Well-posedness of evolution PDEs

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Outline

A confession, An excuse

My aim is to introduce you to the basic techniques used in the analysis of evolution PDEs. Why bother?

- Confession: I love wave equations.
- Want solutions of GR to arbitrary accuracy.
- Possible only if the PDE problem at hand is well-posed.
- NR carried out with ill-posed PDEs for a long-time. Suitable formulations crucial in solution of the two-body problem. Second lecture examines Maxwell equations.

Excuse: Can only hope to scratch the surface. Therefore present the methods as a tool-box containing three key items, plus an adapter for second order systems.

Prototype PDEs

Most crude classification:

- *Elliptic PDEs* have no notion of time. Often arise as the steady-state solutions. Prototype well-posed elliptic problem is the boundary value problem for the Laplace equation.
- *Parabolic PDEs* describe diffusive processes. Intrinsic notion of time. Signals travel at infinite speed. Prototype is IVP for the heat equation.
- *Hyperbolic PDEs* describe causal processes; there is an intrinsic notion of time. Signal speed *finite*. Prototype is IVP the wave equation.

The prototype well-posed problems specify both a simple PDE, the type of data, and the domain that is appropriate.

Outline

The initial value problem

Evolution PDE system,

$$\partial_t u = A^p \partial_p u + B\, u\,, \tag{1}$$

- Highest derivatives are called the principal part. Refer to A^p as the principal matrix.
- Assume matrices A^p and B are constant in both time and space. *Linear, constant coefficient system*.

The initial value, or Cauchy problem, is the following: specify data $u(0, x^i) = f(x^i)$ at time $t = 0$, with spatial coordinates x^i. What is the solution $u(t, x^i)$ later? In other words data is specified *everywhere* in space.

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Well-posedness

If there exist constants K and α, such that for all initial data we have the estimate,

$$||u(t,\cdot)|| \leq K e^{\alpha\, t} ||f(\cdot)||\,,$$

with the L_2 norm,

$$||g(\cdot)||^2 = \int_{\mathbf{R}^3} g^\dagger\, g\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\,,$$

then the initial value problem for (1) is called well-posed. Initial data are L_2.

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Strong hyperbolicity

Unit spatial vector s^i, matrix

$$P^s = A^s = A^p s_p \,,$$

is the principal symbol. System (1) is

- Weakly hyperbolic: if for every s^i, P^s has real eigenvalues.
- Strongly hyperbolic: If weakly hyperbolic, and a constant K exists s.t for every s^i, P^s has a complete set of eigenvectors, with

$$|T_s| + |T_s^{-1}| \leq K \,,$$

  where T_s has eigenvectors of P^s as columns.

Verifying amounts to doing a little linear algebra.

Characteristic variables

Interpretation of strong hyperbolicity?

- Components of the vector $v = T_s^{-1} u$ are called the characteristic variables in the s^i direction.
- Up to non-principal terms and derivatives transverse to s^i direction they satisfy advection equations,

$$\partial_t v = \Lambda_s \partial_s v + (T_s^{-1} A^A T_s) \partial_A v + (T_s^{-1} B T_s) v \,,$$

with speeds equal to the eigenvalues of the principal symbol.

Well-posed $\iff$ Strongly hyperbolic

Strong hyperbolicity is equivalent to well-posedness of the initial value problem. Idea of the proof?

- Fourier transform in space.
- Consider time derivative of $\tilde{u}^\dagger H_s \tilde{u}$ where $H_s = T_s^{-\dagger} T_s^{-1}$.
- Use Parsevals relationship and $K^{-2}\mathbf{1} \leq H_s \leq K^2\mathbf{1}$ to show estimate in physical space.

[Detail in Lecture notes.]

Outline

The initial boundary value problem

Consider now the PDE system similar to what we had before (1),

$$\partial_t u = A^p \partial_p u + F(t, x^i)\,, \tag{2}$$

but now rather than considering solutions on $\mathbb{R}^3$, let us consider trying to find solutions on the half-space $x^1 = x \geq 0$. Specify:

- Initial data $u(0, x^i) = f(x^i)$ on the spatial domain.
- Boundary conditions $L\,u(t, x^i) \hat{=} g(t, x^A)$, where A denotes that data depends only on $x^2 = y$ and $x^3 = z$, with L some matrix whose form we will discuss.

Strong well-posedness

Let $||\cdot||_\Sigma$ denote the L_2 norm on the half-space, and $||\cdot||_{\partial\Sigma}$ in the boundary plane $x=0$. If there exists a constant K_T for every T, independent of the given data and forcing terms, such that for every $0\leq t\leq T$, we have the estimate,

$$
\begin{aligned}
&||u(t,\cdot)||_\Sigma^2+\int_0^t ||u(t',\cdot)||^2_{\partial\Sigma}\,\mathrm{d}t' \\
\leq &K_T^2\left[||f||^2+\int_0^t\left(||F(t',\cdot)||_\Sigma^2+||g(t',\cdot)||^2_{\partial\Sigma}\right)\mathrm{d}t'\right],
\end{aligned}
$$

then we call the problem strongly-well posed. One sometimes sees this definition given without the second term on the left hand side.

Symmetric hyperbolicity

The system is symmetric hyperbolic if there exists a matrix, H such that

- H is Hermitian, positive definite,
- $HA^p s_p$ is Hermitian for every unit spatial vector s^p,

Relationship with strong hyperbolicity? Comparing H with H_s:

- For symmetric hyperbolic systems the symmetrizer may not depend on s^p.
- Every symmetric hyperbolic system is strongly hyperbolic, but not vice-versa.

Maximally dissipative boundary conditions I

Since every symmetric hyperbolic system is strongly hyperbolic,

$$T_x^{-1} P^x T_x = \Lambda_x = \begin{pmatrix} \Lambda_x^I & 0 \\ 0 & \Lambda_x^{II} \end{pmatrix},$$

assume that $\Lambda_x^I > 0$ and $\Lambda_x^{II} < 0$. Partition characteristic variables similarly $v = (v_I, v_{II})^\dagger$. Then,

$$u^\dagger H A^x u = v_I^\dagger H^I \Lambda_x^I v_I + v_{II}^\dagger H^{II} \Lambda_x^{II} v_{II} \geq \gamma\, v_I^\dagger H^I v_I + v_{II}^\dagger H^{II} \Lambda_x^{II} v_{II}\,,$$

for some $\gamma > 0$, and we write,

$$T_x^\dagger\, H\, T_x = \begin{pmatrix} H^I & 0 \\ 0 & H^{II} \end{pmatrix},$$

with $\delta^{-1}\, I \leq H^I, H^{II} \leq \delta\, I$ for some $\delta > 0$. Block diagonal form necessary because $H\, A^x$ is symmetric.

Maximally dissipative boundary conditions II

We restrict from $L\,u \hat{=} g$ to consider boundary conditions of the form

$$v_{II} \hat{=} \kappa\, v_I + g\,,$$

where $\hat{=}$ denotes equality in the boundary. We either assume that

$$H^I \Lambda^I_x + \kappa^\dagger\, H^{II} \Lambda^{II}_x\, \kappa > 0\,,$$

or that

$$|\kappa|^2 \leq \frac{1}{2}\,\gamma\left(2|\Lambda^{II}_x| + \gamma\right)^{-1}.$$

Strong well-posedness of symmetric hyperbolic systems with MDBCs

Consider the time derivative of the energy $E^2 = \int_\Sigma \epsilon \, \mathrm{d}V$ with $\epsilon = u^\dagger H u$, which, using integration by parts,

$$\partial_t E^2 = \int_\Sigma (u^\dagger \, H \, F + F^\dagger \, H \, u) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z - \int_{\partial\Sigma} u^\dagger H A^x u \, \mathrm{d}y \, \mathrm{d}z \,,$$

if we choose:

- the first type of boundary conditions we have the estimate without the boundary term on the left,
- the second type, after messing around a bit, one obtains strong well-posedness.

Discussion:

- The energy method is easier than what follows, so should be used whenever possible.
- It is fantastically powerful, when it applies, and can be used to estimate long-term behavior of solutions to variable coefficient and non-linear problems.
- If system is strongly, but not symmetric hyperbolic, maximally dissipative boundary conditions *do not* guarantee well-posedness.
- Although it is not easy to construct PDEs that are strongly but not symmetric hyperbolic, in GR that *is* the typical situation. Only hope for the IBVP is the Laplace-Fourier method.

Outline

The initial boundary value problem

Consider once again an evolution system of the form (2) ,

$$\partial_t u = A^p \partial_p u + F(t, x^i)\,, \qquad (3)$$

on the half-space $x \geq 0$.

Comparison with energy method:

- Again assume that the system is strongly hyperbolic with non-vanishing speeds. [Convenience only.]
- In contrast to the previous case, we take vanishing initial data $u(0, x^i) = 0$,
- Maintain inhomogeneous boundary conditions of the form $L\, u(t, x^i) \hat{=} g(t, x^A)$.

Strongly well-posedness in the generalized sense

We say the the system is *strongly well-posed in the generalized sense* if there exists a constant K_T for every T, independent of the data and forcing terms, such that for every $0 \leq t \leq T$, we have the estimate,

$$\int_0^t ||u(t',\cdot)||_{\Sigma}^2 \,\mathrm{d}t' + \int_0^t ||u(t',\cdot)||_{\partial\Sigma}^2 \,\mathrm{d}t'$$
$$\leq K_T^2 \int_0^t \left(||F(t',\cdot)||_{\Sigma}^2 + ||g(t',\cdot)||_{\partial\Sigma}^2\right) \mathrm{d}t' \,,$$

for all boundary data g. The terminology *in the generalized sense* means that we have restricted to trivial initial data, and that we have an estimate on the integral in time of the solution on the left hand side of the inequality.

Laplace-Fourier Transform I

Taking the system (3) and Fourier transforming in the y and z gives one-dimensional IBVP for every ω^A. Need representation of the solutions:

- We furthermore Laplace transform in time. The total transformation is,

$$\hat{u}(s,x,\omega^A) = \frac{1}{2\,\pi}\int_0^\infty \int_{\mathbf{R}^2} e^{-s\,t + i\,\omega_A x^A}\, u(t,x,x^A)\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t\,,$$

  with $s = \eta + i\,\xi$ and $\eta > 0$.

- The inverse transform requires a contour integral, along the line $s = \eta + i\,\xi$ with $\eta > 0$ fixed, which fortunately we never have to compute explicitly, because we have Parsevals relation and a theorem to follow.

Laplace-Fourier Transform II

Under this transformation we can rewrite the equations of motion as an ODE system

$$\partial_x \hat{u} = M\,\hat{u} + \hat{G}\,,$$

with symbol and sources,

$$M = (A^x)^{-1}(s\,\mathbf{1} - i\,\omega\,A^{\hat{\omega}})\,, \qquad \hat{G} = (A^x)^{-1}\hat{F}\,,$$

where we write $\omega^A = |\omega|\,\hat{\omega}^A = \omega\,\hat{\omega}^A$, and for later convenience define $k = \sqrt{|s|^2 + \omega^2}$, and the normalized frequencies $s' = s/k$ and $\omega' = \omega/k$.

General L_2 solution

Take ODE without forcing terms $\hat{F}$. Assume that A^x in diagonal form Λ_x. If negative block of the partition, $\Lambda_x^{II} < 0$ has dimensions $(d \times d)$, M has d eigenvalues with negative real part κ_i. General L_2 solution of form,

$$\hat{u}(s, x, \omega^A) = \sum_i^d \sigma_i \, e^{\kappa_i \, x} \, \Phi(x) \, v_i \, ,$$

with v_i the eigenvector or generalized eigenvector associated with κ_i, and $\Phi(x)$ appropriate polynomial.

- Unfortunately strong hyperbolicity does not tell us anything about the eigenvectors of M. Allow for polynomial ansatz.
- The complex coefficients σ_i are to be solved for by plugging general solution into the boundary conditions.

Boundary conditions and boundary stability

Consider same BCs as for symmetric hyperbolic systems. Under Laplace-Fourier transform,

$$\hat{u}^{II} \hat{=} \kappa \, \hat{u}^{I} + \hat{g}.$$

Plugging the general ODE solution into gives a set of linear equations for σ_i,

$$S(s,\omega)^j{}_i \, \sigma_j = \hat{g}_i(s,\omega)\,,$$

for the coefficients σ_i. If they can be solved and there exists a $\delta > 0$ with,

$$|\hat{u}(s,0,\omega^A)| \, < \, \delta \, |\hat{g}(s,\omega^A)|\,,$$

for every s and ω with $\eta \geq 0$ then the system is called boundary stable.

Kreiss's symmetrizer Theorem

The main result of the theory is that if the system is

- symmetric hyperbolic,
- or strongly hyperbolic of constant multiplicity,
- and boundary stable,

then it is strongly well-posed in the generalized sense.

Outline

FT2S Systems I

Very often in physics applications we are not given first order PDE systems like (1), but rather equations that are first order in time and second order in space, like the wave equation,

$$\partial_t \phi = \pi \,, \qquad \partial_t \pi = \Delta \phi \,.$$

Equations of motion from a Hamiltonian fall out this way naturally. To analyze well-posedness of such equations we could reduce them to first order by introducing new variables $d_i = \partial_i \phi$ and rewriting everything as a first order system to which the results we've been discussing apply.

FT2S Systems II

Fortunately that is not necessary, because conditions under which "good" reductions exist have been analyzed. Consider the second order in space evolution system,

$$\partial_t v = A_1^i \partial_i v + A_1 v + A_2 w + F_v \,,$$
$$\partial_t w = B_1^{ij} \partial_i \partial_j v + B_1^i \partial_i v + B_2^i \partial_i w + B_2 w + F_w \,,$$

as in the first order case assume that the coefficient matrices are constant. We call the matrix

$$A^p{}_i{}^j = \begin{pmatrix} A_1^j \delta^p{}_i & A_2 \delta^p{}_i \\ B_1^{pj} & B_2^p \end{pmatrix} ,$$

the principal part matrix of the system.

Strong hyperbolicity and characteristic variables I

Given an arbitrary unit spatial vector s^i, we call the matrix,

$$P^s = S^i A^p{}_i{}^j S_j s_p = \begin{pmatrix} A_1^j s_j & A_2 \\ B_1^{pj} s_p s_j & B_2^p s_p \end{pmatrix}.$$

with the abbreviation

$$S_i = \begin{pmatrix} s_i & 0 \\ 0 & \mathbf{1} \end{pmatrix},$$

the principal symbol of the system.

- Hereon definition of weak and strong hyperbolicity identical.
- This condition is equivalent to well-posedness of the initial value problem. Norm contains spatial derivatives of v.
- Equivalent to strongly hyperbolic first order reduction.

Strong hyperbolicity and characteristic variables I

The characteristic variables of the FT2S system are defined to be the components of

$$u = T_s^{-1} \begin{pmatrix} \partial_s v \\ w \end{pmatrix} .$$

Strong hyperbolicity implies the existence of a complete set of characteristic variables just like in the first order case.

The energy method I

We call a symmetric matrix H^{ij}, independent of s^i, with

$$S_i H^{ij} A^p{}_j{}^k s_p S_k = (S_i H^{ij} A^p{}_j{}^k s_p S_k)^\dagger.$$

for every spatial vector s^i, a candidate symmetrizer. A positive definite candidate symmetrizer is called a symmetrizer. A system with a symmetrizer is called symmetric hyperbolic. The symmetrizer can be used to define a conserved energy, at least up to non-principal terms. The energy density is,

$$\begin{aligned}\epsilon &= (\partial_i v, w)\, H^{ij} (\partial_j v, w)^\dagger \\ &= \partial_i v^\dagger\, H^{ij}_{vv} \partial_j v + \partial_i v^\dagger\, H^i_{vw} w + w\, H^{j\,\dagger}_{vw} \partial_j v^\dagger + w^\dagger\, H_{ww} w\,.\end{aligned}$$

The energy method II

- In simple cases, say for the wave equation, this "PDEs energy" may correspond to a true physical energy.
- In general a Hamiltonian for the system guarantees a candidate symmetrizer, but not a symmetrizer.
- The definition is equivalent to the existence of a symmetric hyperbolic first order reduction. Showing this for higher order derivative systems is tricky.
- Maximally dissipative boundary conditions can be defined for second order in space systems in a similar way to first order systems, and can again be used to guarantee estimates of the solution including boundary data.

The Laplace-Fourier method I

The Laplace-Fourier method applies to the second order in space system straightforwardly. Assume that there are as many v's as w's, and that A_2 is invertible. Grouping the non-principal terms together, we Laplace-Fourier transform,

$$
\begin{aligned}
s^2\,\hat{v} = {} & A^{xx}\partial_x\partial_x\hat{v} + 2\,i\,\omega\,A^{x\hat{\omega}}\partial_x\hat{v} - \omega^2\,A^{\hat{\omega}\hat{\omega}}\,\hat{v} \\
& + s\,B^x\partial_x\hat{v} + i\,\omega\,B^{\hat{\omega}}\,\hat{v} + \hat{F}\,,
\end{aligned}
$$

with

$$
A^{ij} = A_2B_1^{ij} - A_2B_2^{(i}A_2^{-1}A_1^{j)} \qquad B^i = A_1^i + A_2B_2^iA_2^{-1}\,,
$$

and A^{ij} symmetric in i and j.

The Laplace-Fourier method II

Introduce the reduction variables $D\hat{v} = k^{-1}\,\partial_x \hat{v}$,

$$\partial_x \hat{u} = M\,\hat{u} + \hat{G}\,,$$

with the symbol,

$$M(s,\omega^A) = k\begin{pmatrix} 0 & \mathbf{1} \\ A & B \end{pmatrix}.$$

with the lower two blocks given by,

$$A = (A^{xx})^{-1}[s'^2\,\mathbf{1} + \omega'^2\,A^{\hat{\omega}\hat{\omega}}]\,, \qquad B = (A^{xx})^{-1}[s'\,B^x + 2\,i\,\omega'\,A^{x\hat{\omega}}]\,.$$

This type reduction is called a pseudo-differential reduction to first order. Construct the general solution to the ODE and consider boundary stability as before. Norms contain spatial derivatives.

Outline

Summary

I have:

- Given the definitions of well-posedness and hyperbolicity for initial and initial boundary value problems for first order systems,
- Sketched how they are extended to second order in space. I recommend that you take a look at the notes for an expanded version of the lecture, and last years living review article by Sarbach and Tiglio for real detail.

Tomorrow we will see these methods applied to Electromagnetism as a free-evolution system, a satisfactory model for General Relativity.

Some references

The lecture notes contain more references, but here I want to draw your attention to:

- Initial boundary value problems and the Navier-Stokes equations, Kreiss and Lorenz, 1989
- Time dependent problems and difference methods. Gustaffson, Kreiss and Oliger, 1995.
- Various papers. Gundlach and Martín-García. Second order in space systems.
- Continuum and discrete Initial-Boundary Value Problems and Einstein's Field Equations. Sarbach and Tiglio.