1_{23}\,\mu\,s^n & 0 & 0 & h^1_{22}\gamma^{ln} + 2h^2_{22}s^l s^n \\ h^1_{33}\,\mu & 0 & 0 & h^1_{23}\,s^l \\ 0 & h^1_{44}\,\gamma^{ln} & \frac{1}{\mu}\big(h^1_{44}(1-\mu) + h^1_{14}\,\mu\big)s^n & 0 \end{pmatrix},$$

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Symmetric hyperbolicity IV

... for every spatial vector s^i, gives,

$$h^1_{14} = h^1_{23}\,, \qquad h^1_{33} = h^1_{11}\,, \qquad h^2_{22} = 0\,, \qquad h^1_{44} = h^2_{22}\,,$$

for $\mu = 1$, and otherwise,

$$\begin{aligned} h^1_{14} &= h^1_{23}\,\mu\,, & h^1_{33} &= \frac{h^1_{11} + (1-\mu)h^1_{23}}{\mu}\,, & & \\ h^1_{22} &= h^1_{23}\,\mu\,, & h^2_{22} &= 0\,, & h^1_{44} &= h^1_{23}\,\mu\,, \end{aligned}$$

Last thing: positivity. Lorenz;

$$h^1_{11} = 1\,, \qquad h^1_{23} = 0\,, \qquad h^1_{22} = 1\,, \qquad h^1_{22} = 0\,,$$

does the trick. Generic case positive with $h^1_{23} < \frac{1}{2+\mu}$.

Symmetric hyperbolicity V

Comments:

- Could write down MDBCs that render the IBVP well-posed. They would still not be satisfactory. Why?
- There are numerical methods, called summation by parts, that can use the energy to guarantee stability in numerical approximation.
- See `Maxwell_Symmetric.nb`. Needs `xTensor`.

Laplace-Fourier method I

Crucial complication: constraints. Complicates the analysis of the IBVP. MDBCs will pump constraint violation into the domain. Require:

- Well-posedness (goes without saying).
- Constraint preservation. Interested in solutions to Maxwell, so BCs had better respect constraints. In numerics, ok for BCs to cause violation if converges.
- Radiation and gauge control. BCs should control the physical radiation in appropriate way.

Laplace-Fourier method II

Performing the Laplace-Fourier transform gives

$$
\begin{aligned}
s^2\,\hat{\Phi} &= \mu\,[\partial_x^2 - \omega^2]\hat{\Phi}\,,\\
s^2\,\hat{A}_x &= [\partial_x^2 - \omega^2]\hat{A}_x + \left(\tfrac{1}{\mu} - 1\right)s\,\partial_x\hat{\Phi}\,,\\
s^2\,\hat{A}_{\hat{\omega}} &= [\partial_x^2 - \omega^2]\hat{A}_{\hat{\omega}} + \left(\tfrac{1}{\mu} - 1\right)\,i\,\omega\,s\,\hat{\Phi}\,,\\
s^2\,\hat{A}_{\hat{\nu}} &= [\partial_x^2 - \omega^2]\hat{A}_{\hat{\nu}}\,,
\end{aligned}
$$

vector $\hat{A}_i$ has been decomposed

$$\hat{A}_i = \hat{x}_i\hat{A}_{\hat{x}} + \hat{\omega}_i\hat{A}_{\hat{\omega}} + \hat{\nu}_i\hat{A}_{\hat{\nu}}.$$

with $\hat{x}^i$, a unit vector in the x-direction, $\hat{\omega}^i$ a unit vector in the ω^i direction, and $\hat{\nu}^i$ a unit vector orthogonal to both $\hat{x}^i$ and $\hat{\omega}^i$.

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Laplace-Fourier method III

Reduce to first order,

$$
\begin{aligned}
\partial_x \hat{\Phi} &= \kappa\, D\hat{\Phi}\,, \quad \partial_x D\hat{\Phi} = -\kappa\, \tau'_{+\mu}\, \tau'_{-\mu} \hat{\Phi}\,,\\
\partial_x \hat{A}_x &= \kappa\, D\hat{A}_x\,, \quad \partial_x D\hat{A}_x = -\kappa\, \tau'_{+}\, \tau'_{-}\, \hat{A}_x + \kappa \left(1 - \tfrac{1}{\mu}\right) s'\, D\hat{\Phi}\,,\\
\partial_x \hat{A}_{\hat{\omega}} &= \kappa\, D\hat{A}_{\hat{\omega}}\,, \quad \partial_x D\hat{A}_{\hat{\omega}} = -\kappa\, \tau'_{+}\, \tau'_{-}\, \hat{A}_{\hat{\omega}} + i\,\omega' \kappa \left(1 - \tfrac{1}{\mu}\right) s'\, \hat{\Phi}\,,\\
\partial_x \hat{A}_{\hat{\nu}} &= \kappa\, D\hat{A}_{\hat{\nu}}\,, \quad \partial_x D\hat{A}_{\hat{\nu}} = -\kappa\, \tau'_{+}\, \tau'_{-}\, \hat{A}_{\hat{\nu}}\,.
\end{aligned}
$$

with

$$
\tau'_{\pm} = \pm\sqrt{s'^2 + \omega'^2} \qquad \tau'_{\pm\mu} = \pm\sqrt{\frac{s'^2}{\mu} + \omega'^2},
$$

Laplace-Fourier method IV

The general L_2 solution at $x = 0$,

$$\hat{\Phi} = \sigma_\Phi \,, \qquad D\hat{\Phi} = \tau'_{-\mu}\sigma_\Phi \,,$$

$$\hat{A}_x = -\frac{\sigma_Z}{\tau'_-} - \frac{\tau'_{-\mu}\sigma_\Phi}{s'} - \frac{i\,\omega'\,\sigma_{A_{\hat{\omega}}}}{\tau'_-} \,,$$

$$D\hat{A}_x = -\sigma_Z - \frac{\tau'^2_{-\mu}\sigma_\Phi}{s'} - i\,\omega'\,\sigma_{A_{\hat{\omega}}} \,,$$

$$\hat{A}_{\hat{\omega}} = \sigma_{A_{\hat{\omega}}} - \frac{i\,\omega'\,\sigma_\Phi}{s'} \,,$$

$$D\hat{A}_{\hat{\omega}} = \tau'_-\sigma_{A_{\hat{\omega}}} - \frac{i\,\omega'\,\tau'_{-\mu}\,\sigma_\Phi}{s'} \,,$$

$$\hat{A}_{\hat{\nu}} = \sigma_{A_{\hat{\nu}}} \,, \qquad D\hat{A}_{\hat{\nu}} = \tau'_-\sigma_{A_{\hat{\nu}}} \,,$$

Sum of gauge σ_Φ, constraint σ_Z and physical $\sigma_{A_{\hat{\omega}}}, \sigma_{A_{\hat{\nu}}}$ parts.

Laplace-Fourier method V

- Gauge field Φ and the constraint Z satisfy wave equations, the obvious choice is something like a Sommerfeld condition

$$[\partial_t - \sqrt{\mu}\,\partial_x]^2\Phi \hat{=} \partial_t g_\Phi\,, \qquad [\partial_t - \partial_x]Z \hat{=} \partial_t g_Z\,,$$

  on each. Applications choose $g_Z = 0$.

- Electric and magnetic fields gauge invariant, unambiguously represent field strength,

$$[\partial_t - \partial_x]\big(\partial_t A_A + \partial_A\Phi - \partial_x A_A + \partial_A A_x\big) = \partial_t g_A\,.$$

  [ϕ_0 in the terminology of Teukolsky].

Laplace-Fourier method VI

LF transforming and solving gives, for example:

$$\hat{\Phi} = \frac{s'\,\hat{g}_\Phi}{\left(s' + \sqrt{s'^2 + \mu\,\omega'^2}\,\right)^2}\,,$$

$$\hat{A}_x = \frac{i\,\omega'\,\hat{g}_{\hat{\omega}}}{\left(s' + \sqrt{s'^2 + \omega'^2}\,\right)^2} + \frac{\hat{g}_Z}{s' + \sqrt{s'^2 + \omega'^2}} + \frac{\sqrt{s'^2 + \mu\,\omega'^2}\,\hat{g}_\Phi}{\sqrt{\mu}\left(s' + \sqrt{s'^2 + \mu\,\omega'^2}\,\right)^2}\,,$$

$$\hat{A}_{\hat{\omega}} = \frac{\sqrt{s'^2 + \omega'^2}\,\hat{g}_{\hat{\omega}}}{\left(s' + \sqrt{s'^2 + \omega'^2}\,\right)^2} - \frac{i\,\omega'\,\hat{g}_Z}{\left(s' + \sqrt{s'^2 + \omega'^2}\,\right)^2} - \frac{i\,\omega'\,\hat{g}_\Phi}{\left(s' + \sqrt{s'^2 + \mu\,\omega'^2}\,\right)^2}\,,$$

terms like $s' + \sqrt{s'^2 + \omega'^2}$ are bounded away from zero.
See notebook `Maxwell_LF.nb`.

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Laplace-Fourier method VII

Final comments:

- With Lorenz gauge strong well-posedness can be shown for constraint preserving boundary conditions using the energy method with a special symmetrizer,
- Alternatively with the Kreiss-Winicour cascade approach.
- I *think* this is the first time BS shown for our family of gauges.

Outline

Conclusions

Looked at formulations of Electromagnetism suitable for free-evolution:

- For every strongly hyperbolic pure gauge, built a formulation which was itself strongly hyperbolic. Likewise for symmetric hyperbolicity.
- Used Laplace-Fourier method to investigate boundary stability with CPBCs.
- To understand the ins-and-outs I recommend that you study the mathematica notebooks in tandem with the lecture notes.

Some references

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